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REDUCTION OF AN INFINITE-DIMENSIONAL HAMILTONIAN SYSTEM IN CLASSICAL AND QUANTUM MECHANICS

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Abstract *We show that classical or quantum mechanics can both be obtained from an infinite-dimensional Hamiltonian system. The procedure consists in postulating mathematical representatives of the physical observables and performing a reduction of the Hamiltonian system (H, Ω, Φ_t) by the corresponding phase groups as symmetry groups.*

Key words: *infinite Hamiltonian systems, symmetry, reduction*

1. INTRODUCTION

The mathematical frameworks of classical and quantum mechanics are quite different. The first one is usually formulated using Hamiltonian dynamical systems on finite-dimensional manifolds [1], and the second uses unitary linear dynamical systems on infinit-dimensional Hilbert spaces [2]. This makes a comparison between the two theories more difficult. However, it is known that the classical statistical mechanics is related to unitary evolution operators on suitable Hilbert [3], [1], and that, on the other hand, the linear Schrodinger equation can be looked upon as a Hamiltonian dynamical system on an infinite symplectic manifold [4], [5]. The Liouville equation of classical statistical mechanics and the common Schrodinger equation are actually examples of infinite-dimensional linear Hamiltonian systems. Reversing these relations, it would be interesting to establish the precise procedures which would, starting from an abstract infinite-dimensional Hamiltonian system, give as the result the mathematical framework of classical or quantum mechanics.

In this paper we shall show that classical and quantum mechanics can be obtained

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from such Hamiltonian system by reduction using their respective phase-summetry groups. Our main result is the following: An abstract infinite-dimensional linear Hamiltonian system which is invariant under groups G_c or G_q is actually equivalent to a classical mechanical system or to a quantum system respectively. The classical symmetry group G_c is the local version of the summetry group in the quantum case G_q, and both groups will be explicitely formulated in section 3. The fact that the mathematical framework of quantum mechanics can be obtained from an infinite-dimensional Hamiltonian system by the reduction by the group G_q, comes from an application of a well known result [4]. As far as we know, the other, more difficult, part of our result, that is, that classical mechanics also comes from an infinite-dimensional Hamiltonian system by the reduction by the phase-symmetry group G_c is original.

The paper is organised as follows. In the section 2. we recapitulate the standard way of representing an abstract linear Schrodinger equation as an infinite-dimensional Hamiltonian system. We also summarise the procedure due to Marsden and Weinstein [6] known as the reduction of a Hamiltonian system by a symmetry group. In section 3. we prove our main result. We first recapitulate the Koopman formalism of classical statistical mechanics, where the Liouville equation appears as a linear Schrodinger equation on an infinite-dimensional Hilbert space. This also motivates our choice for the phase group of classical mechanics. We then suppose that an abstract linear infinite-dimensional Hamiltonian system is symmetric under either of the groups G_c or G_q, and then perform the corresponding reductions. As the results we obtain the standard mathematical formulations of classical mechanics, in the case of G_c, or quantum mechanics in the case of G_q. In the section 4. we briefly summarise our results and indicate their possible applications.

2. SCHRÖDINGER EQUATION AS A HAMILTONIAN DYNAMICAL SYSTEM

Firstly we consider an abstract linear Schrödinger equation, independently of its physical interpretation, and associate with it an infinite-dimensional Hamiltonian system. Let H be a complex separable Hilbert space, and $\hat{H}$ a self-adjoint operator with some domain $D_H \in H$. The following equation:

$$\frac{1}{i}\frac{d\psi}{dt} = \hat{H}\psi, \quad \psi \in D_H, \quad (h = 1) \tag{1}$$

is called linear Schrödinger equation, and has the following solution, given by a one parameter family of unitary operators $\hat{U}(t)$ on H:

$$\psi(t) = \hat{U}(t)\psi(0) \tag{2}$$

where:

$$\hat{U}(t) = \exp(-it\hat{H}) . \tag{3}$$

As is well known [4] the equation (1) can be looked upon as a Hamiltonian dynamical system (H, Ω, Φ_t) on an infinite-dimensional manifold. The phase space of this system is the space H with the standard symplectic structure, given by the imaginary part of the scalar product:

$$\Omega(\psi,\varphi) = -I\mu < \psi / \phi >, \quad \psi,\phi \in TH \equiv H \times H \tag{4}$$

