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OPTIMAL PIECEWISE UNIFORM PRODUCT QUANTIZATION OF THE MEMORYLESS TWO DIMENSIONAL LAPLACIAN SOURCE

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Abstract. *In this paper simple and complete asymptotical analysis is given for a piecewise uniform product two-dimensional Laplace source quantizer (PUPTDLSQ) with respect to mean-square error (mse). PUPTDLSQ is based on uniform product two-dimensional Laplace source quantizers. Product quantizer optimality conditions and all main equation for a number of phase divisions and optimal number of levels for each partition are presented. These systems, although not optimal, may have asymptotic performance arbitrarily close to the optimum. PUPTDLSQ has complexity implementation between optimal nonuniform quantization (NQ) and uniform product quantization (UPQ). Further more, their analysis and implementation can be simpler than those of optimal systems.*

Key Words: *Uniform Quantization, Two-dimensional Laplace Source, Granular Distortion, Total Distortion*

1. INTRODUCTION

Quantizers play an important role in the theory and practice of modern-day signal processing. The asymptotic optimal quantization problem, even for the simplest case - uniform scalar quantization, is very actual nowadays [1,2]. They do consider the problem of finding the optimal maximum amplitude, so-called, support region for scalar quantizers by minimization of the total distortion D, which is a combination of granular (D_g) and overload (D_o) distortion, $D = D_g + D_o$. Extensive results have been developed on scalar quantization but more on vector quantization. The simplest vector quantization is two-dimensional vector quantization.

The analisys of vector quantizer for arbitrary distribution of the source signal was given in paper [3]. The authors derived the expression for the optimum granular distortion

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and optimum number of output points. However, they did not prove the optimality of the proposed solutions. Also, they did not define the partition of the multidimensional space into subregions. In paper [4], they have derived the expressions for the optimum number of output points, however the proposed partitoning of the multidimensional space for memoryless Laplacian source does not take into consideration the geometry of the multidimensional source. In paper [5], vector quantizers of Laplacian and Gaussian sources were analyzed. The proposed solution for the quantization of memoryless Laplacian source, unlike in [4], takes into consideration the geometry of the source, however, the proposed vector quantizer design procedure is too complicated and unpractical.

In this paper we will give a systematic analysis of piecewise uniform vector quintizer of Laplacian memoryless source. We will give a general and simple way for the piecewise uniform vector quantizer. We will derive the optimum number of output points and the optimality of the proposed solutions will be prooved.

The importance of using the optimal density of points (using rectangular cells) for product quantization and Gaussian source is considered in [6]. The goal of this paper is to solve quantization problem in a case of PUQ and to find corresponding support region. It is done by analytical optimisation of the granular distortion and numerical optimisation of the total distortion. If the distortion is measured by squared error, D becomes the mean squared error (MSE). Distortion mean-squared error (MSE i.e. quantization noise) is used as the criterion for optimisation. In [7] the granular gain (due to cell shape, being 1.53 dB at the maximum) as well as the boundary gain (due to the increase of the dimensions number) was defined showing that the boundary gain dominates. In [5] the uniform cubic quantization is considered for 8 and 16 dimensions.

The MSE of an two-dimensional vector source $\mathbf{x} = (x_1, x_2)$, where x_i are zero-mean statistically independent Laplacian random variables of variance σ^2, is commonly used for the transform coefficients of speech or imagery. The first approximation to the long-time-averaged probability density function (pdf) of amplitudes is provided by Laplacian model [8,p32]. Waveforms are sometimes represented in terms of adjacent-sample differences. The pdf of the difference signal for an image waveform follows the Laplacian function [8,p33]. Laplace source is model for speech [9,p384]. Consider two independent identically distributed Laplace random variables (x_1, x_2) with the zero mean and the unity variance. To simplify the vector quantizer, the Helmert transformation is applied on the source vector giving contours with constant probability densities. The transformation is defined as:

$$r = \frac{1}{\sqrt{2}}(|x_1| + |x_2|)\,,\ u = \frac{1}{\sqrt{2}}(|x_1| - |x_2|)\,.$$

PUQ consists of L optimal uniform vector quantizers. More precisely, our quantizer divides the input plane into L partitions and every partition is further subdivided into L_i $(1 \le i \le L)$ subpartitions. Every concentric subpartition can be subdivided in four equivalent regions, i.e. J-th subpartition in signal plane is allowed to have M_i $(1 \le j \le L_i)$ cells. We perform two-steps optimisation: 1) distortion optimisation (D_i) in every partition under the constraint $4L_i x M_i = N_i$ and 2) optimisation of the total granular distortion $D_g = \sum_{i=1}^{L} D_i$ which achieves the optimal number of points N_i on each partition under the constraint $\sum_{i=1}^{L} N_i = N$.

