UNIVERSITY OF NIŠ
The scientific journal **FACTA UNIVERSITATIS**
Series: **Physics, Chemistry and Technology** Vol. 1, N° 4, 1997 pp. 21 - 27
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EQUIVALENT CRITICAL BEHAVIOUR OF THE PERIODIC HAMILTONIAN MAPS

UDC: 530.1.145

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Abstract. *Recently we have shown that the fractal properties of the critical invariant circles of the standard-map, as summarised by the f(α) spectrum and the generalised dimensions D(q), depend only on the tails in the continued fraction expansion of the corresponding rotation numbers [1]. In this paper this result is extended on the whall class of 2π periodic area-preserving maps. We present numerical evidence that the f(α) and D(q) are the same for all critical invariant circles of any such map which have the rotation numbers with the same tail.*

A typical Hamiltonian system possesses different types of regular orbits, such as: periodic, quasiperiodic and homoclinic. In the case of area-preserving twist maps of a cylinder, given by the following equations:

$$T:\begin{cases} p_{t+1} = p_t - \dfrac{k}{2\pi} f(q_t), \\ q_{t+1} = q_t + p_{t+1}, \end{cases} \qquad q_t \in \mathrm{S}^1,\, p_t \in \mathrm{R} \tag{1}$$

where $f(q_t)$ is any analytic 2π periodic function the set of periodic and quasiperiodic orbits is parametrised by the rotation number ν defined as follows:

$$\upsilon := \lim_{i\to\infty} \frac{\overline{\mathrm{T}}^i \overline{q} - \overline{q}}{i}, \quad \overline{q} \in \mathrm{R}, \tag{2}$$

Received October 27, 1997; in revised form January 16, 1998; accepted January 19, 1998

where $\overline{\mathrm{T}}$ is the lift of the map 1.

The closure of a typical quasiperiodic orbit for a sufficiently small values of the perturbation parameter is an analytic invariant circle. For a large value of the parameter the closure of the quasiperiodic orbit is a more complicated invariant set, called cantorus. It is an invariant Cantor set inbeded in the phase space.

At the critical value of the parameter, which depends on the rotation number and is denoted by $K(\nu)$, the quasiperiodic orbit is still dense on the invariant set which is at least homeomorphic to the circle. However, the invariant measure given by the orbit, that is the density of the orbit points, is singular with the respect to the Lebesque measure on the circle. The function which describes the density of the points of the critical quasiperiodic orbit has an intricate pattern of singularities. Thus, although the critical invariant set is homeomorphic to the circle, the critical quasiperiodic orbit is described by a fractal density function with a nontrivial selfsimilar structure.

It is generally believed that the quasiperiodic orbits on the critical invariant circles, for a large class of area-preserving maps, are all qualitatively similar. The idea that the qualitative behaviour of the critical orbits should depend only on the tail of the c.f.e. of the corresponding rotation number is central in the renormalisation theory for arbitrary rotation numbers [4]. Results of the method of modular smoothing [3] show that the dominant singularities of the critical circles are of the same type for circles with the rotation numbers related by some modular transformation. Such rotation numbers have the same tail in their continued fraction expansion (c.f.e). In the reference [1] we used this as the guideing idea to show that the fractal properties of critical quasi-periodic orbits for a single map (namely the standard map) depends only on the tail of the c.f.e. It is the purpose of this letter to show that the dependance of the fractal properties of the critical orbits on their rotation numbers is universal for a class of 2π periodic area-preserving maps.

