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BOSON BOGOLIUBOV GROUP AND GENERALIZED SQUEEZED STATES

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Abstract: *A convenient choice of the arbitrary phase of the Bogoliubov unitary operator, providing the homogeneous Bogoliubov transformation of the boson annihilation and creation operators, is proposed so that the Bogoliubov transformations form a continuos non-Abelian subgroup of the SL(2,C) group. Two operator identities involving this Bogoliubov operator are established and, with their help, the closed form expressions for the correlation amplitude and the geometric Pancharatnam) phase for ordinary and also generalized squeezed states are obtained. Applications of these general results to certain special cases of practical importance in the realm of quantum optics, such as two-photon interaction process, are briefly discussed.*

Key words: *Boson Bogoliubov transformation, geometric phase, squeezed states*

1. INTRODUCTION

The quantum oscillator is a model of the physical system which is simple, often exactly solvable and, at the same time, nontrivial since it has important applications in quantum electrodynamics, quantum statistical mechanics, molecular physics, and elsewhere. A fundamental theorem [1] asserts in this context that any operator (and in particular the Hamiltonian) may be expressed as a sum of products the boson annihilation and creation operators with suitable non-singular coefficients. The first term in this sum is the usual Hamiltonian for the *simple harmonic oscillator*, and this alone leads to the familiar *number* states. When terms *linear* in annihilation and creation operators are added to this Hamiltonian, a single quantum oscillator initially prepared in the ground state evolves to the corresponding *coherent* state [2] or, if initially prepared in one of the excited number states, the corresponding *displaced number* state is obtained [3]. Further

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inclusion of the terms *quadratic* in annihilation and creation operators leads analogously to the corresponding *ordinary* and *generalized squeezed* states. The importance and certain properties of these states have been discussed in some detail in [4-7]. Ordinary squeezed states have been first introduced and studied in the field of quantum optics with the ultimate aim to obtain a reduced fluctuation in one field quadrature, at the expense of an increased fluctuation in the other, leading to an increase in the signal to noise ratio in suitable experiments ranging from optical communication to detection of gravitational radiation.

The time evolution operator corresponding to the Hamiltonian leading to the squeezed states contains the *Bogoliubov unitary operator* which effects the *homogeneous Bogoliubov transformation* of the boson annihilation and creation operators. This operator is defined by the corresponding transformation up to an arbitrary phase factor. In Section 2 we propose a choice of this phase factor so that the Bogoliubov transformations form a continuous non-Abelian group. This immediately leads to two useful operator identities involving such Bogoliubov operator. The identities are subsequently proved with the help of the smallest faithful 4×4 matrix representation of the *two-photon algebra* h_6 [8]. Next, a faithful 2×2 non-unitary matrix representation, and also an infinite-dimensional *unitary* representation of the boson Bogoliubov group are established (the former, incidentally, demonstrating that the Bogoliubov group is a subgroup of the $SL(2,C)$ group). The relationship between the usual *squeeze* operator and the modified Bogoliubov unitary operator is also briefly discussed. In Section 3, with the help of these new operator identities, the *correlation amplitude* and also the *geometric (Pancharatnam) phase* for a *non*adiabatic and *non*cyclic evolution for ordinary and also generalized squeezed states are found. As is well known, Berry [9] discovered the general existence of an observable phase accumulation in the wave function of a quantum-mechanical system with an adiabatically changing Hamiltonian. Subsequently, the restriction to adiabaticity was lifted by Aharonov and Anandan [10] by removing from the wave function the time integral of the expectation value of the Hamiltonian as a *dynamical phase*. It was shown that once the dynamical phase is removed, the phase difference accumulated during the time evolution of the system has purely geometric origin. Here, in order to determine the geometric phase in the case of ordinary and generalized squeezed states, we follow the approach of Samuel and Bhandari [11] who additionally removed the restriction to cyclic motion. Their work was based on the earlier investigation of Pancharatnam [12]. Finally, possible applications of these new results to certain special cases of practical importance, such as two-photon interaction process, are briefly mentioned. In the view of the current active interest in generalized squeezed states in the realm of quantum optics and elsewhere, these general results appear to be also of some practical importance.

