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EQUIVALENCE OF GEODESIC MOTIONS AND HYDRODYNAMIC FLOW MOTIONS

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Abstract. *We examine the conditions under which the geodesic motions in the interior of a continuous, bounded, perfect-fluid gravitating source are equivalent to its hydrodynamical flow motions. We give some applications revealing the relevance to and usefulness of the result in stellar and galactic dynamics, and cosmology.*

Keys words: *geodesic motion, hydrodynamic flow: equivalence*

1. INTRODUCTION AND MOTIVATION

One of the most profitable ways of galactic and stellar dynamics in studying the dynamics of a bounded, gravitating astrophysical system is by integrating the equations of the geodesic motion, namely the motion of a test particle [1]. Thus, by studying the geodesic motion of test particles and gyroscopes in the exterior of a bounded gravitating source, it is possible to determine the macroscopic physical parameters of the source as e.g. its mass, angular momentum, quadrupole moment e.t.c. Also, by studying the geodesic motions in the interior of a gravitating source, it is possible, using also observational data on the photometry of the source, to conclude about the mass-energy distribution in the source's interior [2-4].

An analogous equally profitable way in studying the dynamics of a continuous bounded gravitating perfect-fluid source is by integrating the Euler equations for the hydrodynamical motion of a finite-volume fluid element [5, 6; see also 7-9].

The question is raised, however, whether or not the geodesic motions in the interior of the continuous source do really correspond to any realistic motions at all [10, 11]. This means that a test particle moving in the interior of a fluid source is not a physical part of the source, at least in the sense that this is true for the fluid volume element. So, the

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hydrodynamic flow motion, being an ideal motion in itself, as the geodesic motion surely is, should, however, be accepted as approximating the real physical motion of the source's constituents, in a much better way than the geodesic motion in the same continuous source could be.

Both kinds of motion, geodesic and hydrodynamical flow, have been used for the physical-dynamical description of the interior of the same source, and the results of both kinds of physical-dynamical description are equally useful, although frequently in different contexts and with different objectives.

With the prospect of checking the possible equivalence, at least in the context of the stellar and galactic dynamics, and cosmology, of the results of both physical-dynamical descriptions of the same continuous source, (namely with the aid of the geodesic motion and of the flow motion), the really interesting question to be answered is not which of the two descriptions is preferable over the other, but, instead, under which conditions, if any, they could be dynamically equivalent to each other. The term *equivalent dynamical description* will mean that the forms of the equations of motion as well as of the integrals of motion (if any) for the geodesic motion and for the flow motion are functionally the same. If such conditions could be stated, the differences between the two methods would reveal themselves once more, showing in this way that in the source's interior the test particle and the finite-volume fluid element can be dynamically equivalent probes. This problem is attacked in the next Section 2, some astrophysical applications are described in Section 3, and the conclusions are summarized in the final Section 4.

2. GEODESIC VERSUS HYDRODYNAMICAL FLOW MOTIONS

In view of the above we state our objective by asking: Is it possible that the equations of the hydrodynamic flow motion and the corresponding integrals of the flow motion take on functional forms identical to the corresponding ones for the geodesic motion in the interior of the same source? Having in mind the relevance of the problem considered to stellar and galactic dynamics, and cosmology we shall concentrate on sources with time-independent and axially-symmetric distributions of mass, internal motions and structure (ASTI sources), and, hence, characterized by at most two typical integrals of motion (geodesic or/and flow motion), namely, energy and angular momentum. From an astrophysical point of view such an assumption, although not necessary, could, however, prove useful, because it is equivalent with the assumption that the source is changing slowly in time (in the sense of e.g. the virial theorem), so that partial time derivatives are negligible compared to the spatial ones. We stress that the fluid source considered is not the most arbitrary one from a theoretical point of view, but, on the other hand, many astrophysically interesting systems are usually assumed to be time-independent and axially-symmetric continuous sources.

Thus, in the case of a bounded gravitating perfect-fluid source with gravitational potential U steming from the mass density ρ through the Poisson equation

$$\vec{\nabla}^2 U = -4\pi G\rho \qquad (1)$$

where G is the gravitational constant, the geodesic equations for the motion of a test particle of constant rest mass, are

$$\dot{\vec{\upsilon}} = \vec{\nabla} U \quad \text{along the test-particle's orbit} \tag{2}$$

where $\vec{\upsilon}$ is the velocity of the test particle, and an overdot denotes total time derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{\upsilon}.\vec{\nabla} \tag{3}$$

with a dot denoting scalar multiplication of vectors, and $\vec{\nabla}$ being the usual *delta operator.* Also in the case of an isotropic perfect-fluid source with pressure p (of thermal, radiative or degenerate-matter nature) the equations of the hydrodynamical motion reduce to the standard Euler equations

$$\dot{\vec{\upsilon}} = \vec{\nabla} U - \frac{1}{\rho}\vec{\nabla} p \quad \text{along the fluid element's orbit} \tag{4}$$

which are usually supplied by the continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}.(\rho \vec{\upsilon}) = 0 \tag{5}$$

expressing the conservation of the fluid element's (and hence of the total fluid source's) rest mass of its baryons.

