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ON THE RELATIVE AND ABSOLUTE TRANSMISSIBILITY OF A VIBRATION ISOLATION SYSTEM SUBJECTED TO BASE EXCITATION

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Abstract. *In this paper a one-degree of freedom passive vibration isolator system which is subject to harmonic base excitation is analyzed. The isolator is modellled as a parallel combination of a stiffness and damping element with cubic non-linearity. The method of averaging is used to obtain the steady-state harmonic response. A parametric analysis is conducted in order to investigate the influence of the system parameters on the relative and absolute transmissibility of the system from the viewpoint of possible improvement of the transmissibility of a system with linear viscous damping.*

Key Words: *Isolation, Base Excitation, Damping, Transmissibility*

1. INTRODUCTION

There are many objects that are vibration-sensitive and have to be protected from vibrations transmitted via a host structure. Such objects are, for example, civil engineering structures or some equipment transported by vehicles. The use of passive isolators is one of the most widely applied approaches for the vibration protection of these systems.

The performance of passive isolators is usually evaluated by considering displacement transmissibility, which is a measure of the reduction of transmitted motion afforded by an isolator. Two types of displacement transmissibility indices are defined: absolute transmissibility and relative transmissibility. The absolute transmissibility is the ratio of the vibration amplitude of the equipment to the vibration amplitude of the host structure (base) [1], [2], which should be as small as possible. The other characteristic is the relative transmissibility. It represents the ratio of the relative deflection amplitude of the iso-

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lator to the displacement amplitude imposed at the base [1], [2], which is expected to be as close to unity as possible. Passive linear isolators are usually modelled as one-degree-of-freedom (1-DOF) discrete systems under harmonic base excitation. The performance characteristics of these isolators with various types of damping forces have been widely reported in the literature [1]-[5]. However, we must take into account the non-linearity present in the damping mechanisms is also important, because it can have an important influence on the outcome.

A common type of damping in isolation systems is viscous damping. In this case the damping force is dependent on the relative velocity across the damper. If the damping force is proportional to this velocity, the damping is linear. This type of damping is used to characterize some elastomeric materials or the effects of air damping at low velocities. One type of non-linear viscous damping occurs when the damping force is proportional to the square of the relative velocity, and this is attainable from a turbulent flow of a fluid through an orifice [6], [7]. A generalized non-linear damping force can be described as being proportional to the *p*th power of the relative velocity across the damper. The case $p=0$ represents a Coulomb damper; the values $p=1$ and $p=2$ represent linear viscous and quadratic damping, respectively. Other values of the parameter p characterize damping properties in some particular system configuration, such as pipes and U-tubes (see [6] and references cited therein). The damping of certain alloys and composite materials is sometimes described by a power law as well [8].

Ruzicka and Derby [6] considered a 1-DOF system with a velocity-*p*th power damping mechanism by using the concept of equivalent viscous damping. Varying the value of the exponent p between 0.5 and 5, they presented its effect on transmissibility and amplification factor characteristics graphically. Ravindra and Mallik [9] analyzed a 1-DOF isolation system with a symmetric and asymmetric restoring force and with velocity-*p*th power damping for the cases $p=1$, 1.5, 2, 3, where the second and the last value represents fractional and cubic damping models. These authors demonstrated that the increase in the damping index p and a damping coefficient can eliminate the appearance of the undesirable jump in the transmissibility curve or reduce the jump width. Xiong et al. [10] investigated a nonlinear interactive system consisting of a piece of equipment, a traveling flexible ship excited by waves and a nonlinear isolator placed between them. Three cases regarding non-linearities of the isolator were considered: non-linear *p*th-power damping but linear stiffness; *q*th-power stiffness but linear damping; and the combination of *p*th-power damping and *q*th-power stiffness. They gave practical guidelines for the design of that vibration isolation system, which includes the fact that an increase of the nonlinear damping power p provides substantial reductions in the power transmission to the equipment at the resonance frequency of the isolator.

