UNIVERSITY OF NIŠ
The scientific journal **FACTA UNIVERSITATIS**
Series: **Physics, Chemistry and Technology** Vol. 1, N° 5, 1998 pp. 97 - 100
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PANCHARATNAM PHASE AND GALILEAN TRANSFORMATION

UDC: 514.84

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Abstract. *Galilean transformation properties of the total, dynamical and geometric phases for generally nonadiabatic and noncyclic evolution of a quantum system are obtained in the context of non-relativistic quantum mechanics. In particular, it is shown that whether the evolution of the quantum system in the interval* $[t_1,t_2]$ *is cyclic does depend on the state of its (uniform) motion.*

Key words: *quantum mechanics, Pancharatnam (geometric) phase, Galilean transformation*

Since the discovery by Berry [1], in the context of non-relativistic quantum mechanics, of the general existence of an observable phase accumulation in the wave function of a quantum-mechanical system with an adiabatically changing Hamiltonian, the understanding of this phase has gained greatly in its deepness. The Berry phase has attracted great theoretical interest and it has been repeatedly corroborated by experiment. The restriction to adiabaticity was lifted by Aharonov and Anandan [2] by removing from the wave function the time integral of the expectation value of the Hamiltonian as a *dynamical phase*. It was shown that once the dynamical phase is removed, the phase difference accumulated during the time-evolution of the system has purely geometric origin. Finally, the restriction to cyclic motion, and also unitary evolution, was removed by Samuel and Bhandari [3]. Their work was based on the earlier investigation of Pancharatnam [4] on the interference of polarized light. A recent resource letter on geometric phases is found in [5]. In particular, the geometric phase for *coherent*, *displaced number* and also *squeezed states* of the quantum oscillator has been discussed repeatedly (see *e.g.* [6-8]). In this paper, the Galilean transformation properties of the total, dynamical and geometric phases for generally nonadiabatic and noncyclic evolution

Received February 20, 1998; in revised form July 10, 1998; accepted August 14, 1998

of a quantum system, are established. In particular, it will be shown that whether the evolution of the system in the interval $[t_1, t_2]$ is cyclic depends on the state of its uniform motion.

Consider, in an inertial frame of reference S, a quantum system undergoing generally nonadiabatic and noncyclic evolution during the interval $[t_1, t_2]$, so that its initial and final states are $|\psi, t_1\rangle$ and $|\psi, t_2\rangle$, respectively (we are assuming throughout that the states are non-orthogonal and also normalized). Such evolution defines a curve in projective Hilbert space. Following the work of Samuel and Bhandari [3], it is known that the total phase $\Phi(t_1, t_2)$, appearing in

$$\langle\psi, t_1 | \psi, t_2\rangle \equiv \rho(t_1, t_2)e^{i\Phi(t_1,t_2)}, \quad \rho(t_1, t_2) > 0, \tag{1}$$

can be decomposed into a dynamic and geometric part. The dynamic part, denoted by $\delta(t_1, t_2)$, is given by the time integral of the expectation value of the Hamiltonian generating the evolution of the system ($\hbar = 1$)

$$\delta(t_1, t_2) = -\int_{t_1}^{t_2} \langle\psi, t | \hat{H}(t) | \psi, t\rangle dt \tag{2}$$

The geometric part of the phase, denoted by $\beta(t_1,t_2)$, is determined here indirectly by removing the accumulation of local phase changes from the global phase, i.e. $\beta(t_1, t_2) = \Phi(t_1, t_2) - \delta(t_1, t_2)$. The geometric phase is reparametrisation invariant, i.e. independent of the speed at which the path in projective Hilbert space is traversed; it depends only on this path [3]. Now consider another inertial frame S', moving relative to S with constant velocity $\mathbf{V}$ so that $\mathbf{r}' = \mathbf{r} - \mathbf{V}t$ and $t' = t$ (here we consider only the Galilean subgroup consisting of boosts in a constant direction). Galileo's relativity principle asserts the invariance of the mechanical equations, and in particular the Schrödinger equation, under such transformation. As is well known [9], the effect the Galilean transformation produces, in this context, is to multiply the wave function by the phase factor, $\Psi(\mathbf{r},t) = \Psi'(\mathbf{r}',t')\exp(i\chi)$, with

$$\chi = \chi(\mathbf{r}', t') \equiv m\left(\mathbf{V}\cdot\mathbf{r}' + \frac{V^2 t'}{2}\right). \tag{3}$$