The global self-similar structure of fractals with the nontrivial scaling is usually described by the spectrum of singularities $f(\alpha)$, related to the spectrum of generalised (Reneyi) dimensions [2] (the definitions are briefly recapitulated later in the text).Our numerical calculations, presented in [1], show that the singularity spectrum $f(\alpha)$ and the spectrum of fractal dimensions $D(q)$ of the invariant measure $\mu(\nu)$ are the same for all orbits of the SM with the rotation numbers with the same tails in the c.f.e., and are different if the tails of the c.f.e are different. Thus, the critical circles of the SM can be devided in equivalent classes with respect to their fractal properties. Members of the same class have the same $f(\alpha)$, $D(q)$ and the tail in the c.f.e. and different tails imply different $f(\alpha)$ and $D(q)$. In this paper we shall present the numerical evidance that the functions $f(\alpha)$ and $D(q)$ are the same for all critical circles of all 2π periodic area-preserving maps provided that the coresponding frequencies have the same tail in the c.f.e.

The fractal properties of the critical tori were described by $f(\alpha)$ and $D(q)$ for the first time in the reference [5]. However Osbaldestin and Sarkis calculated $f(\alpha)$ and $D(q)$ for a few critical circles without sistematic exploration and discussion of the dependence on the number-theoretical properties of the rotation numbers. The result that the information dimension D_1 is the same for all orbits with the same tail in the c.f.e. of their rotation numbers is implicitly contained in the recent work of Hunt et. al. [6]. Our results show that not only D_1 but the whole spectrum $D(q)$ does not depend neither on the map, provided it is 2π periodic, nor on the detailes of the rotation number but depends only on the tail of the rotation number. This extends results of the reference [6] were only the information dimension D_1 was considered.

As pointed out before the critical orbit can still be considered as an orbit of a homeomorphism of the circle. To define the function $f(\alpha)$ one considers an infinite set of partitions of the circle. The N-th partition contains N pieces labeled by an index i: $1 \leq i \leq N$. The size of the i-th piece is denoted by l_i, and the probability that a point of the orbit is in the i-th piece is denoted by p_i. One then assumes that p_i scales as $p_i \approx l_i^{\alpha}$, and defines the function $f(\alpha)$ as the Hausdorf dimension of the set of points having exponent α. If the partitions and the measure are appropriate for the considered self-similar structure then the function $f(\alpha)$ can be calculated from the properties of the limit of the sequence of partitions.

To compute $f(\alpha)$ of the critical circle with an arbitrary rotation number ν we follow the procedure used in reference [7] for computation of $f(\alpha)$ for the critical circle with the rotation number equal to the golden mean $\nu = [0,1^{\infty}]$. Similar, but not the same, partitions which gave the same asymptotic results were used in the references [5,6]. In our computations the partitions of the circle are given by the points of the periodic orbits which approximate the critical quasi-periodic orbit. The rotation numbers of the set of periodic orbits are chosen as the successive continued fraction approximants of the irrational $\nu = [0, a_1, a_2...]$. Thus the i-th partition of the circle is given by the n points of the periodic orbit with the rotation number $m/n = [0, a_1, a_2,...a_i]$ at the parameter value $k = K(\nu)$. There are two such orbits, one elliptic and one hyperbolic, but our results for $f(\alpha)$ and $D(q)$ did not depend on which of the two types of orbits were used to generate the partitions. If (p_i,q_i) and (p_{i+1},q_{i+1}) are coordinates of the two neighbouring points on the periodic orbit then

$$l_i(n) = [(p_i - p_{i+1})^2 + (q_i - q_{i+1})^2]^{1/2}\,. \tag{3}$$

The measure of $l_i(n)$ is defined as $p_i(n)=1/n$, and the partition function of the partition with n points is then given by:

$$\Gamma_n(q_n,\tau_n) = \sum_{i=1}^{i=n} \frac{p_i^{q_n}}{l_i^{\tau_n}} = n^{-q_n} \sum_{i=1}^{i=n} l_i^{\tau_n} \tag{4}$$