To characterize the influence of non-linear damping and extend some of the results found by Ruzicka and Derby [6], a 1-DOF vibration isolation system with linear stiffness and pure cubic damping is examined in this paper. The system is analyzed analytically from the viewpoint of displacement transmissibility. The results for cubic damping are compared with the case of linear viscous damping, to highlight the positive and negative effects of the non-linear system considered.

2. The Model of the System and its Equation of Motion

The system considered in this paper is shown in Fig. 1. It comprises an isolated mass m and a harmonically moving base; the isolator is modelled by a linear spring with stiffness k_1 and a non-linear damper. The coefficient of linear viscous damping is c_1 and the coefficient of cubic damping is c_2.

The equation of motion with respect to the relative displacement $z = x - y$ and for the harmonic base excitation $y = Y\sin(\omega t)$ is given by:

$$m\ddot{z} + c_1\dot{z} + c_2\dot{z}^3 + k_1 z = m\omega^2 Y \sin(\omega t). \quad (1)$$

Introducing the following non-dimensional parameters:

$$u = \frac{z}{Y}, \quad \omega_0 = \sqrt{\frac{k_1}{m}}, \quad \Omega = \frac{\omega}{\omega_0}, \quad \tau = \omega_0 t, \quad \zeta_1 = \frac{c_1}{2\sqrt{k_1 m}}, \quad \zeta_2 = \frac{c_2}{2m}\sqrt{\frac{k_1}{m}}Y^2, \quad \text{(2a-g)}$$

the equation of motion transforms to:

$$u'' + 2\zeta_1 u' + 2\zeta_2 (u')^3 + u = \Omega^2 \sin(\Omega\tau). \quad (3)$$

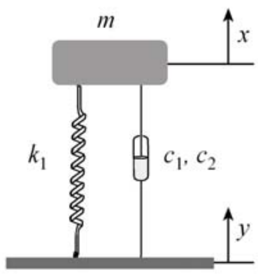

Fig. 1 A vibration isolation system with a spring and a viscous damper subjected to base excitation

2.1 Solution procedure

In order to find a steady-state response of the system, the method of averaging is used [11]. The solution of Eq. (3) is assumed in the form

$$u = U(\tau)\sin(\varphi(\tau)), \quad (4)$$

with

$$\varphi(\tau) = \Omega\tau + \psi(\tau), \quad (5)$$

where U is the amplitude of the relative motion and ψ is the phase.

The first time derivative of the solution (4) has the form

$$u' = U(\tau)\Omega\cos(\varphi(\tau)), , \quad (6)$$

with the constraint:

$$U'\sin\varphi + U\psi'\cos\varphi = 0. \quad (7)$$

The derivative of the expression (6) is:

$$u'' = \Omega U'\cos\varphi - \Omega U(\Omega + \psi')\sin\varphi. \quad (8)$$

Substituting Eqs. (4)-(6) and (8) into (3), and using Eq. (7) yields:

$$\Omega U' = -[U(1-\Omega^2)\sin\varphi + 2\Omega U\zeta_1\cos\varphi + 2\Omega^3 U^3\zeta_2\cos^3\varphi - \Omega^2\sin(\varphi-\psi)]\cos\varphi, \quad (9)$$

$$\Omega U\psi' = [U(1-\Omega^2)\sin\varphi + 2\Omega U\zeta_1\cos\varphi + 2\Omega^3 U^3\zeta_2\cos^3\varphi - \Omega^2\sin(\varphi-\psi)]\sin\varphi. \quad (10)$$

