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AN OPERATOR CORRESPONDING TO THE PHASE OF HARMONIC OSCILLATOR

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Abstract. *An quantum mechanical operator corresponding to the tangent of the classical phase variable of harmonic oscillator is constructed together with the related phase states. The properties of these states and the way they relate to the Pegg-Barnett Hermitian phase operator theory are investigated.*

Key words: *quantum harmonic oscillator, phase operator*

1. Introduction

The question of the proper dynamical variable corresponding to the phase of a quantum field has been the subject of discussion for a long time. The problem appeared to be solved for the first time by Dirac in 1927 only to be refuted much later, in 1964, by Susskind and Glogower [1]. They introduced two Hermitian dynamical variables analogous to the sine and cosine of the phase, but as the two variables do not commute this has also been regarded as an unsatisfactory solution. There have been numerous attempts to construct other, more satisfactory phase operators [2-6]. These are all based on the well known fact that for *any* superposition of number states of harmonic oscillator, the position and momentum expectation values follow *exactly* the corresponding classical motion, $x(t) = X\cos(\varphi - \omega t)$ and $p(t) = m\omega X\sin(\varphi - \omega t)$, leading to the supposition that the classical phase φ-ωt remains a well defined concept in the quantum domain. Phase calculations, based on various phase formalisms, have been carried out for coherent [7], squeezed [8], displaced number [9] and generalized squeezed states [10]. Several experiments were also reported in which phase differences and their fluctuations were measured as a function of average photon number, and attempts were made to test some of the definitions against experiments [11] but no clear conclusion emerged. All this illustrates difficulties in formulating a simple and consistent operator of the phase of a

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quantum oscillator and even questions the status of the phase as an observable in the framework of the orthodox quantum mechanical theory. This long-standing problem is of considerable theoretical and also experimental importance since the number-phase uncertainty relation depends on it. In this paper the matrix representation, in the number base, of an operator corresponding to the *tangent* of the classical phase variable is postulated in the framework of the conventional quantum mechanics. The exact eigenkets of this operator are obtained, and an approximate solution, valid for $n >> 1$ (n denoting the eigenvalue of the number operator), is also found. Based on the exact solution, the corresponding *phase states* are then introduced. It is found that the approximate expansion coefficients (valid for $n >> 1$) are closely related to the expansion coefficients of the phase states postulated in the Pegg-Barnett model [3-4, 6]. The completeness and orthonormality of the phase states associated with our postulated operator are examined, and it is demonstrated that these phase states indeed form a complete set, but *fail* to fulfill the orthonormality condition in its usual form. The consequences of this failure are discussed. Additionally, the modifications leading to the Pegg-Barnett Hermitian phase operator theory are investigated.

2. OPERATOR CORRESPONDING TO THE TANGENT OF THE CLASSICAL PHASE

One starts from the requirements that the quantum-mechanical phase should have the same significance as the classical phase in the appropriate limit, and that it should be associated with well behaved Hermitian operator so that it is, at least in principle, an observable quantity. Classically, the tangent of the phase angle for the simple harmonic oscillator is

$$\tan(\varphi - \omega t) = \frac{1}{m\omega}\frac{p(t)}{x(t)} \tag{2.1}$$

From this we postulate the quantum mechanical operator in the Heisenberg picture to be

$$\hat{\tau}(t) \equiv \frac{1}{2m\omega}(\hat{p}(t)\hat{x}^{-1}(t) + \hat{x}^{-1}(t)\hat{p}(t)) \tag{2.2}$$