2. OPERATOR IDENTITIES

The homogeneous Bogoliubov transformation of the boson annihilation $\hat{a}$ a and creation $\hat{a}^+$ operators, $\hat{b} = \mu\hat{a} + \nu\hat{a}^+$, for a pair of complex parameters $\mu \equiv |\mu|\exp(i\phi)$ and $\nu \equiv |\nu| exp(i\theta)$, obeying additionally $|\mu|^2 - |\nu|^2 = 1$, is canonical since it leaves the commutator invariant, $[\hat{a},\hat{a}^{+}]=[\hat{b},\hat{b}^{+}]=\hat{1}$. A theorem of von Neumann asserts that every canonical transformation can be represented as a unitary transformation; thus in particular $\hat{b}=\hat{B}\hat{a}\hat{B}^{+}$. The Bogoliubov unitary operator $\hat{B}=\hat{B}(\mu,v)$ is defined by this relation up to arbitrary phase factor. A convenient choice then leads to the normal form

$$\hat{B}(\mu,v)=\mu^{-\frac{1}{2}}\exp\left(-\frac{v}{2\mu}\hat{a}^{+2}\right)\exp(-\ln\mu\hat{a}^{+}\hat{a})\exp\left(\frac{v^{*}}{2\mu}\hat{a}^{2}\right), \quad (1)$$

or, in terms of the squeeze operator, $\hat{S}(\xi)\equiv\exp\frac{1}{2}(\xi^{*}\hat{a}^{2}-\xi\hat{a}^{+2})$, to

$$\hat{B}(\mu,v)=\exp[-i\phi(\hat{a}^{+}\hat{a}+\frac{1}{2})]\hat{S}(\xi), \quad (2)$$

with

$$\xi\equiv|\xi|\exp[i(\phi+\theta)],\ \ |\xi|\equiv\tanh^{-1}\left(\frac{|v|}{|\mu|}\right). \quad (3)$$

Equation (2) shows that, in the general case of a *complex* μ, the Bogoliubov operator is *not* identical to the squeeze operator. Now we note that, with the proposed choice of the phase factor, the Bogoliubov transformations *form a continuous non-Abelian group* since, for any μ', v' and μ'', v'' (with $|\mu'|^2-|v'|^2=1$ and $|\mu''|^2-|v''|^2=1$, one has

$$\hat{B}(\mu',v')\hat{B}(\mu'',v'')=\hat{B}(\mu,v), \quad (4)$$

with

$$\mu\equiv\mu'\mu''+v'^{*}v'',\ \ v\equiv\mu'^{*}v''+v'\mu'' \quad (5)$$

and for which again $|\mu|^2-|v|^2=1$. The unit element of the group is $\hat{B}(1,0)$, and the inverse of $\hat{B}(\mu,v)$ is $\hat{B}^{-1}(\mu,v)=\hat{B}^{+}(\mu,v)=\hat{B}(\mu^{*},-v)$. Thus, any two Bogoliubov transformations applied in succession are equivalent to a single Bogoliubov transformation. This is to be compared with e.g. the case of the displacement operator, $\hat{D}(\delta)\equiv\exp(\delta\hat{a}^{+}-\delta^{*}\hat{a})$ (with δ denoting a complex parameter), for which one has

$$\hat{D}(\delta')\hat{D}(\delta'')=\exp\left[\frac{1}{2}(\delta'\delta''^{*}-\delta'^{*}\delta'')\right]\hat{D}(\delta'+\delta'') \quad (6)$$

or, more importantly, with the case of the closely related squeeze operator for which

$$\hat{S}(\xi')\hat{S}(\xi'')=\exp\left[-i\phi(\hat{a}^{+}\hat{a}+\frac{1}{2})\right]\hat{S}(\xi) \quad (7)$$

with

$$\xi'\equiv|\xi'|\exp(i\theta'),\ \ \mu'\equiv\cosh|\xi'|,\ \ v'\equiv\exp(i\theta')\sinh|\xi'|, \quad (8)$$

and with analogous relations for ξ'', μ'' and v''. In (7), ξ is obtained from (3), with μ and v as defined by (5). Equation (7) shows that, in the general case, any two successive squeeze transformations are, in fact, equivalent to a single *Bogoliubov* transformation, as defined by (2). This demonstrates that the squeeze operator does *not* provide sufficiently general and complete picture. The subset of all Bogoliubov transformations, with μ real (i.e. $\phi = 0$), consists of all squeeze transformations, but this subset does not form a group. Consistent use of the Bogoliubov operator, instead of the squeeze operator, leads to simplification and, more importantly, to certain new results. In this Section, the proofs of the operator identity (4), and another useful identity involving Bogoliubov and displacement operators, namely