2. OPTIMALITY CONDITIONS FOR UNIFORM TWO-DIMENSIONAL QUANTIZATION AND QUANTIZER DESIGN

Probability density function (*pdf*) of a zero mean, unit variance Laplacian random varible is given as follows:

$$f(x) = \frac{1}{\sqrt{2}} e^{-\sqrt{2}|x|} \tag{1}$$

From equation (1), joint pdf function of two independent, identically distributed Laplace random variables (x_1,x_2) can be written as:

$$f_{1,2}(x_1, x_2) = \frac{1}{2} e^{-\sqrt{2}(|x_1|+|x_2|)} \tag{2}$$

To simplify the vector quantizer, the Helmert transformation is applied on the source vector giving contours with constant probability densities. This orthogonal transformation is defined as:

$$r = \frac{1}{\sqrt{2}}(|x_1| - |x_2|) \ , \ u = \frac{1}{\sqrt{2}}(|x_1| - |x_2|) \tag{3}$$

The obtained probability density function is:

$$f(r,u) = \frac{1}{2} e^{-2r} \tag{4}$$

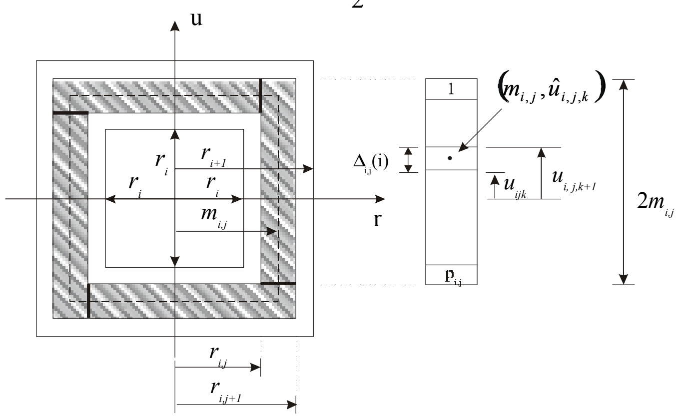

Fig. 1. Two-dimensional space partitioning.

The contour of constant pdf function given by equation (4) represents square line in two-dimensional *ru* system. This square surface representing dynamic range of a two dimensional quantizer, can be partitioned into L concentric domains as shown in Fig. 1. The number of output points in each domain is denoted by N_i, where $N = \Sigma^{L}_{i=1} N_i$ represents the total number of output points. Every concentric domain can be further partitioned into L_i concentric subdomains of equal width. Every subdomain is divided into four regions each containing M_i rectangular cells. An output point is placed in the centre of each cell. Coordinates of the *k*th output point in *j*th subdomain (in first region) of the *i*th domain in *ru* coordinate system are $(m_{i,j}, \hat{u}_{i,j,k})$.

The quantising cells are rectangular and the representation vector is $c_{ijk} = (m_{ij}, \hat{u}_{ijk})$. The quality of a quantizer can be measured by the goodness of the resulting reproduction in comparison to the original. One way of accomplishing this is to define a distortion measure $d(\mathbf{r},\mathbf{c}_{ijk})$ that quantifies cost or distortion resulting from reproducing $\mathbf{r}(r,u)$ as $c_{ijk} = (m_{ij}, \hat{u}_{ijk})$ and to consider the average distortion as a measure of the quality of a system, with smaller average distortion meaning higher quality. The most common convenient and widely used measure of distortion between an input vector $\mathbf{r}$ and quantised vector $\mathbf{c}_{ijk}$, is the squared error [9,p325] Euclidean distance between two vectors defined as $d(\mathrm{r}, \mathrm{c}_{ijk}) = |\mathrm{r} - \mathrm{c}_{ijk}|^2 = (r - m_{ij})^2 + (u - \hat{u}_{ijk})^2$.