The symmetrization is introduced in (2.2), as usual, for two reasons: to obtain Hermitian operator and to resolve the ordering ambiguity. This is a "natural" quantization of the classical observable (2.1). In fact, by considering other, more general ordering rules one can construct (infinitely) many different phase operators. The properties of the phase operator (2.2) have not been explored and the purpose of this paper is to fill this gap. The matrix representations of the $\hat{x}(t)$ and $\hat{p}(t)$ in the number base are well known [12]. From the matrix representing $\hat{x}(t)$ one finds, after some straightforward algebra, its inverse $\hat{x}^{-1}(t)$ and, from (2.2), the matrix representation of the operator corresponding to the tangent of the classical phase

$$\hat{\tau}(t)=\begin{pmatrix} 0 & 0 & \sqrt{\frac{2}{1}}\varepsilon_1^* & 0 & -\sqrt{\frac{2\cdot4}{1\cdot3}}\varepsilon_2^* & 0 & \sqrt{\frac{2\cdot4\cdot6}{1\cdot3\cdot5}}\varepsilon_3^* & \cdots \\ 0 & 0 & 0 & \sqrt{\frac{2}{3}}\varepsilon_1^* & 0 & -\sqrt{\frac{2\cdot4}{3\cdot5}}\varepsilon_2^* & 0 & \\ \sqrt{\frac{2}{1}}\varepsilon_1 & 0 & 0 & 0 & \sqrt{\frac{4}{3}}\varepsilon_1^* & 0 & -\sqrt{\frac{4\cdot6}{3\cdot5}}\varepsilon_2^* & \\ 0 & \sqrt{\frac{2}{3}}\varepsilon_1 & 0 & 0 & 0 & \sqrt{\frac{4}{5}}\varepsilon_1^* & 0 & \\ -\sqrt{\frac{2\cdot4}{1\cdot3}}\varepsilon_2 & 0 & \sqrt{\frac{4}{3}}\varepsilon_1 & 0 & 0 & 0 & \sqrt{\frac{6}{5}}\varepsilon_1^* & \\ 0 & -\sqrt{\frac{2\cdot4}{3\cdot5}}\varepsilon_2 & 0 & \sqrt{\frac{4}{5}}\varepsilon_1 & 0 & 0 & 0 & \\ \sqrt{\frac{2\cdot4\cdot6}{1\cdot3\cdot5}}\varepsilon_3 & 0 & -\sqrt{\frac{4\cdot6}{3\cdot5}}\varepsilon_2 & 0 & \sqrt{\frac{6}{5}}\varepsilon_1 & 0 & 0 & \\ \vdots & & & & & & & \ddots \end{pmatrix}$$

Here $\varepsilon_k \equiv i\exp(2ik\omega t)$ with $i=\sqrt{-1}$ and $k = 1,2,...$. The regularities along non zero sub and super diagonals of this matrix are evident. The matrix is manifestly Hermitian. If we denote an eigenket of the operator $\hat{\tau}\equiv\hat{\tau}(0)$, at some initial instant of time $t = 0$, by its eigenvalue τ', then $\hat{\tau}|\tau'\rangle=\tau'|\tau'\rangle$ leads to the following three-term recurrence relation ($n = 2,3,...$)

$$(\tau'+i)\sqrt{n(n-1)}t_n(\tau')=-[(2n-3)-2i]t_{n-2}(\tau')-(\tau'-i)\sqrt{(n-2)(n-3)}t_{n-4}(\tau'), \quad (2.3)$$

for the expansion coefficients $t_n(\tau')\equiv\langle n|\tau'\rangle$ of the eigenkets $|\tau'\rangle$ in the number base, $|\tau'\rangle=\sum_n t_n(\tau')|n\rangle$. From the matrix representation of the $\hat{\tau}$ operator it is apparent that there are *two* linearly independent eigenkets corresponding to the same eigenvalue τ' (this being, therefore, two-fold degenerate)

$$|\tau',+\rangle\equiv\sum_{n=0}^{\infty}t_{2n}(\tau')|2n\rangle\ ,\quad |\tau',-\rangle\equiv\sum_{n=0}^{\infty}t_{2n+1}(\tau')|2n+1\rangle. \quad (2.4)$$

These two eigenkets are mutually orthogonal. The three-term recurrence relation (2.3) leads to the following closed form expressions for the expansion coefficients

$$t_{2n}(\tau')=\frac{c_{2n}}{(1-z)^{n+3/4}}\,{}_2F_1(\tfrac{1}{2}-n,-n,\tfrac{5}{4}-n,z)\ , \quad (2.5)$$

and

$$t_{2n+1}(\tau')=\frac{c_{2n+1}}{(1-z)^{n+5/4}}\,{}_2F_1(-n,-n-\tfrac{1}{2},\tfrac{3}{4}-n,z)\ , \quad (2.6)$$

with.