The operator $\hat{H}$ defines the corresponding vector field $X_{\hat{H}}$ via:

$$X_{\hat{H}}\psi = i\hat{H}\psi \tag{5}$$

which is a Hamiltonian vector field with respect to the symplectic structure Ω. Thus, there is a function $H(\psi)$ on the phase space H given by:

$$H(\psi) = \frac{1}{2} < \psi \mid \hat{H} \mid \psi > \tag{6}$$

such that: $\Omega(X_{\hat{H}},.) = dH$.

Let us now briefly sketch a procedure for reducing the dimension of the phase space by using the symmetry of the Hamiltonian system. The general form of the procedure is due to Marsden and Weinstein [6]. MW reductions in the two cases which are relevant for us are disscused in the next section.

Let G be a group acting by symplectic diffeomorphisms on a symplectic manifold M. If G is a symmetry group of a Hamiltonian system on M then the generators of the group give functions f on M which are first integrals of the Hamiltonian system. They are not necessarely in involution since the group need not be abelian. Common level sets of these functions M_f are invariant submanifolds of M for the Hamiltonian flow. Denote by G_f a subgroup of G which leaves M_f invariant and thus is acting on M_f. A factor manifold $M_r \equiv M_f/G_f$ is the set of orbits of the subgroup G_f on M_f. It is often a symplectic manifold, and the Hamiltonian system on M gives a Hamiltonian system on M_r. The manifold M_r is called the reduced phase-space and the Hamiltonian system on M_r is called the reduced system (of the original system with respect to the symmetry group G). As a common example consider the one-parameter group given by the flow $G \equiv F_t$ of the Hamiltonian h itself. Level sets of the Hamiltonian M_h are $2N$-1 dimensional manifold invariant under the flow F_t. The set of orbits M_f/F_t is a $2N$-2 symplectic manifold. It is the reduced phase-space in this case.

3. REDUCTIONS BY THE PHASE-SYMMETRY GROUPS OF CLASSICAL AND QUANTUM MECHANICS

We shall now outline the MW reductions of the system (H, Ω, Φ_t) corresponding to the phase groups of classical and quantum mechanics G_c and G_q. First we shall consider the classical case. We shall introduce and discuss the group G_c and then perform the corresponding reduction. We shall then repeat the analysis in the quantum case.

Classical Mechanics

The classical mechanics is obtained if the observables are taken to be represented by the operators of multiplication by real functions acting on the Hilbert space $H = L_{2C}(\mathrm{P})$, where P is finite-dimensional symplectic manifold. An operator $\hat{A}$ on $L_{2C}(\mathrm{P})$, satisfies the invariance condition:

$$\hat{A}' = \exp(i\lambda)\hat{A}\exp(-i\lambda) = \hat{A}, \quad \lambda : \mathrm{P} \to \mathrm{R} \tag{7}$$

if $\hat{A}$ is a multiplication by some function $a(x)$ on P: $(\hat{A}\psi)(x) = a(x)\psi(x)$. Thus the phase group of classical mechanics is:

$$G_c = \{\exp i\lambda \,/\, \lambda : \mathrm{P} \to \mathrm{R}\} \ . \tag{8}$$

The self-adjoint operators which commute with the action of this group on (H, Ω, Φ_t), that is the multiplications by real functions on P, are the physical observables of the classical mechanics.