$$\hat{D}(\delta)\hat{B}(\mu,\nu) = \hat{B}(\mu,\nu)\hat{D}(\mu\delta + \nu\delta^*) \tag{9}$$

will be outlined. In Section 3, with the help of these new operator identities, the *correlation amplitude* for ordinary and also *generalized squeezed states* will be found. Finally, general expressions for the *total* and *dynamical* phases will be obtained yielding, from their difference, the *geometric* (*Pancharatnam*) *phase* for a nonadiabatic and noncyclic evolution from an initial generalized squeezed state at time t' to the final state at a time t''.

The two operator identities, (4) and (9) (together with (7)), can be established (laboriously) with the help of the *normal ordering method* [13], using the normal ordered forms of the Bogoliubov and displacement operators. Simpler approach uses coherent states to establish, in these identities, equality of the left and right hand sides of the matrix elements, $\langle\alpha\,|\,l.h.s.\,|\,\beta\rangle = \langle\alpha\,|\,r.h.s.\,|\,\beta\rangle$, between two arbitrary coherent states $|\,\alpha\rangle$ and $|\,\beta\rangle$. The proofs are then accomplished using (over)completeness of the coherent states and some straightforward integration. However, once the operator identities (4) and (9) are established, the simplest proof is obtained within the *two-photon algebra* h_6 [8]. As is well known, this algebra is spanned by the six operators $\{\hat{a}^+\hat{a}+1/2, \hat{a}^{+2}, \hat{a}^2, \hat{a}^+, \hat{a}, \hat{1}\}$ with familiar commutation relations. In physical processes, this algebra acts on the harmonic-oscillator Hilbert space, and the matrix elements of these operators are computed with respect to the number states $|\,n\rangle$ that form a basis for this space. However, many computational simplifications can be obtained in the smallest faithful (4×4) matrix representation of this algebra (this is given explicitly in [8]). This finite-dimensional representation is *not* Hermitian (nor unitary) and thus it is not directly useful for computing Hilbert space matrix elements. Nevertheless, the 4×4 matrix representation is useful for disentangling exponential operator products. In the case of the identities (4) and (9), one requires the matrix representations of $\hat{B}(\mu,\nu)$ and $\hat{D}(\delta)$. We find

$$\hat{B}(\mu,\nu) \doteq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mu^* & -\nu & 0 \\ 0 & -\nu^* & \mu & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \hat{D}(\delta) \doteq \begin{pmatrix} 1 & 0 & 0 & 0 \\ \delta & 1 & 0 & 0 \\ \delta^* & 0 & 1 & 0 \\ 0 & \delta^* & -\delta & 1 \end{pmatrix}, \tag{10}$$

where the symbol $\doteq$ stands for "is represented by". The proofs of the operator identities (4) and (9) then boil down to simple matrix multiplication. Incidentally, equation (10) shows that a faithful 2×2 non-unitary matrix representation of the boson Bogoliubov group is

$$\hat{B}(\mu,v) \doteq \begin{pmatrix} \mu^* & -v \\ -v^* & \mu \end{pmatrix}, \ |\mu|^2 - |v|^2 = 1, \tag{11}$$

the latter being a subgroup of the $SL(2,C)$ group. Additionally, the Hilbert space matrix elements of the Bogoliubov operator, $B_{mn}(\mu,v) \equiv \langle m|\hat{B}(\mu,v)|n\rangle$, already obtained earlier in [7], provide an infinite-dimensional *unitary* representation of the boson Bogoliubov group (one should only take into account a different phase factor used there in the definition of the Bogoliubov operator). These matrix elements are expressed in terms of the polynomials which are solution of the linear second-order differential equation given explicitly in [7]. Finally in this section we remark that the boson Bogoliubov group is, in fact, the inhomogeneous metaplectic group generated by $\hat{a}^2, \hat{a}^{+2}, \hat{a}^+\hat{a}, \hat{a}, \hat{a}^+$, and $\hat{1}$, and that the subgroup generated by $\hat{a}^2, \hat{a}^{+2}$ and $\hat{a}^+\hat{a} + 1/2$ is isomorphic to $SU(1,1)$ as it can be seen from (11).

