UNIVERSITY OF NIŠ
The scientific journal **FACTA UNIVERSITATIS**
Series: **Working and Living Environmental Protection** Vol. 1, No 2, 1997, pp. 65 - 72
Editor of series: Ljiljana Rašković, e-mail: ral@kalca.junis.ni.ac.yu
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http:// ni.ac.yu/Facta

DYNAMICS OF THE DUFFING OSCILLATOR WITH IMPACTS

UDC:534.64

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Abstract. *Single degree of freedom system, which is described with the Duffing differential equation, with rigid arrester was studied on the basis of the non-linear systems theory and stereo-mechanical impact theory. The impact vibrations in Duffing oscillator subjected to harmonics excitation in the case of the unilateral rigid arrester and in the case of the symmetrical bilateral arrester were determined numerically by approximate method Runge - Kutta of the fourth order. Turbo pascal programs were composed and the computer graphics was used. The qualitative analyses of the global dynamic stability of the non-linear impact systems having unilateral or symmetrical bilateral arrester was carried out on the basis of phase portraits and of two-dimensional mapping.*

Key words: *vibroimpact motion, unilateral and symmetrical rigid arrester, stereo-mechanical impact theory, phase portrait, two dimensional mapping.*

1. INTRODUCTION

Vibroimpact motion is used in many machines for carrying out work processes. Some typical machines in which the vibroimpact regime forms the basis of work processes are hammers, presses, rammers, stampers, etc. For the vibroimpact machines' work processes it is important to achieve regulated periodic motion, that is, stationary periodic vibroimpact regimes. The transition motion regimes occur from the moment a machine is switched on to the moment of establishing a stationary regime; it is what makes their examination worth while. The dynamic model used for describing these machines is an oscillator with the rigid fixed unilateral arrester. Of the same interest is the study of the dynamic oscillator model that has an impact upon the rigid fixed bilateral arrester used for modeling vibroimpact models characteristic for moveable mechanisms such as joints, kinetic couples and other transmission connectors in which clearances appear.

Received September 28, 1998

The vibroimpact processes in dynamic systems described by the linear differential equations with adjoined impact conditions, according to the stereomechanical theory of impact, have been studied by "accurate" analytical method (adjustment method) [1, 2, 5], by approximate analytical methods (method of equivalent linearization, by transformation of variables, etc.) [1,13]. The application of the analytical methods is very complex and restricted in view of the fact that the solutions are of transcendent type, that is, they cannot be found in a closed form since the oscillatory motions are interrupted by impacts. The modern methods for solving these problems are based upon the application of computer graphics' numerical methods with geometrical data interpretation [9,10,11,12]. The referential literature shows that the vibroimpact systems in which oscillatory motions are described by the linear differential equations and the impact conditions adjoined to them according to the stereomechanical theory of impact have been studied.

The problem of a non-linear oscillator, described by the Duffing differential equation, excited by the periodic force, in the case of unilateral and bilateral symmetrical rigid arrester has been discussed in this paper together with inclusion of respective impact conditions according to the stereomechanical impact theory of impact. The differential equations' system is written in the form of autonomous differential equations' system depending on dimensionless parameters. By means of the Runge-Kutt approximate method the phase portraits are obtained by means of the computer graphics in the Turbo-Pascal; they serve as the basis from which it is possible to examine dynamic behavior of the described vibroimpact systems.

2. DYNAMIC MODELS

This paper deals with the Duffing oscillator that consists of oscillatory mass m, a spring of non-linear elasticity force $F_e = -(kz + nz^3)$ and a damper of the attenuating force $F_B = -cz'$, where $z' = dz/d\tau$ (τ- time). The oscillatory mass is acted upon by the periodic force $F\cos(\nu\tau)$. In the first case, mass of the non-linear oscillator strikes upon a rigid fixed unilateral arrester placed at distance z_0 from its equilibrium position - the dynamic model is represented in Fig. 1a, while in the second case (b), in Fig. 1b, mass of the non-linear oscillator strikes upon a rigid fixed bilateral arrester placed symmetrically with respect to the equilibrium position of the oscillatory mass at distance $\pm z_0$.