$$z\equiv\frac{1+i\tau'}{2},\ \ c_{2n}\equiv\frac{\Gamma(3/4)}{2\pi^{3/4}}\frac{(-2)^n(-\frac{1}{4})_n}{[(2n)!]^{1/2}},\ \ c_{2n+1}\equiv\frac{\Gamma(5/4)}{2^{1/2}\pi^{3/4}}\frac{(-2)^n(1/4)_n}{[(2n+1)!]^{1/2}}\ . \quad (2.7)$$

In (2.7), $(a)_n = a(a+1)...(a+n-1)$ denotes the Pochhammer symbol. Since in (2.5) and (2.6), $n = 0,1,2,...$, the hypergeometric functions appearing there are polynomials of order n, so that expansion coefficients $t_n(\tau')$ are simply algebraic functions of the auxiliary complex variable z. The τ' values range in the infinite interval $\tau' \in (-\infty,+\infty)$. The constants c_{2n}, and c_{2n+1} are determined from the normalization condition. In fact, any two even expansion coefficients are mutually orthogonal

$$\int_{-\infty}^{+\infty} d\tau' t_{2n}^*(\tau') t_{2n'}(\tau') = \delta_{nn'}, \tag{2.8}$$

and similarly in the odd case one has

$$\int_{-\infty}^{+\infty} d\tau' t_{2n+1}^*(\tau') t_{2n'+1}(\tau') = \delta_{nn'}. \tag{2.9}$$

In the $n >> 1$ limit the three-term recurrence (2.3) becomes simply

$$(\tau'+i)t_n(\tau') \cong -2\tau' t_{n-2}(\tau') - (\tau'-i)t_{n-4}(\tau'), \tag{2.10}$$

with the relevant solution

$$t_n(\tau') = g(\tau')e^{in\arctan\tau'}. \tag{2.11}$$

This expression is exact solution of (2.10) and only approximate solution of (2.3), valid for $n >> 1$. In (2.11), $g(\tau')$ represents a function of τ'. In particular, the choice $g(\tau') \equiv \pi^{-1/2}(1+\tau'^2)^{-1/2}$ leads to the approximate solution that agrees with the $n >> 1$ limit of (2.5) and (2.6) and that also satisfies (2.8) and (2.9).

3. PHASE STATES

Suppose that the (initial) phase angle φ' ranges generally in the interval from φ_0 to $\varphi_0 + 2\pi$, with φ_0 representing a constant phase, and define $\tau' \equiv \tan\varphi'$. Then for each phase φ' from the first or fourth quadrant, there is another phase, $\varphi' + \pi$, that gives the same τ' value (this is causing, incidentally, the two-fold degeneracy of the τ' eigenvalues). Therefore, the states $|\tau',\pm\rangle$ are certain linear combinations of the corresponding *phase states* $|\varphi'\rangle$ and $|\varphi'+\pi\rangle$, and vice versa. Now, for the $n >> 1$ part of the states $|\tau',\pm\rangle$, the approximate solution (2.11) holds good. This in turn shows that the state proportional to $|\tau',+\rangle + |\tau',-\rangle$ corresponds to the phase state $|\varphi'\rangle$, and similarly that $|\tau',+\rangle - |\tau',-\rangle$ corresponds to the state $|\varphi'+\pi\rangle$. This motivates the following definition of the phase states

$$|\varphi'\rangle \equiv \sum_{n=0}^{+\infty} f_n(\varphi')|n\rangle, \tag{3.1}$$

with $\varphi' \in [\varphi_0, \varphi_0 + 2\pi)$, $\tau' \equiv \tan\varphi'$, and

$$f_n(\varphi') = \langle n|\varphi'\rangle \equiv \left(\frac{1+\tau'^2}{2}\right)^{\frac{1}{2}} t_n(\tau'), \tag{3.2}$$

for φ' from the first or fourth quadrant, and

$$f_n(\varphi') \equiv (-1)^n \left(\frac{1+\tau'^2}{2} \right)^{\frac{1}{2}} t_n(\tau'), \tag{3.3}$$

for φ' from the second or third quadrant. (Note that in this case, one has $f_n(\varphi') = (-1)^n f_n(\varphi' - \pi)$).