In the special case of an ASTI source, scalar and vectorial multiplication of both members of the geodesic equations of motion (2) with the position vector $\vec{x}$ yields the energy (per unit mass) and angular-momentum (per unit mass) integrals, in cylindrical coordinates *r*, *φ*, *z*,

$$E_g = \frac{1}{2}\upsilon^2 - U, \quad S_g = r^2 \dot{\varphi} \tag{6}$$

On the other hand the equations of hydrodynamical motion (4) imply quite generally (namely, not for an ASTI source necessarilly) that

$$\frac{d}{dt}\left(\frac{\upsilon^2}{2} - U\right) = -\frac{1}{\rho}\dot{p} \tag{7.1}$$

and

$$\frac{d}{dt}(\vec{x} \times \vec{\upsilon}) = \vec{x} \times \left(\vec{\nabla} U - \frac{1}{\rho}\vec{\nabla} P\right) \tag{7.2}$$

with the symbol *x* denoting vectorial multiplication, and so no integrals of the hydrodynamic flow motion exist at all. In the trivial case of the *isobaric motion*

p= constant along the flow motion

Eq. (4) takes on the form of its geodesic analogue, Eq. (3), and Eq. (7.1) reduces to the geodesic energy integral E_g. Moreover, for an ASTI source, Eq. (7.2) reduces to the geodesic angular-momentum integral S_g. Hence isobaricity and ASTI fluid source imply that the test particle and the fluid volume element are dynamically equivalent probes.

However, isobaricity of the hydrodynamic flow motions is not always physically necessary. On the other hand, especially for isolated systems, like e.g. the astrophysical ones, *adiabaticity* of the hydrodynamical flow motions could be physically necessary and useful. This is due to the fact that, for adiabatic processes, the fluid-volume element's thermodynamic content remains constant during its motion, beyond, of course, and independently, in the Newtonian theory of gravity, of the constancy of its rest mass dictated by the continuity equation (5). For adiabatic processes, the first law of classical thermodynamics

$$d\Pi + pd\left(\frac{1}{p}\right) = d(Heat\ quantity) = 0 \tag{8}$$

where ρΠ is the internal specific-energy density associated with the fluid's expansions and contractions, reduces to

$$\frac{1}{\rho}\vec{\nabla}p = \vec{\nabla}\left(\Pi + \frac{p}{\rho}\right) \tag{9}$$

Thus the equations of the hydrodynamic motion (4) take on the geodesic form

$$\dot{\vec{\upsilon}} = \vec{\nabla} V \tag{10}$$

where the generalized potential V is defined via

$$V := U - \left(\Pi + \frac{p}{\rho}\right) \tag{11}$$

In complete analogy to Eq. (2), Eq. (10), under adiabatic processes in an ASTI source, admit (per unit mass) the energy (or Bernoulli) and the angular-momentum integrals

$$E_f = \frac{\upsilon^2}{2} - V, \qquad S_f = r^2\dot{\varphi} \tag{12}$$

Consequently adiabatic flow motions in an ASTI fluid source are equivalent to the geodesic motions in it. We remark that the constancy of the test-particle's rest mass is directly corresponded to the continuity equation, because the latter's integrating over any subvolume of the fluid source's three-dimensional volume results in the constancy of the subvolume's rest mass.

3. SOME ASTROPHYSICAL APPLICATIONS

As an example of the usefulness of the above result to stellar and galactic dynamics, and cosmology, we recall that in their context we usually assume a certain gravitational potential, and intergrating the geodesic equations (2) we find the position of the test particle as a function of the time $\vec{x} = \vec{x}(t)$ .On the basis of the distribution of the orbits $\vec{x} = \vec{x}(t)$ so derived, we estimate the matter density $\overline{\rho}$ implied by them. Obviously the self consistency of the method demands that $\overline{\rho}$ must be equal to the density ρ producing the realistic potential U, namely

$$\overline{\rho} = \rho = -\frac{1}{4\pi G}\vec{\nabla}^2 U \tag{13}$$

Next we assume an equation of state for the fluid-source's interior

$$p = p(\rho, T) \tag{14}$$

where for generality we included also the matter's temperature T. Under adiabatic processes Eq. (8) defines, to the approximation of an additive constant, the quantity

$$\Pi + \frac{p}{\rho} = \int \frac{dp(\rho,T)}{\rho} := 4\pi G f\left[\rho(\vec{x})\right] \tag{15}$$

as a known function of the matter density ρ. Thus the potential V satisfies the Poisson-type equation

$$\vec{\nabla}^2 V = -4\pi G \rho_V \tag{16}$$

where, obviously, the differential equation for ρ is

$$\rho_V = \rho + \vec{\nabla}^2 f \tag{17}$$

In analogy to the classical case, integrating the geodesic-type equation of motion (10) for obtaining the orbits $\vec{x} = \vec{x}(t)$, and comparing the derived matter distribution with observational results on the photometry of the fluid source we estimate a source density $\overline{\rho}_V$ which, for consistency reasons, must satisfy

$$\overline{\rho_V} = \rho_V = -\frac{1}{4\pi G}\vec{\nabla}^2 V \tag{18}$$

After checking the appropriateness of $\overline{\rho}_V$ we then solve the differential equation (17) to find ρ. We remark that ρ_V incorporates the contribution of the thermodynamic content of the fluid source (namely, its thermal motions), but not of its ordered internal motions (of velocities $\vec{\upsilon}$).