3. CORRELATION AMPLITUDE AND GEOMETRIC PHASE

Now we focus on the case of the general time dependent Hamiltonian containing terms linear and quadratic in annihilation and creation operators [4, 7, 8] ($\hbar = 1$)

$$\hat{H}(t) = \omega(t)\left(\hat{a}^+\hat{a} + \frac{1}{2}\right) + [h(t)\hat{a} + g(t)\hat{a}^2 + H.c.] \tag{12}$$

One can think, for deffinitiveness, of a single quantum oscillator, with a variable frequency $\omega(t)$, driven by a transient external force represented in the Hamiltonian by the two given complex functions $h(t)$ and $g(t)$. In order for $\hat{H}$ to be interpreted as an energy, it should be bounded below. This requires $\omega > 2|g|$. This Hamiltonian has important applications in quantum optics and molecular dynamics [8]. The corresponding time evolution operator is

$$\hat{U}(t,0) = \exp(i\varepsilon)\hat{B}(\mu,v)\hat{D}(\delta) \tag{13}$$

with three complex and one real time dependent parameters μ, v, δ and ε respectively, satisfying a well known set of coupled differential equations of motion [4, 7, 8]. Their integration (numerical in the general case) leads to the functions $\mu(t)$, $v(t)$, $\delta(t)$ and $\varepsilon(t)$ which determine completely the time evolution operator (13). A single quantum oscillator, prepared initially ($t = 0$) in the m-th number state $|m\rangle$ (with $m = 0,1,2,...$) evolves then to the corresponding generalized squeezed state (GSS), $|GSS,m,t\rangle \equiv U(t,0)|m\rangle$. The special case $m = 0$ produces the ordinary squeezed states. These states have been first studied in the field of quantum optics with the ultimate aim to obtain a reduced fluctuation in one field quadrature, at the expense of an increased fluctuation in the other, leading to an increase in the signal to noise ratio. In order to find how the generalized squeezed states at different times, t' and t'', are correlated with each other, one constructs the inner product

$$\begin{aligned} C(t',t'') &\equiv \langle GSS,m,t'|GSS,m,t''\rangle = \\ &= \exp[i(\varepsilon''-\varepsilon')]\langle m|\hat{D}^+(\delta')\hat{B}^+(\mu',v')\hat{B}(\mu'',v'')\hat{D}(\delta'')|m\rangle . \end{aligned} \tag{14}$$

Here the notation $\varepsilon' \equiv \varepsilon(t')$, $\varepsilon'' \equiv \varepsilon(t'')$, and analogous, is used. With the help of the operator identities (4), (9) and (6), and noting that $\hat{D}^{+}(\delta) = \hat{D}(-\delta)$ in the case of (9), we obtain the *correlation amplitude*

$$C(t',t'') = \exp[i\chi(t',t'')]\langle m \,|\, \hat{B}(\bar{\mu},\bar{\nu})\hat{D}(\bar{\delta}) \,|\, m\rangle \,, \tag{15}$$

with

$$\bar{\mu} \equiv \mu'^{*}\mu'' - \nu'^{*}\nu'' ,\;\; \bar{\nu} \equiv \mu'\nu'' - \nu'\mu'' ,\;\; \bar{\delta} \equiv \delta'' - \bar{\mu}\delta' - \bar{\nu}\delta'^{*} , \tag{16}$$

and

$$\chi(t',t'') \equiv \varepsilon'' - \varepsilon' - Im\{(\delta'' - \bar{\delta})\delta''^{*}\} \,. \tag{17}$$

It is seen that the correlation amplitude is effectively determined by the diagonal matrix element $\overline{U}_{mm}$ of the time evolution operator (13), in which the substitutions $\mu \to \bar{\mu}$, $\nu \to \bar{\nu}$, and similar, are made. The explicit expression for this matrix element can be found in [7]. The *modulus* of the correlation amplitude provides a quantitative measure of the resemblance between the generalized squeezed states at different times. The *phase* $\phi_m(t',t'')$ of the correlation amplitude

$$\phi_m(t',t'') = \chi(t',t'') - \bar{\varepsilon} + \arg\{\overline{U}_{mm}\} \,. \tag{18}$$

together with the dynamical phase [10]