Using (2.8) and (2.9) one verifies that the expansion coefficients $f_n(\varphi')$ are orthonormal for *any n* and *n*'

$$\int_{\varphi_0}^{\varphi_0+2\pi} d\varphi' f_n^*(\varphi') f_{n'}(\varphi') = \delta_{nn'} . \tag{3.4}$$

Thus, the phase states (3.1) are complete

$$\int_{\varphi_0}^{\varphi_0+2\pi} d\varphi' |\varphi'\rangle\langle\varphi'| = \hat{1} , \tag{3.5}$$

so that in particular (cf. (3.1))

$$|n\rangle = \int_{\varphi_0}^{\varphi_0+2\pi} d\varphi' f_n^*(\varphi') |\varphi'\rangle . \tag{3.6}$$

In the case when $n >> 1$ one gets from (2.11), (3.2) and (3.3), the approximate expansion coefficients

$$f_n(\varphi') \cong \frac{\exp(in\varphi')}{(2\pi)^{1/2}} \quad \varphi' \in [\varphi_0, \varphi_0 + 2\pi) . \tag{3.7}$$

These coefficients are closely related to the expansion coefficients of the phase states postulated in the Pegg-Barnett theory.

In order to examine the orthonormality of the exact phase states, it is *not* permissible to substitute (3.6) into (3.1), *invert the order* of integration over φ' and summation over *n*, and conclude (erroneously in this case) that $\langle\varphi|\varphi'\rangle = \delta(\varphi - \varphi')$. As is well known [13], this manipulation is *not* always justified and, in fact, precisely this point needs to be investigated. One proceeds by using the completeness of the position states to write

$$\langle\varphi|\varphi'\rangle = \int_{-\infty}^{+\infty} dx \langle\varphi|x\rangle\langle x|\varphi'\rangle . \tag{3.8}$$

Now, Eqs. (3.1)-(3.3) and (2.5)-(2.6) imply that

$$\langle x|\varphi\rangle = \frac{(1+\tan^2\varphi)^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}\sigma} |x|^{\frac{1}{2}} \exp\left[i\left(\frac{x}{2\sigma}\right)^2 \tan\varphi \right] \theta(\pm x) . \tag{3.9}$$

Here, $\sigma \equiv (\hbar/2m\omega)^{1/2}$, $\theta(\pm x)$ denotes the Heaviside step function, and the upper (lower) sign is used when $\cos\varphi > 0$ (< 0). The last two equations lead to [14]

$$\langle\varphi|\varphi'\rangle = \frac{1}{2}\delta(\varphi-\varphi') + \frac{1}{2\pi}\sqrt{1+\tan^2\varphi}\sqrt{1+\tan^2\varphi'}\,P.v.\left(\frac{i}{\tan\varphi'-\tan\varphi}\right), \qquad (3.10)$$

whenever $\cos\varphi$ and $\cos\varphi'$ have the same sign, and $\langle\varphi|\varphi'\rangle = 0$ otherwise. Thus $\langle\varphi|\varphi'\rangle \neq \delta(\varphi-\varphi')$. It is of interest to note that in certain integrals the quantity $\langle\varphi|\varphi'\rangle$, as given by Eq.(3.10), behaves effectively as a δ-function. For example, if one forms $\langle m|n\rangle$ with the help of (3.6)

$$\langle m|n\rangle = \int_{\varphi_0}^{\varphi_0+2\pi} d\varphi f_m(\varphi) \int_{\varphi_0}^{\varphi_0+2\pi} d\varphi' f_n^*(\varphi')\langle\varphi|\varphi'\rangle, \qquad (3.11)$$

then $\langle\varphi|\varphi'\rangle = \delta(\varphi-\varphi')$ would, together with (3.4), immediately lead to δ_{mn} as needed. Now, $\langle\varphi|\varphi'\rangle \neq \delta(\varphi-\varphi')$; instead, one finds numerically from (3.10) that the principal part gives the same contribution to the double integral in (3.11) as the first term, $\frac{1}{2}\delta(\varphi-\varphi')$, leading again to $\langle m|n\rangle = \delta_{mn}$ thus preserving consistency.