Here, m is the mass of the system. This, together with (1) implies that the total phases in the two frames are related via

$$\Phi(t_1, t_2) = -\frac{mV^2}{2}(t_2' - t_1') + \theta'(t_1', t_2') \tag{4}$$

and also that

$$\rho(t_1, t_2) = R'(t_1', t_2') \tag{5}$$

Here, $R'(t_1', t_2')$ and $\theta'(t_1', t_2')$ are the modulus and phase of the following expression

$$R' e^{i\theta'} \equiv \int d^3r' \psi'^*(\mathbf{r}', t_1')\psi'(\mathbf{r}' - \mathbf{R}_{12}', t_2')$$

with $\mathbf{R}_{12}' \equiv \mathbf{V}(t_2' - t_1')$. In special case $\mathbf{R}_{12}' \equiv 0$, $R' e^{i\theta'}$ reduces exactly to $\rho' e^{i\Phi'}$, so that one can, in (4), delineate in the frame S' the contribution of the total phase, Φ', to the phase θ'. Therefore, eq. (4) provides in effect the Galilean transformation for the total phase.

Similarly, from (2) one obtains for the dynamical phases in the two frames

$$\delta(t_1,t_2)=\delta'(t_1',t_2')-\frac{mV^2}{2}(t_2'-t_1')-\mathbf{V}\cdot\int_{t_1'}^{t_2'}\left\langle\hat{\mathbf{p}}'\right\rangle_{t'}dt' \qquad (6)$$

Here $\left\langle\hat{\mathbf{p}}'\right\rangle_{t'}\equiv\left\langle\psi',t'|\hat{\mathbf{p}}'|\psi',t'\right\rangle$ is the expectation value of the momentum operator with respect to state $|\psi',t'\rangle$, in S'. Finally, eqs. (4) and (6) yield Galilean transformation of the Pancharatnam phase

$$\beta(t_1,t_2)=\beta'(t_1',t_2')+\theta'(t_1',t_2')-\Phi'(t_1',t_2')+\mathbf{V}\cdot\int_{t_1'}^{t_2'}\left\langle\hat{\mathbf{p}}'\right\rangle_{t'}dt' \qquad (7)$$

Obviously, the geometric phase in non-relativistic quantum mechanics is not Galilean invariant.

The question now arises whether this Pancharatnam phase, which is well-defined in all frames, leads in special case of *cyclic evolution in one frame* to the cyclic evolution in *all* frames. It is demonstrated in the following that this is *not* the case. Consider a system undergoing cyclic evolution during the time interval $[t_1, t_2]$. Classically there is no difference between the initial and final states, and the system in its final state appears as if it has not undergone any evolution. From the quantum viewpoint, the final and initial states coincide *up to a global phase*, $|\psi,t_2\rangle=\exp[i\Phi(t_1,t_2)]|\psi,t_1\rangle$ [10], so that the memory of the cyclic evolution is completely contained in the phase factor. This, together with (1), leads to the equivalent condition, $\rho(t_1,t_2) = 1$, for the evolution to be cyclic. Since generally eq. (5) holds good, and since $\rho'(t_1',t_2')\neq R'(t_1',t_2')$, the condition $\rho(t_1,t_2) = 1$, in S does *not* imply $\rho'(t_1',t_2')=1$ in S', one finds that cyclic evolution in non-relativistic quantum mechanics is *not* a Galilean invariant notion. This, in fact, should not come as a surprise if one recalls that analogous situation arises even in the classical mechanics. Indeed, there a cyclic evolution in the frame S implies a closed orbit in the corresponding phase space. Due to the spatial displacement of the frames S and S', this closure property of the phase curve in S is not preserved in S' so that the evolution is not cyclic in the latter frame.

In conclusion, the Galilean transformation properties of the total, dynamical and geometric phases for generally nonadiabatic and noncyclic evolution of a quantum system are obtained in the context of non-relativistic quantum mechanics. In particular, it is shown that whether the evolution of the quantum system in the interval $[t_1,t_2]$ is cyclic does depend on the state of its motion.

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GALILEJEVE TRANSFORMACIJE PANČARATNAMOVE FAZE

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Galilejeve transformacije ukupne, dinamičke i gemetrijske faze za opšti slučaj neadijabatske i neciklične evolucije kvantnog sistema su nađene u kontekstu nerelativističke kvantne mehanike. Posebno, pokazano je da cikličnost evolucije kvantnog sistema tokom intervala [t_1,t_2] zavisi od stanja njegovog (uniformnog) kretanja.

Ključne reči: *kvantna mehanika, Pančaratnamova (geometrijska) faza, Galilejeva transformacija*