Averaging the right-hand sides of Eqs. (9) and (10) over a period of 2π

$$\Omega U' = -\frac{1}{2\pi}\int_0^{2\pi}\left[U\left(1-\Omega^2\right)\sin\varphi + 2\Omega U\zeta_1\cos\varphi + 2\Omega^3 U^3\zeta_2\cos^3\varphi - \Omega^2\sin\left(\varphi-\psi\right)\right]\cos\varphi\, d\varphi, \quad (11)$$

$$\Omega U\psi' = \frac{1}{2\pi}\int_0^{2\pi}[U(1-\Omega^2)\sin\varphi + 2\Omega U\zeta_1\cos\varphi + 2\Omega^3 U^3\zeta_2\cos^3\varphi - \Omega^2\sin(\varphi-\psi)]\sin\varphi\, d\varphi, \quad (12)$$

they are simplified to:

$$\Omega U' = \frac{1}{2\pi}(-\pi\Omega^2\sin\psi - 2\pi\zeta_1\Omega U - \frac{3}{2}\pi\zeta_2\Omega^3 U^3), \quad (13)$$

$$\Omega U\psi' = \frac{1}{2\pi}(\pi U - \pi\Omega^2 U - \pi\Omega^2\cos\psi). \quad (14)$$

The steady-state solutions satisfy $U' = \psi' = 0$, i.e.:

$$\Omega^2\sin\psi = -2\zeta_1\Omega U - \frac{3}{2}\zeta_2\Omega^3 U^3, \quad (15)$$

$$\Omega^2\cos\psi = U - \Omega^2 U. \quad (16)$$

Combining Eqs. (15) and (16), the following implicit amplitude-frequency equation is obtained:

$$36\zeta_2^2 U^6\Omega^6 - 16(1 - U^2 - 6U^4\zeta_1\zeta_2)\Omega^4 - 32(U^2 - 2U^2\zeta_1^2)\Omega^2 + 16U^2 = 0. \quad (17)$$

3. On the Displacement Transmissibility

Due to the definition of the amplitude U and Eqs. (4) and (2a), there is equivalence between the relative transmissibility T_r and this amplitude:

$$T_r \equiv U. \tag{18}$$

The absolute displacement transmissibility T_a is equal to the amplitude V of the absolute displacement $v = x/Y$:

$$T_a \equiv V. \tag{19}$$

where:

$$V = \sqrt{U^2 + 2U\cos\psi + 1}, \tag{20}$$

with the term $\cos\psi$ being found from Eq. (14).

The displacement transmissibility characteristics are presented below in dB, i.e. as $20 \cdot \log_{10} T_r$ and $20 \cdot \log_{10} T_a$.

3.1 Linear viscous damping

For the case of linear viscous damping, the non-linear damping ratio is equal to zero $\zeta_2=0$, so that Eq. (17) simplifies to

$$(1-U^2)\Omega^4 + 2(1-2\zeta_1^2)U^2\Omega^2 - U^2 = 0. \tag{21}$$

Solving it for U and using Eq. (18), the well-known expression for the relative transmissibility of a system with linear damping [1] ,[2] is derived:

$$T_r^L = \frac{\Omega^2}{\sqrt{(1-\Omega^2)^2 + 4\zeta_1^2\Omega^2}}. \tag{22}$$

The relative transmissibility is less than unity regardless of the value of the linear damping ratio ζ_1 for $\Omega < 1/\sqrt{2}$. Also, it is less than unity for all the values of the frequency ratio Ω if $\zeta_1 > 1/\sqrt{2}$. If damping is increased, the resonant frequency is increased as well, while the relative transmissibility decreases. The low frequency attenuation rate is independent of the value of ζ_1 and it is equal to 40 dB per decade. The high-frequency relative transmissibility asymptotes to unity, i.e. 0 dB [6].

By using Eqs. (16), (18)-(20) and (22), the expression for the absolute transmissibility of a system with linear damping is obtained:

$$T_a^L = \frac{\sqrt{1 + 4\zeta_1^2\Omega^2}}{\sqrt{(1-\Omega^2)^2 + 4\zeta_1^2\Omega^2}}. \tag{23}$$

It is well-known [1], [2], [6] that isolation of vibration occurs for $\Omega > \sqrt{2}$, and amplification takes place for $\Omega < \sqrt{2}$. If damping is increased, the resonant frequency and the corresponding absolute transmissibility are decreased. The high-frequency amplification rate is 20 dB per decade.