4. Discussion and conclusion

It is seen that the phase states (3.1) form a complete set but *fail* to fulfill the orthonormality condition in its usual form. It is known that all Hermitian operators do *not* posses a complete, orthonormal set of eigenstates and that only those that do, represent the physical quantities (observables) [15]. This suggests that, in the framework of the conventional quantum mechanics, and in agreement with [1, 16], the phase of the harmonic oscillator is *not* an observable. Indeed, if an phase observable had existed, one could have defined a time operator via the advance in the phase of the oscillator. This, in turn, would effectively make the time a *random variable* (rather than a parameter) thus depriving the standard quantum-mechanical theory of its smooth, monotonically increasing evolution parameter needed to formulate the Schrödinger equation and leading to grave difficulties. On the whole, one is left in the unenviable position of having a classical observable (the phase) without a satisfactory quantum counterpart. Apparently, the phase of the oscillator is an essentially *classical* notion and it can be assigned to a quantum state *only* if the position and momentum expectation values are localized within intervals δx and δp such that the box $\delta x \cdot \delta p$ subtends a *small* angle at the origin of the x - p plane.

Now we turn our attention to the recent important theoretical work of Pegg and Barnett [3-6] concerning the phase operator of harmonic oscillator. They introduced a mathematical model of the single-mode electromagnetic field which involves a finite but arbitrary large state space. The dimensionality of this space, $N + 1$, is allowed to tend to infinity only after calculation of expectation values is made. The finiteness of the state space means that the operators involved may have *different* properties than those of their infinite space counterparts. The distinctive feature of their model lies in the fact that the form of the phase states

$$\left|\varphi_k\right\rangle \equiv \frac{1}{\sqrt{N+1}} \sum_{n=0}^{N} \exp(in\varphi_k)\left|n\right\rangle , \tag{4.1}$$

with

$$\varphi_k \equiv \varphi_0 + \frac{2\pi k}{N+1}, \quad (k = 0,1,2,...N) , \tag{4.2}$$

is, in fact, postulated, and then the Hermitian phase operator is introduced as the operator which has eigenstates that are these phase states. The existence and form of the Pegg-Barnett Hermitian phase operator follow directly and uniquely from these phase states. Conventionally, one would expect the reverse. In fact, if one were to take the infinite-N limit of the phase states (4.1), which is explicitely not done in the Pegg-Barnett formalism, this limit would be the simultaneous eigenstates of the Susskind-Glogower Hermitian sine and cosine operators [1, 17-18]. As these two operators do not commute their use has been regarded as an unsatisfactory solution.

It is apparent that the phase states (4.1), used in the Pegg-Barnett model, are closely related to the phase states (3.1). Indeed, the use of the *approximate* $f_n(\varphi)$, Eq.(3.7), together with truncation of the Hilbert space and entailed use of equidistant discrete values of the angle variable, in order to obtain complete (but not *over*complete) and mutually orthogonal set of states, lead from (3.1) and (3.7) directly to the states (4.1). Under these assumptions the Pegg-Barnett Hermitian theory, together with its predictions, is recovered from the present approach. The enumerated approximations allow one to find a well behaved Hermitian phase operator. Indeed, one reads in [6] that "this result contradicts the well established belief that no such operator can be constructed". Having in mind the results of Sect.3, it is apparent that the Pegg-Barnett model represents a nontrivial departure from the principles of the orthodox quantum mechanics. If corroborated by the experiment, the model would require modification of the standard quantum mechanical approach, namely substitution of an infinite state space of the harmonic oscillator by the one of finite but arbitrarily large dimensions, since only the latter approach leads to the well behaved Hermitian phase operator. (The impossibility of distinguishing, *by physical experiments*, the difference between the two state spaces was discussed by Böhm [19]). Perhaps it is worth emphasizing once again that the Pegg-Barnett model depends critically on the exactness of the adopted phase states (4.1), and these have a contribution from the states with small quantum number which is different from that for the states of this paper. This would be important in the case of fields with very small mean photon numbers, where non-classical effects might be expected to be observable.