Vectors of the Hilbert space $L_{2C}(\mathrm{P})$ do not have direct physical interpretation, but the following expression:

$$<\psi \,|\, f \,|\, \psi> = \int_{\mathrm{P}} f \,|\, \psi \,|^2 \, d\mu \ , \tag{9}$$

which is invariant under the action of the phase group G_c, is interpreted as the mean value of the observable f in the statistical ansamble described by the distribution $\rho = |\psi|^2$. Thus, the definition of physical observables introduces an equivalence relation on $H = L_{2C}(\mathrm{P},\mu)$:

$$\psi \approx \psi' \ \text{ iff } \ |\psi|^2 = |\psi'|^2 \ . \tag{10}$$

Consider now the MW reduction of the system (H, Ω, Φ_t) by the symmetry group G_c, which, as we shall show, results in a finite-dimensional Hamiltonian system $(\mathrm{P}, \omega, F_t)$. Since $G_c = \{\exp(i\alpha) |\ \alpha\colon \mathrm{P}\to\mathrm{R}\}$, where P is $2N$ dimensional real manifold, the algebra of G_c is: $G_c = \{\alpha |\ \alpha\colon \mathrm{P}\to\mathrm{R}\}$, the abelian algebra of real functions on P. The symplectic action of the group G_c on $H = L_{2c}(\mathrm{P},\mu)$ is given by: $\exp(it\alpha)\colon \psi\to\exp(it\alpha)\psi$. The Hamiltonian $H_\alpha\colon H\to\mathrm{R}$ corresponding to an element $\alpha\in G_c$ is given by:

$$H_\alpha(\psi) = \frac{1}{2} <\psi \,|\, \hat{\alpha} \,|\, \psi> = \frac{1}{2}\int_{\mathrm{P}} \alpha(x) \,|\, \psi(x) \,|^2 \, d\mu(x) \,, \tag{11}$$

where the operator $\hat{\alpha}$ is given by:

$$(\hat{\alpha}\psi)(x) = \alpha(x)\psi(x) \ . \tag{12}$$

The group G_c is abelian so that the reduction procedure gives as the reduced manifold the set of equivalence classes:

$$H_r = L_{2c}(\mathrm{P},\mu) / \approx \,, \tag{13}$$

where $\approx$ is the equivalence relation given by $\psi \approx \phi$ iff $|\psi|^2 = |\phi|^2$. The algebra of real functions on the set of equivalence classes H_r is identical with the algebra of real functions on P, where the identification is given by the following rule:

$$F_f(\tilde{\psi}) = \int_{\mathrm{P}} f \,|\, \psi \,|^2 \, d\mu, \quad \psi \in H, \tilde{\psi} \in H_r \ . \tag{14}$$

Finally because of duality between the algebra of real functions on P and the manifold P itself we can identify H_r with P.

The symplectic structure and the Hamiltonian on P follow from those on H, using the equation (2) as the definition of the Hamiltonian flow F_t on P.

Quantum mechanics

Let us now turn onto the MW reduction which gives the quantum mechanics. All objects in the Hilbert space formulation of quantum mechanics are invariant under the global S_1 group of transformations, the circle group. Physical observables are represented by self-adjoint operators on the complex Hilbert space and the states are represented by projection operators i.e. by unit rays in the Hilbert space. Thus the phase group of quantum mechanics is the circle group S^1:

$$G_q = \{\exp i\alpha \mid \alpha \in R\}, \tag{15}$$

and is much smaller than the phase group of classical mechanics. The phase group G_c can be obtained from G_q if the real constants α are replaced by real functions on a finitely dimensional manifold. We shall not consider larger groups corresponding to additional kinematical symmetries, which might also involve some superselection rules.

The group S^1 acts symplectically on H by: $\psi \rightarrow \exp(i\alpha)\psi$. S^1 is a one-parameter group with the generator given by some real constant, let say unity. The Hamiltonian of the action is given by:

$$H_1(\psi) = \frac{1}{2} < \psi \mid \psi >, \tag{16}$$

so that the level set of H_1 is the unit sphere:

$$H_1^{-1}(\psi) = \{\psi : \|\psi\| = 1\}. \tag{17}$$

The reduced phase space is:

$$H_r = H_1^{-1}(\psi)/S^1 = \mathrm{P}(H). \tag{18}$$

the projective Hilbert space.