$$\delta_m(t',t'') = \int_{t'}^{t''} \langle \hat{H}(t)\rangle dt \,, \tag{19}$$

lead, additionally, to the *geometric* (*Pancharatnam*) phase, $\beta_m(t',t'') = \phi_m(t',t'') - \delta_m(t',t'')$, for a nonadiabatic and noncyclic evolution from an initial generalized squeezed state at time t' to the final state at a time t'' [11]. In (19), the expectation value of the Hamiltonian, at a time t, is obtained from

$$\begin{aligned}\langle \hat{H}(t)\rangle &= \langle GSS,m,t \,|\, \hat{H}(t) \,|\, GSS,m,t\rangle = \\ &= \langle m \,|\, \hat{D}^{+}(\delta)\hat{B}^{+}(\mu,\nu)\hat{H}(t)\hat{B}(\mu,\nu)\hat{D}(\delta) \,|\, m\rangle \,.\end{aligned} \tag{20}$$

Using $\hat{B}\hat{a}\hat{B}^{+} = \mu\hat{a} + \nu\hat{a}^{+}$ (so that $\hat{B}^{+}\hat{a}\hat{B} = \mu^{*}\hat{a} - \nu\hat{a}^{+}$), and also $\hat{B}^{+}\hat{a}^{2}\hat{B} = \mu^{*2}\hat{a}^{2} + \nu^{2}\hat{a}^{+2} - \mu^{*}\nu(2\hat{a}^{+}\hat{a} + 1)$, we find

$$\langle \hat{H}(t)\rangle = (\omega - 2Im\{\dot{\nu}\nu^{*}\})\left(m + |\delta|^{2} + \frac{1}{2}\right) - 2\dot{\varepsilon} + Im\{\delta^{*2}(\mu\dot{\nu} - \dot{\mu}\nu)\} \,. \tag{21}$$

Equations (19) and (21) provide, then, the required dynamical phase. All phases are *global* quantities (the same at different points in the configurational space), depend linearly on the quantum number m, and all are uniquely defined *up to* $2\pi n$ (n = integer). It is apparent that the total phase (and therefore the Pancharatnam phase as well), is *not* an additive quantity (the phase for evolution $t' \to t'' \to t'''$ is not the sum of the phases for the evolution $t' \to t''$ and then $t'' \to t'''$, with $0 \le t' < t'' < t'''$). Different limiting cases of β_m can be observed, e.g. when $g(t) \equiv 0$ and $m = 0$, the Pancharatnam phase for *coherent* states is obtained [14].

In various special cases of some practical importance, for example when $\omega(t) = \omega_0 =$

const, $h(t) \equiv 0$ and $g(t) \equiv G\exp(2i\omega_0 t)$, with G representing a real quantity characterizing the strength of the interaction in (12) (in such case the first term in the Hamiltonian describes e.g. the free energy of the single-mode electromagnetic field while the remaining terms describe a two-photon interaction process such as *parametric amplification* when a classical wave of frequency $2\omega_0$ generates two photons with frequency ω_0), the equations of motion can be integrated in closed form. Finally, we remark that the ordinary squeezed states have been already produced [15], and that the generalized squeezed states can, in principle, be obtained in much the same way by driving two-photon processes with a classical source. With the experimental realization of generalized squeezed states, in the realm of quantum optics, it will hopefully become possible to study these states in more detail.

In summary, a general boson Bogoliubov transformation operator which forms a continuous non-Abelian group was introduced. On this basis the identities of this operator were studied and the general expressions for the correlation amplitude and geometric phase for ordinary and generalized squeezed states were found.

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BOZONSKA BOGOLJUBOVLJEVA GRUPA I GENERALIZOVANA SAŽETA STANJA

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Predložen je izbor proizvoljne faze Bogoljubovljevog unitarnog operatora koji obezbeđuje da skup svih homogenih Bogoljubovljevih transformacija bozonskih anihilacionih i kreacionih operatora čini kontinualnu neabelovu podrgupu SL(2,C) grupe. Ustanovljena su dva identiteta koja zadovoljava ovakav Bogoljubovljev operator i pomoću njih su nađeni izrazi za korelacionu amplitudu i geometrijsku (Pančaratnamovu) fazu generalizovanih sažetih stanja. Kratko je prodiskutovana primena ovih opštih rezultata na dvofotonske interakcione procese u kvantnoj optici.

Ključne reči: *Bozonska Bogoljubovljeva transformacija, geometrijska faza, sažeta stanja*