If the partitions and measures are appropriate for the considered fractal then the partition function is of the order unity when:

$$\tau_n = (q_n - 1)D_n(q_n) \tag{5}$$

and

$$D(q) = \lim_{n\to\infty} D_n q(n) \tag{6}$$

where $D(q)$ is the set of generalised (Reneyi) dimensions of the fractal. $f(\alpha)$ is then given by:

$$f_n(\alpha_n) = q_n\alpha_n(\tau_n) - \tau_n, \quad \alpha_n(\tau_n) = \frac{d\tau_n}{dq_n} \tag{7}$$

and

$$f(\alpha) = \lim_{n\to\infty} f_n(\alpha_n)\,. \tag{8}$$

If the results are convergent then the choice of measure and the partitions are indeed appropriate for the considered asymptotically self-similar structure. In summary, the procedure to calculate $f(\alpha)$ and $D(q)$ for a critical quasiperiodic orbit consists of the following steps: Estimate the critical value $K(\nu)$, using for example Greene criterion [8], and then calculate $\tau_n(q_n)$ for the partitions generated by the periodic orbits with the rotation numbers given by the successive continued fraction convergents. Each $\tau_n(q_n)$ gives the corresponding α_n, $f_\mathrm{n}(\alpha)$ and $D_n(q_n)$. Finally estimate the limit of $f_\mathrm{n}(\alpha)$ and $D_n(q_n)$. The convergence can be enhanced by the usual ratio trick, and the condition that $\Gamma(\tau_n,q_n)/\,\Gamma(\tau_{n+1},q_{n+1})$ can be used to provide explicit formulas for q_n and α_n as functions of τ_n [7].

Since any 2π periodic function can be writen as its Fourier series it is enough to consider the maps of the form (1), where the function $f(q)$ is given by:

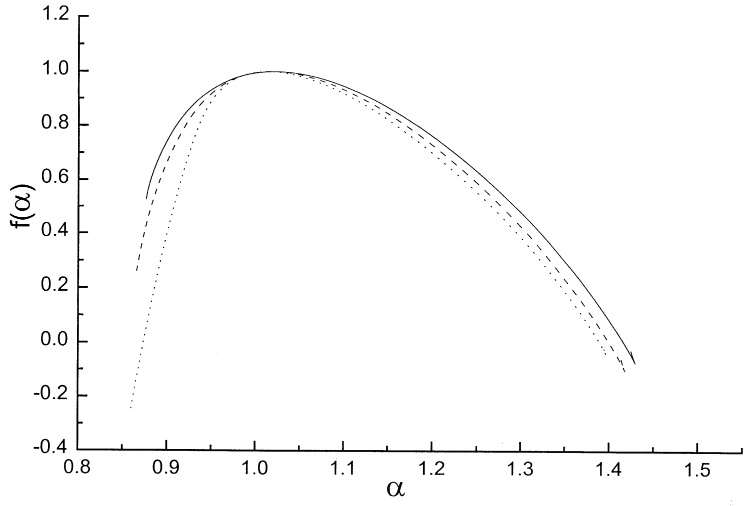
$$f(q) = \sin(2\pi q) + \sin(4\pi q) + \sin(6\pi q) + \ldots \tag{9}$$

We present our results using three maps of such form, with the functions $f(q)$ given by one, two or three terms in the Fourier expansion. The calculations for maps with more Fourier terms are consistent but incomplete becouse the numerical calculations of the critical periodic orbits become much more difficult.