The otput point coordinates are given by the equations:

$$m_{i,j} = \frac{r_{i,j+1} + r_{i,j}}{2} \text{ and } \hat{u}_{i,j,k} = \frac{u_{i,j,k} + u_{i,j,k+1}}{2} \quad (5)$$

Rectangular cell dimensions are:

$$\Delta = \frac{r_{\max}}{L}, \ \Delta_i = \frac{\Delta}{L_i} \text{ and } \Delta_{ij} = \frac{r_{i,j} + r_{i,j+1}}{M_i} \quad (6)$$

The range of the quantizer is $r_{\max}$. We can write:

$$r_i = (i-1)\cdot\Delta, \ r_{i,j} = r_i + (j-1)\cdot\Delta_i, \ i = 1,\ldots,L+1, \ j = 1,\ldots,L_i + 1 \quad (7)$$

The distortion is a sum of granular and overload distortion $D = D_g + D_o$:

$$\begin{aligned} D = \sum_{i=1}^{L} D(i) + D_o = \{\frac{1}{2}\sum_{i=1}^{L}\sum_{j=1}^{L_i}\sum_{k=1}^{M_i}\int_{r_{i,j}}^{r_{i,j+1}}\int_{u_{i,j,k}}^{u_{i,j,k+1}} [(r-m_{i,j})^2 + (u-\hat{u}_{i,j,k})^2]\frac{1}{2}e^{-2r}\,dr du\} \\ + \{2m_{L,L_L}\int_{r_{\max}}^{\infty}(r-m_{L,L_L})^2 e^{-2r}dr + \frac{2}{3}\frac{m_{L,L_L}^3}{M_L^2}\int_{r_{\max}}^{\infty} e^{-2r}dr\} \end{aligned} \quad (8)$$

After integration over u and the reordering, D_g becomes

$$D_g = \sum_{i=1}^{L}\sum_{j=1}^{L_i}\left(2m_{i.j}\frac{\Delta_i^2}{12} + \frac{2m_{i,j}^3}{3M_i^2}\right)P_r(m_{i,j})$$

where $\Delta_i = \dfrac{\Delta}{L_i}$ and $P_r(m_{i,j}) = 2e^{-2m_{i,j}}\Delta_i$.

After the reordering of sum and integration over r it obtains

$$D(i) = \frac{1}{2}\left(\frac{\Delta_i^2}{12}P_i + \frac{2}{3M_i^2}I_i\right)$$

where $P_i = 4\int_{r_i}^{r_{i+1}} re^{-2r}dr$, $I_i = 2\int_{r_i}^{r_{i+1}} r^3 e^{-2r}dr$, $M_i = \dfrac{N_i}{4L_i}$ and $D_g = \sum_{i=1}^{L} D(i)$.

The i-th partition distortion is

$$D_i = \frac{1}{2}\left(\frac{1}{12}\frac{\Delta^2}{L_i^2}P_i + \frac{32L_i^2}{3N_i^2}I_i\right)$$

After solving $\frac{\partial D_i}{\partial L_i} = 0$ we obtain

$$L_{iopt} = \sqrt[4]{\frac{\Delta^2 P_i N_i^2}{128 I_i}}$$

and $M_i = N_i / 4L_{iopt} = \sqrt[4]{\frac{I_i N_i^2}{2\Delta^2 P_i}}$.

Optimal solution is found applying the method of Lagrange multipliers.

$$J = \sum_{i=1}^{L} D(i) + \lambda \sum_{i=1}^{L} N_i \ , \ \frac{\partial J}{\partial N_i} = 0$$

yielding : $N_i = N \frac{\sqrt[4]{P_i I_i}}{\sum_{j=1}^{L} \sqrt[4]{P_j I_j}}$ for fixed N. The final expression for D_g is:

$$D_g = \frac{2\sqrt{2}\Delta}{3N}\left(\sum_{i=1}^{L} \sqrt[4]{P_i I_i}\right)^2 \qquad (9)$$

3. Numerical analysis and results

a) Optimization obtained with simple rounding of optimal real values

As an illustration of the PUPTDLSQ performance, we show the gain $G(dB) = 10\log(D_{gskal} / D_{2Dl})$ as a function of the number of bits per sample R.

Table 1.