REFERENCES

1. P. Carruthers and M. M. Nieto, Phase and Angle Variables in Quantum Mechanics, *Rev. Mod. Phys.*, **40**, 411-440 (1968).
2. R. Lynch, The quantum phase problem: a critical review, *Phys. Rep.* **256**, 367-436 (1995).
3. D. T. Pegg and S. M. Barnett, Unitary Phase Operator in Quantum Mechanics, *Europhys. Lett.* **6(6)**, 483-487 (1988).
4. D. T. Pegg and S. M. Barnett, Phase properties of the quantized single-mode electromagnetic field, *Phys. Rev. A* **39**, 1665-1675 (1989).
5. J. A. Vaccaro and D. T. Pegg, Phase properties of squeezed states of light, *Opt. Commun*, **70**, 529-534

(1989).
6. S. M. Barnett and D. T. Pegg, On the Hermitian phase operator, *J. Mod. Opt.* **36**, 7-19 (1989).
7. P. Carruthers and M.M. Nieto, Coherent states and the number-phase uncertainty relation, *Phys. Rev. Lett.*, **14,** 387-389 (1965).
8. N. Grønbech-Jensen, P. L. Christiansen and P.S. Ramanujam, Phase properties of squeezed states, *J. Opt. Soc. Am.* B **6**, 2423-2429 (1989).
9. I. Mendaš and D. B. Popović, Number-phase uncertainty product for displaced number states, *Phys. Rev. A* **50,** 947-953 (1994).
10. I. Mendaš and D. B. Popović, Number-phase uncertainty product for generalized squeezed states arising from the Pegg-Barnett Hermitian phase operator formalism, *Phys. Rev. A* **52,** 4356-4364 (1995).
11. J. W. Noh, A. Fougères and L. Mandel, Measurement of the Quantum Phase by Photon Counting, *Phys. Rev. Lett.* **67,** 1426-1429 (1991).
12. E. Merzbacher, *Quantum Mechanics* , John Wiley&Sons, New York, 1970, 2nd ed., p.361.
13. A. Messiah, *Quantum Mechanics*, North-Holland, Amsterdam, 1961, Vol. I, p.181.
14. W. Pauli, *Selected Topics in Field Quantization*, The MIT Press, Cambridge, Massachusetts, 1973, p.56 and 96.
15. Ref. [13], p.188.
16. R. Jackiw, Minimum Uncertainty Product, Number-Phase Uncertainty Product, and Coherent States, *J. Math. Phys.*, **9(3)**, 339-346 (1968).
17. E. C. Lerner, H. W. Huang and G. E. Walters, Some Mathematical Properties of Oscillator Phase Operators, *J. Math. Phys.* **11(5),** 1679-1684 (1970).
18. R. Loudon, *The Quantum Theory of Light*, Oxford University Press, Oxford, 1973,1st ed., p143.
19. A. Böhm, *The Rigged Hilbert Space and Quantum Mechanics*, Springer-Verlag, Berlin, 1978, p.4.

OPERATOR FAZE HARMONIJSKOG OSCILATORA

I. Mendaš

Konstruisani su kvantnomehanički operator koji odgovara tangensu klasične faze harmonijskog oscilatora kao i odgovarajuća fazna stanja. Razmotrene su osobine ovih stanja i njihova veza sa Peg-Barnetovom teorijom za hermitski operator faze.

Ključne reči: *kvantni harmonijski oscilator, operator faze*