3.2 Pure cubic damping

The case of pure cubic damping corresponds to ζ_1=0 and $\zeta_1 \neq 0$. Equation (17) reduces to

$$36\zeta_2^2 U^6 \Omega^6 - 16(1-U^2)\Omega^4 - 32U^2\,\Omega^2 + 16U^2 = 0. \quad (24)$$

This equation is cubic in U^2. Thus Eq. (18) yields the expression for the relative transmissibility:

$$T_r^C = \sqrt{\frac{6^{1/3} I^{2/3} - 288^{1/3}(\Omega^2 - 1)^2}{9\Omega^3 \zeta_2 I^{1/3}}}, \quad (25)$$

where

$$I = 27\Omega^7 \zeta_2 + \sqrt{3}\sqrt{16(\Omega^2 - 1)^6 + 243\Omega^{14}\zeta_2^2}. \quad (26)$$

In Fig. 2 the relative transmissibility is presented for different values of the parameter ζ_2. It illustrates that the low-frequency attenuation rate is equal to 40 dB per decade, as in the case of a system with linear viscous damping. Further, the high-frequency relative transmissibility becomes less than unity for different values of Ω, which depends on the non-linear damping ratio ζ_2. Here the attenuation rate is approximately 20/3 dB per decade, which agrees with the results given in [6], where it is found to be 2dB per octave. The additional difference between a linear system and a system with pure cubic damping has to do with the value of the relative transmissibility for very high values of Ω. For the latter it slowly tends to zero, i.e. to –∞ dB, not to unity, as does the former.

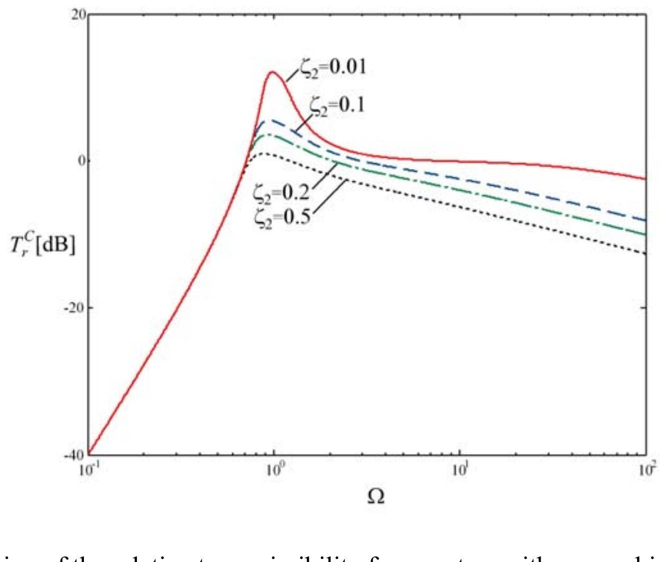

Fig. 2 Variation of the relative transmissibility for a system with pure cubic damping for different values of the non-linear damping ratio ζ_2

The absolute transmissibility, obtained on the basis of Eqs. (16), (18)-(20) and (25) is given by:

$$T_a^C = \sqrt{1+\left(\frac{2}{\Omega^2}-1\right)\frac{6^{1/3}I^{2/3}-288^{1/3}(\Omega^2-1)^2}{9\Omega^3\zeta_2 I^{1/3}}}. \tag{27}$$

Fig. 3 shows how the absolute transmissibility changes with Ω for various values of ζ_2. It can be seen that as ζ_2 increases, the maximum of this transmissibility decreases, moving towards lower frequencies. For $\Omega = \sqrt{2}$, the absolute transmissibility is equal to unity (0 dB), which holds also for the systems with linear viscous damping. This implies that the region of amplification is equal for both the systems with linear viscous and cubic damping.