$L = 8$	$R = 4$, $r_{max} = 4.4$, $Gmax = 3.43$dB		$R = 6$, $r_{max} = 6.6$, $Gmax = 6.05$dB		$R = 8$, $r_{max} = 8.2$, $Gmax = 8.32$dB	
i	L_i	M_i	L_i	M_i	L_i	M_i
1	5	6	16	20	63	76
2	2	11	11	31	43	123
3	1	9	6	32	26	125
4			3	31	15	110
5			2	20	9	85
6			1	16	5	68
7					3	48
8					2	30

By exceeding L better performances can be achieved but complexity becomes greater.

For $L = 8$ and rates $R = (4,6,8)$, it obtained optimal value of r_{max}, maximal value of G_{max} and optimal integer values of (L_i,M_i) are given in Table 1. For all that it is very important that $L_i,M_i \geq 1$ for $i = 1,\ldots,L$. Greater gain can be obtained for greater L and R. For comparing the obtained results to the previous ones, r_{max} from [5] is used, being obtained for $1-D$ approach. The corresponding D_g^{scal} is compared to the obtained result using the following gain definition $G = 10\log(D_{gskal} / D_{opt})$. The performance gain obtained by our method over the uniform scalar quantization for different rates can be presented in this manner: for $R = 4$. $G = 3.43$dB; for $R = 6$, $G = 6.05$dB and for $R = 8$, $G = 8.32$dB.

b) Choosing the best integer combination

We took in consideration all possible combinations within one partition and as an optimal combination take the one with minimal distortion.

The rounding of real values of the number of points to integer values results in the change of the overall number of points which results in the change of rate.

As a criteria for finding the optimal combination for integer values of the number of points we use the minimal difference of distortion in the case of our combination and the theoretical distortion, for the same rate.

$$\Delta D(R_i) = D(R_i)^{PUPTDLSQ} - D(R_i)^{THEOR}$$

$$D(R_i)^{theor} = \frac{e}{\pi e^{2R}}$$

Table 2. Integer values for $L = 2$

$L = 2$	$L1$	$M1$	$L2$	$M2$
$R = 2$	1	2	1	2
$R = 3$	3	3	2	3
$R = 4$	7	5	3	7
$R = 5$	15	11	6	12
$R = 6$	31	24	11	23
$R = 7$	68	49	18	41
$R = 8$	146	96	31	72

Table 3. Integer values for $L = 3$

$L = 3$	$L1$	$M1$	$L2$	$M2$	$L3$	$M3$
$R = 2$	1	1	1	1	1	1
$R = 3$	3	1	2	3	1	4
$R = 4$	5	5	3	8	2	6
$R = 5$	11	11	6	15	3	12
$R = 6$	24	23	12	28	5	21
$R = 7$	52	49	23	54	8	36
$R = 8$	113	99	42	104	14	54

4. Conclusion

In this paper the simple and complete asymptotical analysis of two-dimensional Laplace source piecewise uniform product quantization is given. The existence of a single minimum depending on the number of points on various levels is proven. Simple expression for granular distortion in closed form is obtained. The results obtained by the asymptotic analysis demonstrate the significant performance gain over the uniform scalar quantization (even 8.32 dB for $R = 8$). By choosing the best integer combination in b) we obtain a small gain in performances in regards the case a).

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OPTIMALNA PO DELOVIMA UNIFORMNA PRODUKTNA KVANTIZACIJA DVODIMENZIONALNOG LAPLASOVOG IZVORA BEZ MEMORIJE

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U ovom radu prosta i kompletna asimptotska analiza je data za uniformni produktni dvodimenzionalni kvantizer Laplasovog izvora (PUPTDLSQ) sa stanovišta srednje kvadratne greške. PUPTDLSQ je zasnovana na uniformnim produktnim dvodimenzionalnim kvantizerima Laplasovog izvora. Optimalni uslovi i sve jednačine za broj faznih podela i optimalni broj nivoa produktnog kvantizera za svaku particiju su predstavljeni. Ovaj sistem nije optimalan, medjutim može imati asimptotske osobine bliske optimalnim. PUPTDLSQ ima kompleksnost primene izmedju optimalne neuniformne kvantizacije i uniformne produktne kvantizacije. Analiza i primena je prostija od optimalnih sistema.

Ključne reči: *uniformna kvantizacija, dvodimenzionalni laplasov izvor, granularno izvrtanje / iskrivljenost, potpuno izvrtanje*