We note that in the case of an adiabatic polytrope with an equation of state

$$p = A\rho^{\gamma+1}, \; A, \gamma : constants \tag{19}$$

Eqs. (15) and (17) reduce to

$$\Pi + \frac{p}{\rho} = A\left(1 + \frac{1}{\gamma}\right)\rho^{\gamma} + const. \tag{20}$$

$$\rho_V = \rho + \frac{A(\gamma+1)}{4\pi G}\rho^{\gamma-2}[\rho\vec{\nabla}^2\rho + (\gamma-1)(\vec{\nabla}\rho).(\vec{\nabla}\rho)]. \tag{21}$$

A solution to Eq. (21) of the form

$$\rho = \rho_0 + \delta\rho(r,z), \quad |\delta\rho| \leq \rho_0 \tag{22}$$

implies, to linear order in $\delta\rho/\rho_0$,

$$\rho_0 = \rho_v, \quad \vec{\nabla}^2(\delta\rho) + \kappa^2(\delta\rho) = 0, \quad \kappa^2 := \frac{4\pi G}{4(\gamma+1)\rho_0^{\gamma+1}} \tag{23}$$

with a solution (for $\gamma+1>0$)

$$\delta\rho(r,z) = \frac{e^{-i\kappa R}}{R}, \quad R = (r^2+z^2)^{1/2} \tag{24}$$

Such results can in principle be checked observationally. Actually, as it has been recently reported [12], the inclusion, via Eq (7), of the source's internal characteristics induce, beyond the standard classical proper-mass density, ρ, an additional internal mass-density depending on the source's proper mass density and pressure. This extra density and the corresponding extra mass, due explicitly to the fluid nature of the gravitating source and its assumed adiabatic flow motions, could be of importance to the problem of the determination of the mass of the nuclear galactic regions and its nature. Such total dynamical masses are usually determined on the basis of Doppler shift measurements and Newtonian geodesic motion using general distribution laws for the total density ρ_v [13]. However, as it is generally known [14], the observed motions in the nuclear regions are not geodesic (namely the motion of a star-test particle) but instead the motion of maser-emitting fluid regions (namely, hydrodynamical flow motions). So the problem of the equivalence of the geodesic and hydrodynamical flow motions, addressed here, seems to be important in central current astrophysical-cosmological issues like the masses of galaxies and clusters of galaxies [15], the existence of black holes in centers of galaxies, both spiral and elliptical [16-19; see also 20], and the existence and nature of dark matter [20, 21]. Such problems are currently under investigation.

4. CONCLUSIONS AND OUTLOOK

We proved that, in the context of the Newtonian theory of gravity, the geodesic motions in the interior of a continuous, bounded, perfect-fluid gravitating source are equivalent to the adiabatic, hydrodynamical flow motions in the same source. Also we examined some possible astrophysical applications of the result of interest to the case of the determination of galactic masses, the existence of black holes in the centers of galaxies, and the existence and nature of the dark matter.

At the outcome we note that it could present some interest to examine the relevance and importance of the above method to the case of a fluid source endowed with a magnetic and an electric field, so that the geodesic motion in the source be equivalent to the hydromagnetic flow motion, and also to the post-Newtonian approximation of general relativity for both the magnetized and non-magnetized fluid sources, so that all the internal characteristics of the fluid source are properly taken into consideration. Related astrophysical phenomena are the solar magnetosphere, the magnetospheres of magnetic stars, the polar-cap regions of pulsars, the linear jets in galaxies and binary (or not) stars, and the surface of magnetized black holes in the centers of giant galaxies. The mathematical treatment of such problems is described in [10, 11].

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EKVIVALENCIJA GEODEZIJSKOG KRETANJA I HIDRODINAMIČKOG TOKA KRETANJA

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U radu su prikazana istraživanja uslova pod kojima su kretanja po geodezijskoj unutrašnjosti kontinualnog ograničenog idealno fluidnog izvora gravitacije ekvivalentna sa njihovim hidrodinamičkim tokovima kretanja. Date su neke primene koje ukazuju na relevantnost i korisnost rezultata u zvezdanoj i galaktičkoj dinamici i kosmologiji.

Ključne reči: *geodezijsko kretanje, hidrodinamički tok: ekvivalencija*