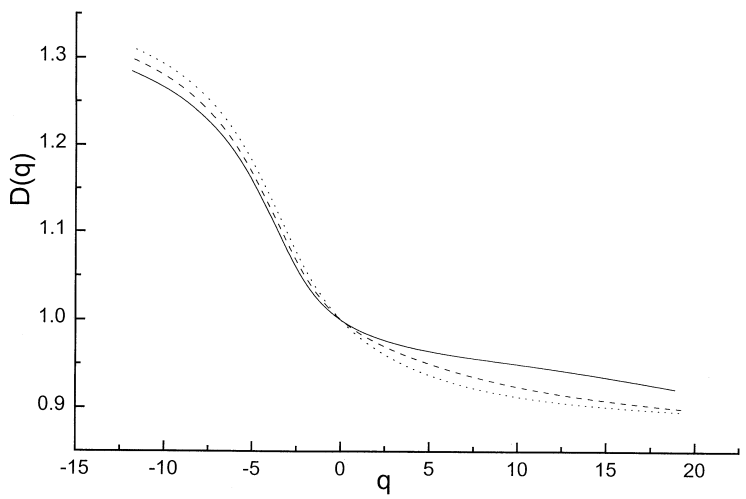
Our main result is illustrated in figures 1a,b and 2a,b. The figures represent $f(\alpha)$ and $D(q)$ for the critical quasiperiodic orbits with specially selected rotation numbers, and for the three 2π periodic maps. In the previous paper [1] we demonstrated that in the case of the standard map $f(\alpha)$ and $D(q)$ are the same for all rotation numbers with the same tail in the c.f.e. Figures 1 a, b showh that this is also thrue for the map with three Fourier terms. The curves represented by circles illustrate $f(\alpha)$ and $D(q)$ for a class of quasi-periodic orbits with the same tail in the c.f.e. as the tail of the golden mean, that is $[1^\infty]$. The curves for quasiperiodic orbits with different rotation numbers in this class are indistinguishable. The same result is true for the classes given by other tails. The curves given by crosses (x), represent $f(\alpha)$ and $D(q)$ for the class with the tail equal to $[2^\infty]$, the triangles represent the class with the tail equal to $[3^\infty]$ and the pluses represent the class with the tail $[4^\infty]$. The same result was obtained for other considered maps.

Figures 2a, b show that for various 2π periodic maps the functions $f(\alpha)$ and $D(q)$ for the critical quasi-periodic orbits are the same provided that the rotation numbers are the same. Combined with the previous results this gives our main conclusion: The functions $f(\alpha)$ and $D(q)$ for different classes of the critical quasi-periodic orbits are different, but within the class, given by the tail of the c.f.e. of the rotation number, these functions are the same for all considered 2π periodic maps.

Our calculations of $f(\alpha)$ and $D(q)$ require knowledge of quite long periodic orbits at the critical values of the parameter. These orbits are used to determine the values of $K(\nu)$ and to calculate the partition functions (4). The final results, that is $f(\alpha)$ and $D(q)$, are extremely sensitive on the value of the parameter k. In order to establish our conclusions we needed to calculate $K(\nu)$ with at least five significant digits, which implies calculations of very long periodic orbits. If one of the initial coefficients in the c.f.e. of the rotation number is large then the long periodic orbit is in a small neighbourhood of a very unstable periodic orbit. Calculation of such orbits is obviously a very difficult numerical problem. This has limited us to a relatively small set of about five quasiperiodic orbits in each considered class and to the maps with only a few Fourier terms. However we believe that the results are typical to put confidence in our conclusions.

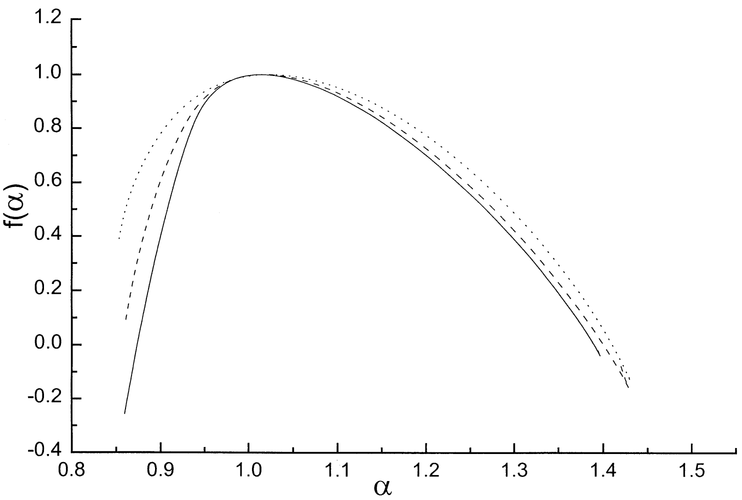


(a)