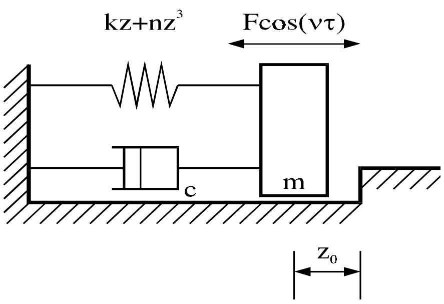


Fig. 1a

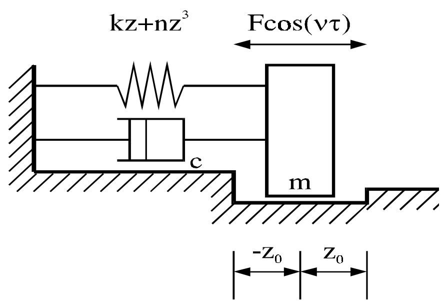


Fig. 1b

The differential equation of the Duffing oscillator motion is of the form:

$$mz'' = F_B + F_e + F\cos(\nu\tau) \tag{1}$$

that is,

$$mz'' = -kz - nz^3 - cz' + F\cos(\nu\tau) \tag{2}$$

For the permissible regions of motion, in the case (a) $z < z_0$ and in the case (b) $|z| < z_0$, $z'' = d^2z/d\tau^2$ (τ - time). In position $z = z_0$ (case a) and $|z| = z_0$ (case b) there is an impact upon the rigid fixed arrester causing a momentous change of intensity and of the direction of velocity z' of moveable mass m of the non-linear oscillator. In accordance with the stereomechanical theory of impact - with some approximation - it can be taken that the value of the velocity immediately after the impact is related to the value of the velocity immediately before the impact - which is momentous - by the following relation:

$$\begin{aligned} &z'(\tau^+) = -Rz'(\tau^-) \\ &\text{case a}: z(\tau^+) = z(\tau^-) = z_0 \\ &\text{case b}: |\,z(\tau^+)\,| = |\,z(\tau^-)\,| = z_0 \\ &(\tau^+ - \tau^-) \to 0 \end{aligned} \tag{3}$$

where τ^+ is time immediately after the impact and τ^- is time immediately before the impact.

Introducing dimensionless parameters $x = z/z_0$, $t = \omega\tau$, where $\omega^2 = k/m$, from differential equation (2) the dimensionless differential equation is obtained of the following form:

$$\ddot{x} = -\delta\dot{x} - x - \gamma x^3 + f\cos(\Omega t) \tag{4}$$

where

$$\delta = \frac{c}{(km)^{1/2}}, \quad \gamma = \frac{nz_0^2}{k}, \quad f = \frac{F}{kz_0}, \quad \Omega = \frac{\nu}{\omega}, \quad \left(\dot{\ }\right) = \frac{d}{dt} \tag{5}$$

Impact condition (3) related to the momentous change of the oscillatory mass velocity due to the impact against the rigid fixed arrester, after introducing dimensionless parameters is transformed into the form:

$$\begin{aligned} &x'(t^+) = -Rx'(t^-) \\ &\text{case a}: x(t^+) = x(t^-) = 1 \\ &\text{case b}: |\,x(t^+)\,| = |\,x(t^-)\,| = 1 \\ &(t^+ - t^-) \to 0 \end{aligned} \tag{6}$$

where is $R \in [0,1]$ the coefficient of restitution (establishing) velocity. In differential equations (4) and (6) differentiation is done with respect to dimensionless time t.