MW reduction of the infinite-dimensional Hamiltonian system (H, Ω, Φ_t) by the group G_q as the symmetry group results in another infinite-dimensional Hamiltonian system (P(H), Ω, Φ_t) . The latter is defined on a projective space P(H) as the symplectic manifold. However the sole functions on P(H) which are physically interpreted are quadratic functions of the local symplectic coordinates. Let us briefly describe the Hamiltonian system (P(H), Ω, Φ_t).

The standard symplectic structure on P(H) follows from that on the original H and the reduction procedure. A state $s_0 \in \mathrm{P}(H)$ is represented by a fixed $|s_0> \in H$. Open neighbourhoods of the point s_0 are sets of s such that $< s_0|s > \neq 0$. We can write: $|s >= \lambda^0|s_0> +| p>$, where $s_0 \mid p >= 0$, and $Im\lambda^0 = 0$, $\lambda^0 > 0$. Let $\{|s_k>\}$ be a bases in $\perp_{s0} H$, where $\perp_{s0} H$ denotes the ortocomplement of $|s_0>$ in H. Then the real and the imaginary parts of the expansion coefficients λ^k in $|p >= \Sigma_k\lambda^k|s_k>$ are the real local coordinates in the neighbourhood of s_0. Thus P(H) has the structure of a smooth manifold. The tangent space to P(H) at s_0: $c(H)$ can be identified with $\perp_{s0} H$. Indeed, for every $\upsilon \in T_{s0}\mathrm{P}(H)$ such that $s = s_0 + \delta t/\upsilon$ there is the corresponding $|\upsilon > \in \perp_{s0} H$ such that $|s >= |s_0> +\delta t|\upsilon$, where $|s >$ is the vector corresponding to s. The vector $|\upsilon>$ is fixed by υ up to the common phase factor, i. e. moduo the phase group G_q. Finally the symplectic structure on P(H) is well defined by:

$$\Omega_s(\upsilon_1, \upsilon_2) = -2\,\mathrm{Im} < \upsilon_1 \mid \upsilon_2 > \quad \upsilon_1, \upsilon_2 \in T_s\mathrm{P}(H). \tag{19}$$

The local real coordinates ($Re\lambda_k$, $Im\lambda_k$) are the symplectic coordinates with respect to this symplectic structure. The flow $\Phi_t \equiv \hat{U}(t)$ on H gives afther the reduction by G_q the

Hamiltonian flow on P(H).

4. CONCLUSIONS

The main result of our analyses can be summarised as follows: Reductions by

$$G_c = \{\exp i\lambda \mid \lambda : \mathrm{P} \to \mathrm{R}\} \tag{20}$$

or

$$G_q = \{\exp i\alpha \mid \alpha \in \mathrm{R}\} \tag{21}$$

of a linear infinite-dimensional Hamiltonian system gives as the result the mathematical frameworks of classical or quantum mechanics, respectively. Thus, both classical or quantum evolution can be studied using the same infinite-dimensional Hamiltonian system, and the results can be specified to the classical or the quantum case by the reduction by the corresponding phase-symmetry groups.

We believe that our result helps to understand better the relation between classical and quantum dynamical systems. This result could be useful for example in analysing problems related to coupled quantum and classical systems [7], transition to classical limits of quantum systems [8] and the problem of quantum integrability [9], [10].

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REDUKCIJA BESKONAČNO-DIMENZIONOG HAMILTONOVOG SISTEMA U KLASIČNOJ I KVANTNOJ MEHANICI

N . Burić

Pokazano je da se klasična i kvantna mehanika mogu dobiti redukcijom istog beskonačno-dimenzionog Hamiltonovog sistema. Procedura se sastoji u postuliranju matematičkih reprenzenata fizičkih veličina i u redukciji Hamiltonovog sistema (H, Ω, Ψ_t) pomoću odgovarajuće fazne grupe kao grupe simetrije.