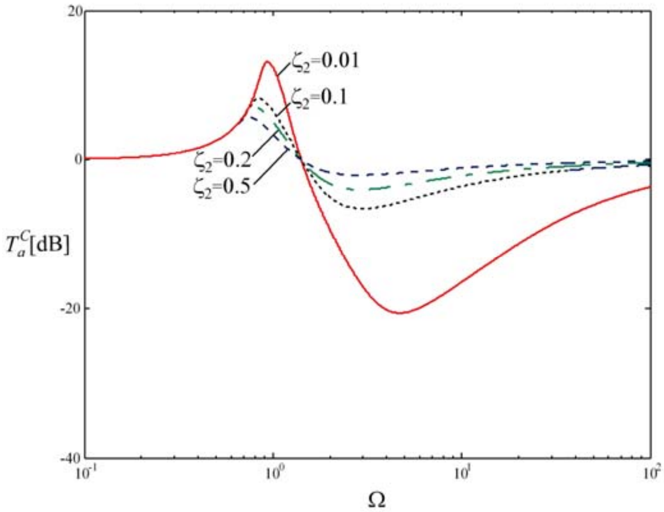

Fig. 3 Variation of the absolute transmissibility for a system with pure cubic damping for different values of the non-linear damping ratio ζ_2

What is also similar to the case of linear damping is the beneficial influence of damping in the region of amplification, and the detrimental influence in the region of isolation. However, unlike in the case of linear systems, for which the absolute transmissibility tends to zero (-∞ dB) as Ω →∞, for the system with pure cubic damping it tends to unity (0 dB), which corresponds to a rigidly connected system. In the case of cubic damping, the absolute transmissibility has a local minimum, the existence of which was also observed by Ruzicka and Derby [6]. Fig. 3 additionally shows that this local minimum shifts towards higher frequencies as damping is decreased.

3.3 Comparison: linear and pure cubic damping

In order to compare the absolute transmissibility of a system with linear viscous damping given by Eq. (23) and cubic damping given by Eqs. (25) and (26) for the equal values of the linear and non-linear cubic damping ratio $\zeta \equiv \zeta_1 = \zeta_2$, the ratio between the corresponding absolute transmissibility ΔT_a is calculated and shown in Fig. 4. When the corresponding curve is above the horizontal axis, the absolute transmissibility of a linearly damped system is higher than the absolute transmissibility of a system with pure cubic damping, i.e. the latter has better performance characteristics with respect to the absolute transmissibility.

It is possible to find the critical value of the damping ratio ζ_c for which a linear system has lower absolute transmissibility for all frequencies. It corresponds to the case when (Ω, ΔT_a) = ($\sqrt{2}$,0) is a local maximum. Finding the first derivative of ΔT_a, equating it with zero for $\Omega = \sqrt{2}$, one obtains:

$$\zeta_c \equiv \zeta_1 = \zeta_2 = \frac{\sqrt{5}}{2\sqrt{2}} \approx 0.79 \,. \tag{28}$$

The graph of ΔT_a that corresponds to this value is plotted in Fig. 4 as a red solid line.

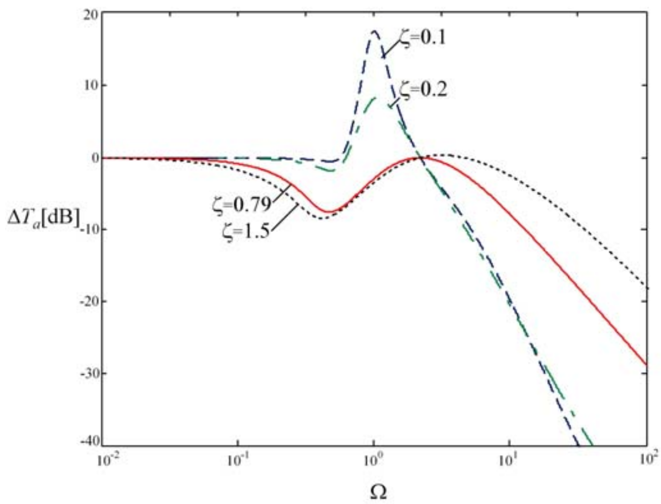

Fig. 4 Ratio between the absolute transmissibility of a system with linear viscous and cubic damping ΔT_a for the same values of the damping ratio $\zeta \equiv \zeta_1 = \zeta_2$