Non-linear differential equation (4) can be written in the form of autonomous differential equations of the first order where (x, y, t) are coordinates of three-dimensional space $R^2 \times I$:

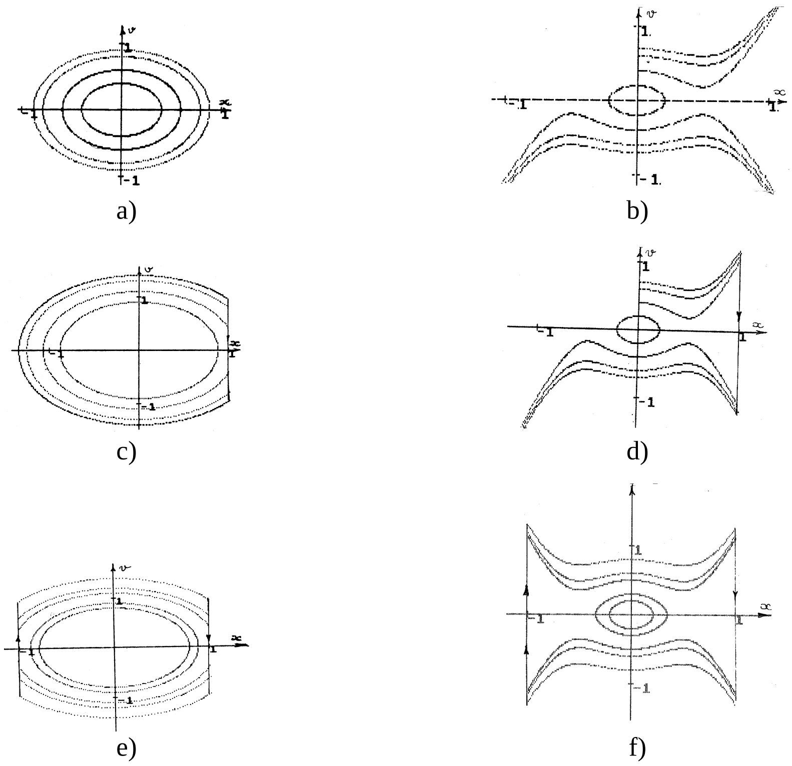
$$\dot{x} = y$$
$$\dot{y} = -\delta y - x - \gamma x^3 + f\cos(\Omega t) \tag{7}$$
$$\dot{t} = 1$$

Corresponding impact conditions (6) can be written in concise form:

$$(x, y, t) \to (x, -Ry, t) \quad \text{za} \quad x = 1,$$
$$\text{case (a)} \quad x = 1 \tag{8}$$
$$\text{case (b)} \quad |x| = 1$$

3. Phase portraits

Duffing oscillator defined by a system of differential equation (7) with no damping and with no external excitation force has one stationary point $(x,y) = (0,0)$ as a stable center in the case of a strong spring ($\gamma > 0$). In the case of a mild spring ($\gamma < 0$) there are three stationary points, namely, a stable center at $(x,y) = (0,0)$ and two unstable saddles $(x,y) = (\pm 1 / |\gamma|^{1/2}, 0)$. In the case of the strong spring, the oscillatory processes are taking



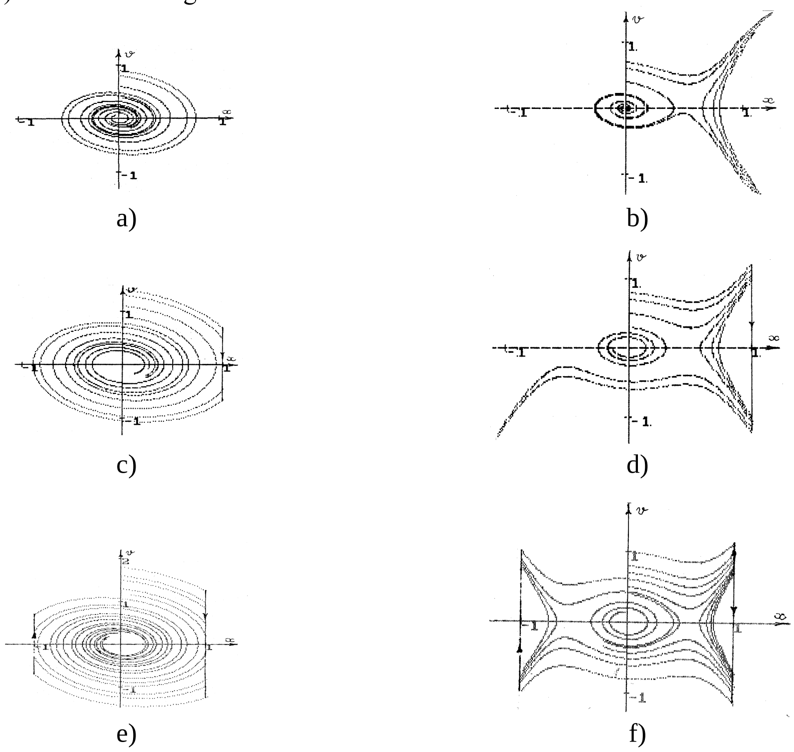
a) b) c) d) e) f)