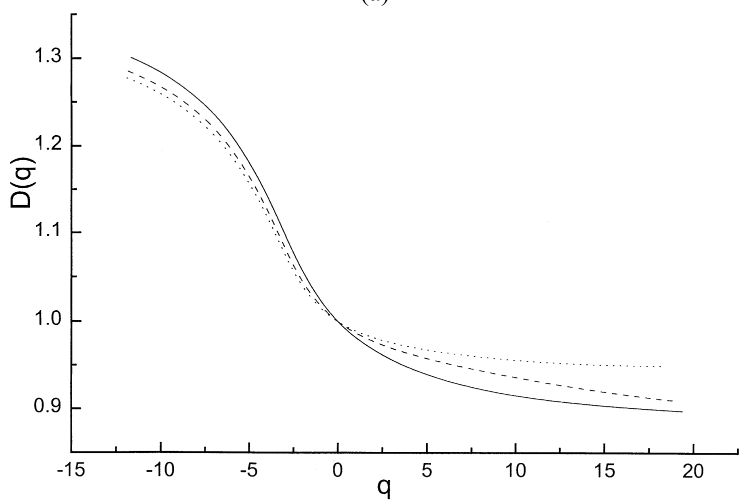


(b)

Fig.1. a, b $f(\alpha)$ (a) and $D(q)$ (b) curves for the critical circles in several equivalence classes. Shown are the classes corresponding to the tails equal to $[1^\infty]$ (full), $[2^\infty]$ (dashed), $[3^\infty]$ (dotted) and $[4^\infty]$ (dot-dashed).



(a)



(b)

Fig.2. a, b $f(\alpha)$ (a) and $D(q)$ (b) curves for the critical circles for the three 2πperiodic maps and the representative rotation numbers in the characteristic classes. Shown are the classes corresponding to the tails equal to $[1^{\infty}]$ (full), $[2^{\infty}]$ (dashed) and $[3^{\infty}]$ (dotted). The curves corresponding to different maps are the same within the numerical limitations.

REFERENCES

1. N. Burić, N, M. Mudrinić and K. Todorović, Equuivalent classes of critical circles, J. Phys. A.: Math. and Gen. 30, L161 (1997).

2. T. C. Halsey, M. H. Jansen, L. P. Kadanoff, I. Procaccia and B. Shraiman, Fractal measures and their singularities: the characterization of strange sets, Phys. Rev. A. 33, 1141 (1986).
3. N. Burić and I. C. Percival, Modular smoothing and KAM tori, Physica 71D, 37 (1994).
4. R. S. MacKay, Renormalisation in Area-Preserving Maps, World Scientific, Singapore (1993).
5. G. H. Osbaldestin and N. Y. Sarkis, Singularity spectra of a critical circle, J. Phys. A: Math. and Gen. 20 L963 (1987).
6. B. R. Hunt, K. N. Khanin, Y. G. Sinai and J. A. Yorke, Thermodynamic formalism for critical circles, J. Stat. Phys. 85, 261 (1996).
7. H. Bambi and S. Jicong, Nonanalitic twist maps and Frankel-Kantorova model, Physica 71D 23 (1994).
8. J. M. Greene, A method for determining a stochastic transition, J. Math. Phys. 20 1183 (1979).

EKVIVALENTNE KLASE PONAŠANJA PERIODIČNIH HAMILTONOVIH PRESLIKAVANJA

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Nedavno smo pokazali da fraktalne osobine kritičnih invarijantnih krugova standardnog preslikavanja, opisane $f(\alpha)$ spektrom i spektrom generalisanih dimenzija $D(q)$, zavise samo od repa u verižnom razlomku odgovarajućeg rotacionog broja [1]. U ovom članku je taj rezultat proširen na čitavu klasu 2π-periodičnih preslikavanja koja očuvavaju površinu. Dati su numerički rezultati koji potvrđuju da su $f(\alpha)$ i $D(q)$ isti za sve kritične invarijantne krugove ma kog takvog preslikavanja ukoliko imaju isti rep u verižnom razlomku rotacionog broja.