The shaded region shown in Fig. 5 represents a region in which $\Delta T_a > 0$ for different values of the damping ratio. This is the region, when, for a fixed Ω and $\zeta \equiv \zeta_1 = \zeta_2$, a system with pure cubic damping has smaller absolute transmissibility than a system with linear damping. If the damping ratio is smaller than ζ_c, the system with pure cubic damping yields smaller absolute transmissibility in the region defined by the intersection of a horizontal line through some value $\zeta_1 = \zeta_2$ (for example, 0.1 or 0.2 as indicated in Fig. 5) with the left boundary of the shaded region and the value $\Omega = \sqrt{2}$. If the damping ratio is higher than ζ_c (for example, 1.5 as shown in Fig. 5), it holds for the regions of Ω, the lower boundary of which is $\Omega = \sqrt{2}$, while the upper boundary is the intersection of the horizontal line with the curvilinear line in Fig. 5.

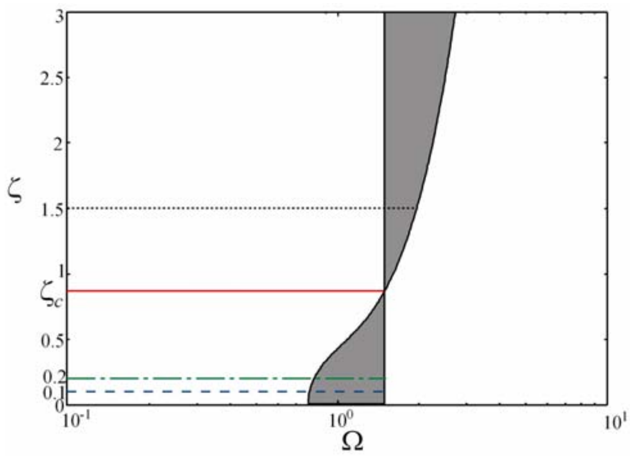

Fig. 5 Regions (shaded) in which $\Delta T_a > 0$ for a range of the values of the damping ratio $\zeta \equiv \zeta_1 = \zeta_2$

4. Conclusions

It can be concluded that a system with pure cubic damping has better performance characteristics just for a relatively small number of combinations of the damping parameter and the excitation frequency ratio Ω. However, its major advantage has to do with the cases with small damping. Then it gives smaller absolute transmissibility in the resonance region than a linear system with the same value of the damping ratio. However, it has a detrimental effect on the absolute vibration transmissibility in the isolation region.

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O RELATIVNOJ I APSOLUTNOJ PRENOSIVOSTI VIBRO-IZOLACIONOG SISTEMA SA KINEMATSKOM POBUDOM

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U ovom radu se razmatra pasivni izolator vibracija sa jednim stepenom slobode sa harmonijskom pobudom osnove. Izolator se modeluje paralelno vezanom linearnom oprugom i prigušnicom kubne nelinearnosti. Metod osrednjavanja se koristi za određivanje stacionarnog harmonijskog odziva. Sprovodi se analiza uticaja parametara sistema na relativnu i apsolutnu prenosivost sa stanovišta mogućeg poboljšanja ovih karakteristika u odnosu na izolacione karakteristike sistema sa linearnim viskoznim prigušenjem.

Ključne reči: *Izolacija, Kinematska pobuda osnove, Prigušenje, Prenosivost*