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p-ADIC HARMONIC OSCILLATOR WITH TIME-DEPENDENT FREQUENCY

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Abstract: Classical and quantum properties of the one-dimensional p-adic harmonic oscillator with time-dependent frequency are considered. A p-adic phase space is used to present classical evolution. The kernel of quantum evolution operator is found and the corresponding eigenvalue problem is formulated. Under definite conditions some vacuum states are obtained. As an illustration, examples of $\omega = \omega_0$ and $\omega = \omega_0/(1-at)^2$ are taken.

1. Introduction

It is well known that experimental and observational data always belong to the field of rational numbers $\mathbb{Q}$, and theoretical physics traditionally use the field of real $\mathbb{R}$ and complex numbers $\mathbb{C}$. $\mathbb{R}$ can be regarded as completion of $\mathbb{Q}$ with respect to the standard absolute value ($|\ |_\infty$) which, in addition to p-adic norms ($|\ |_p$), is only one of possible nontrivial norms on $\mathbb{Q}$. Completions of $\mathbb{Q}$ with respect to p-adic norms ($p =$ a prime number) give the fields of p-adic numbers $\mathbb{Q}_p$ ($p = 2,3,5,...$). On p-adic numbers and p-adic analysis one can see Refs. [1-5].

Since $\mathbb{R}$ and $\mathbb{Q}_p$ have the same origin in $\mathbb{Q}_p$, and have $\mathbb{Q}$ as their subsets, it has been for a long time a challenge to find some application of p-adic numbers in theoretical physics. First considerable p-adic models have been constructed in 1987. for string amplitudes [6]. After that p-adic numbers have been applied in different topics of theoretical and mathematical physics (for a review, see Refs. [7-8]).

The metric induced by p-adic norm is the ultrametric (non-archimedean) one. Possible existence of such spaces at the Planck length ($l_{pl} \approx 10^{-33}$cm) [6] is one of the main motivations to investigate p-adic quantum models, as a first step to obtain relativistic p-adic quantum mechanics. According to the Vladimirov-Volovich (VV) approach [9-10], p-adic quantum mechanics is a triple

$$(L_2(\mathbb{Q}_p),\ W_p(z),\ U_p(t)),$$

where z is a point of a p-adic classical phase space and t is a p-adic time. $L_2(\mathbb{Q}_p)$ is the Hilbert space of complex square-integrable functions on $\mathbb{Q}_p$, $W_p(z)$ is a unitary representation of the Heisenberg-Weyl group on $L_2(\mathbb{Q}_p)$, and $U_p(t)$ is a unitary representation of the evolution operator on $L_2(\mathbb{Q}_p)$. Similar approach to p-adic quantum mechanics is presented in Refs. [11-14].

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The Weyl quantization procedure, which is used in the above formulation of p-adic quantum mechanics can be successfully applied to all systems with quadratic Hamiltonians. However, in literature only the simplest one-dimensional systems are considered: a free particle, a harmonic oscillator (compact and non-compact), and a particle in a constant field. One of the very attractive physical systems whose kernel of the evolution operator can be exactly given is a harmonic oscillator with time-dependent frequency. In standard quantum mechanics this time-dependent oscillator is of interest to quantum optics [15] and cosmology [16-17].

In this paper we consider some classical and quantum aspects of the one-dimensional p-adic harmonic oscillator with time-dependent frequency.

2. p-Adic Mathematics

Every rational number $x=0$ can be written as $x=p^{\gamma}\frac{m}{n}$, where $\gamma,\ m,\ n\in\mathbb{Z}$ (integers); p being an arbitrary prime number wich neither divides m nor n. The $p-$adic norm $|x|_p$ of x is defined as follows

$$|x|_p=p^{-\gamma},\ |0|_p=0, \tag{2.1}$$

and strong triangle inequality holds

$$|x+y|_p\leq\max(|x|_p,|y|_p). \tag{2.2}$$

Norms with property (2.2) are called non-archimedean or ultrametric. Any p-adic number can be uniquely represented by the canonical series

$$x=p^{\gamma}\sum_{i=0}^{\infty}x_i\,p^i,\ \ 0\leq x_i\leq p-1,\ x_0\neq 0. \tag{2.3}$$

It is possible to define metrics on $\mathbb{Q}_p$, induced by p-adic norm $|\ |_p$. The distance between $x,y\in\mathbb{Q}_p$ is $d_p(x,y)=|x-y|_p$. The metric space $(\mathbb{Q}_p,d_p)$, as well as $(\mathbb{R},d)$, is locally compact, complete and separable. Call

$$B_{\gamma}(a)=\{x:|x-a|_p\leq p^{\gamma}\}\ ,\ \gamma\in\mathbb{Z}, \tag{2.4}$$

a p-adic "ball" ("disk") with center a, and

$$S_{\gamma}(a)=\{x:|x-a|_p=p^{\gamma}\}\ ,\ \gamma\in\mathbb{Z}, \tag{2.5}$$

a p-adic "sphere" ("circle"). The p-adic balls are both open and closed sets

$$B_{\gamma}(a)=\{x:|x-a|_p\leq p^{\gamma}\}=\{x:|x-a|_p<p^{\gamma+1}\}, \tag{2.6}$$

therefore $\mathbb{Q}_p$ is totally disconnected.