$\delta = 0$; $\gamma = 0.1$; $R = 1$ $\delta = 0$; $\gamma = -4.0$; $R = 1$

Fig. 2.

place with respect to the stationary periodic phase trajectories whose position depends on the choice of initial values (Fig.2a). For the case of the mild spring depending on the choice of initial values either a stationary periodic oscillating regime with respect to parts of the closed curves or a non-stationary one with respect to unstable branches, separators or open branches (Fig. 2b) can occur.

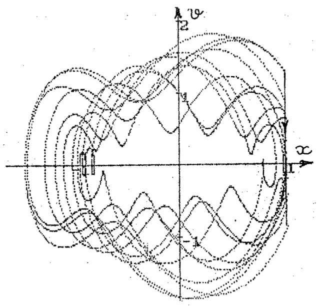
The impact influence on the basic oscillatory system causes a change of the phase portrait in the sense of cutting off parts of the phase trajectories that are outside the permissible region of motion as illustrated in Fig. 2c-f for the case of an ideally elastic impact when $R = 1$. In the case of damping (δ), the stationary points' coordinates do not change while the phase trajectories deform into spirals due to energy dissipation (Fig. 3a and b). The corresponding phase portraits for the adjoined impact conditions for the cases (a) and (b) are shown in Fig. 3c-f.



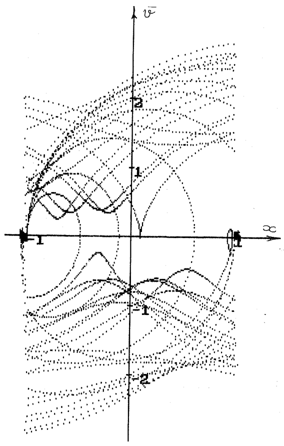
$\delta = 0.2$; $\gamma = 0.1$; $R = 1$ $\delta = 0.2$; $\gamma = -4.0$; $R = 1$

Fig. 3.

If the Duffing oscillator is acted upon by the periodic excitation force the response is of periodic, subharmonic or chaotic type depending on the system parameters (γ,δ), amplitude and frequency of the compulsive force as well as the initial values of displacement and velocity of the oscillatory mass. Global behavior of the vibroimpact systems of the arresters' types (a) and (b) is observed on the basis of the phase portraits. The phase portraits, described in Fig. 4a and b, for the cases of unilateral and bilateral arresters, are obtained by solving a system of differential equations (7) along with inclusion of the corresponding impact conditions (8), the Runge-Kuta numerical method of the forth order according to the Turbo-Pascall program by means of the computer graphics. The phase portraits in Fig. 4 correspond to chaotic regimes followed by the phenomena of "pasting" and "scratching" impacts.



a)



b)

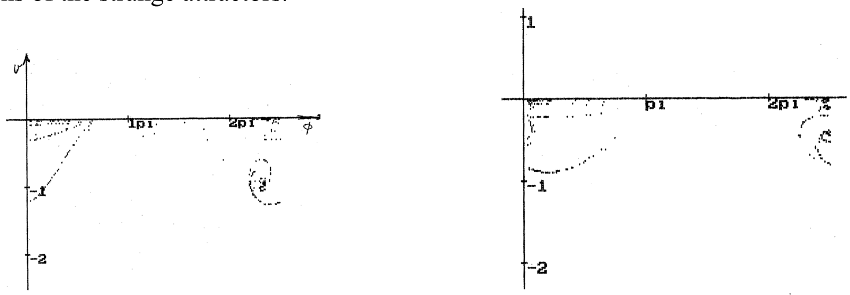
$\delta = 0$; $\gamma = 0.1$; $f = 2.0$; $R = 0.7$

Fig. 4.

4. Two-dimensional dotted mapping

The vibroimpact oscillators belong to the group of discrete dynamic models. Information about the vibroimpact oscillators' behavior is obtained in the projected phase plane by the numerical simulation in the form of phase trajectories. The time evolution of the impact occurrence described by the phase trajectories, that is, phase portraits for varied initial velocities gives an idea about the transition regimes that take the form of strange attractors in the case of stochastic response. It is usually important to examine asymptotic behavior of the phase trajectories as well as existence of stationary motion; in relation to this, it is also necessary to give an answer to the question about "where do the phase trajectories end?" The places where they end are called asymptotic restricted sets "attractors." The attractors' geometrical structure can be simple (a fixed equilibrium point for dissipative systems without excitation forces, the boundary circle corresponding to periodic motion) or very complex for the cases of chaotic motion that is neither equilibrium nor periodic. The attractors of complex geometric structure as qualitative indicators of the process stochacity are called strange attractors. The basic characteristic of strange attractors is their sensitivity to the initial conditions. In Poincare section the boundary circle generates only one point whereas chaotic motion generates a great number of points filling up the section place with no rule. If assumed that $f: X \rightarrow X$ is a discrete dynamic system of the differential flow X in the section chosen in the arrester's plane due to the very nature of the process, the answer to the above question ("where do the phase trajectories end?") can be more precisely formulated on the basis of accumulating the flow X points in the impact section. In that sense mapping is done in the chosen impact section for a very long iterative procedure; thus the information about the vibroimpact oscillator behavior is completed.

For the observed vibroimpact systems (cases a and b), the two-dimensional mapping has been done for varied values of the initial velocities while, at the same time, the sections are separated of the phase attractors in the rectilinear coordinate systems in which the abscissa represents the phase $\varphi = t \mod(2\pi/\Omega)$, while the ordinate gives velocity v immediately after the oscillator mass impact against the arrester's right side. Figs. 5a and b give sections of strange attractors for the vibroimpact systems in the cases (a) and (b) of the arrester for the same values of the parameters, amplitude and compulsion force phase as well as the initial values. This example provides for observing the chaotic regime of oscillation with very similar point's distribution in corresponding sections of the strange attractors.



a) b)

$\delta = 0.1$; $\gamma = 0.1$; $f = 1.0$; $R = 0.5$

$\Omega = 0.8$; $x_0 = 1.0$; $v_0 = 1.0 + 0.2k$; $k = 1,2,...,10$

Fig. 5.

5. Conclusion

This paper presents a procedure for global dynamics of the Duffing oscillator's vibroimpact process, with unilateral and bilateral symmetrical rigid arrester as well as with periodic excitation force. The created programs provide for efficient drawing of phase portraits and phase attractors' sections upon which dynamic behavior of the described virboimpact systems with given parameters can be observed. The examination of the vibroimpact dynamics for varied parameters' values requires inclusion of the modern bifurcation theory that would surely be of interest for further research.

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DINAMIKA DUFFING-OVOG OSCILATORA SA UDARIMA

Slavka Mitić

Oscilatorni sistem sa jednim stepenom slobode, opisan Duffing-ovom diferencijalnom jednačinom, sa krutim ograničivačem, izučavan je na osnovu nelinearne teorije i stereomehaničke teorije udara. Udarne oscilacije Duffing-ovog oscilatora, sa harmonijskom pobudom, u slučaju jednostranog krutog ograničivača i u slučaju simetričnog dvostranog krutog ograničivača, odredjene su numerički pomoću metode Runge-Kutta četvrtog reda. Sastavljeni su odgovarajući Turbo-pascal programi i korišćena je kompjuterska grafika. Kvalitativna analiza globalne dinamičke stabilnosti nelinearnih udarnih sistema sa jednostranim ili sa simetričnim dvostranim ograničivačem sprovedena na osnovu faznih portreta i dvodimenzionih mapa.