The algebraic structures of $\mathbb{R}$ and $\mathbb{Q}_p$ are considerably different. $\mathbb{R}$ has a unique quadratic extension obtained by adding $\sqrt{-1}$, wich is the solution of the equation $x^2+1=0$, i.e.

$$\mathbb{C}=\mathbb{R}(\sqrt{-1})=\{z=x_1+ix_2:x_1,x_2\in\mathbb{R}\}.$$

Squaring a generic element x of $\mathbb{Q}_p$, as given by Eq. (2.3), leads to

$$x=p^{2\gamma}(x_0^2+2\,x_0x_1p+\,x_1^2p^2+\cdots). \tag{2.7}$$

Comparing this with the expansion of an arbitrary p-adic number y

$$y=p^{\gamma}(y_0+y_1p+\ldots) \tag{2.8}$$

it follows that two necessary conditions for y to be a square are:

$\gamma(y) = 2\gamma(x)$ and its first digit y_0 is a quadratic residue modulo p.

A number $a \in \mathbb{Z}$ is a quadratic residue modulo p if equation

$$x^2 \equiv a(\mathrm{mod} p), \ (x^2 = kp + a, \ k \in \mathbb{Z}), \tag{2.9}$$

has a solution $x \in \mathbb{Z}$. It is useful to introduce Legendre symbol

$$\left(\frac{a}{p}\right) = \begin{cases} 1, \text{ if } a \text{ is a quadratic residue modulo } p, \\ -1, \text{ if } a \text{ is not a quadratic residue modulo } p, \\ 0, \text{ if } a \equiv 0(\mathrm{mod}\, p). \end{cases} \tag{2.10}$$

$\mathbb{Q}_p$ $(p \neq 2)$ has three distinct quadratic extensions while for $p = 2$, $\mathbb{Q}_2$ has seven of them.

Function $f(x)$ is called analytical in B_γ if it can be represented in that ball by convergent power series

$$f(x) = \sum_{k=0}^{\infty} f_k \ x^k \ , \ f_k \in \mathbb{Q}_p . \tag{2.11}$$

Elementary p-adic functions are given by series of the same form as in the real case. Thus, we have

$$\exp x = \sum_{k=0}^{\infty} \frac{x^k}{k!} , \tag{2.12}$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k \ x^{2k+1}}{(2k+1)!} , \tag{2.13}$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k \ x^{2k}}{(2k)!} . \tag{2.14}$$

However the domain of convergence is different from that in the real case. Functions $\exp x$, $\sin x$, $\cos x$ have identical radius of convergence: $\mathrm{R} = 1/p$ for $p \neq 2$, and $\mathrm{R} = 1/4$ for $p = 2$ (B_{-1} for $p \neq 2$, and B_{-2} for $p = 2$, are commonly designated by G_p). It is worth noting that

$$|\sin x|_p = |x|_p \ , \ |\exp x|_p = |\cos x|_p = 1. \tag{2.15}$$

Additive character $\chi(x)$ of the field $\mathbb{Q}_p$, which is a continuous, complex function of p-adic argument, represents a character of additive group $\mathbb{Q}_p^+$. Function $\chi_p(\xi x)$ given by

$$\chi_p(\xi x) = \exp(2\pi i \{\xi x\}_p), \tag{2.16}$$

is a character of the field $\mathbb{Q}_p$ for any fixed $\xi \in \mathbb{Q}_p$. $\{\xi x\}_p$ in the above expression is a fractional part of ξx. By definition, fractional part of number $x \in \mathbb{Q}_p$ is

$$\{x\}_p = \begin{cases} 0 \text{ for } \gamma(x) \geq 0 \text{ or } x = 0 \\ p^\gamma (x_0 + x_1 p + \ldots + x_{-\gamma-1} p^{-\gamma-1} , \text{ for } \gamma(x) < 0. \end{cases} \tag{2.17}$$

Definition (2.16) gives a general representation of additive characters of $\mathbb{Q}_p$. Additive character $\chi_p(x)$ on the ball B_γ (2.4) is represented by

$$\chi_p(x) = \chi_p(\xi x), \ \xi \in \mathbb{Q}_p, \ |\xi|_p > p^{-\gamma}, \ |\xi x|_p > 1. \tag{2.18}$$

The field $\mathbb{Q}_p$ is locally compact commutative group with respect to addition, therefore, there exists a positive Haar measure dx on $\mathbb{Q}_p$, which is translation invariant: $d(x+a)=dx$. Its normalization is fixed by the requirement for this measure to be equal to unity on the set $\mathbb{Z}_p$,

$$\mu(\mathbb{Z}_p)=\int_{\mathbb{Z}_p} dx=\int_{|x|_p\leq 1} dx=1. \tag{2.19}$$

Of particular interest to us will be Gauss's integral

$$\int_{\mathbb{Q}_p}\chi_p(ax^2+bx)\,dx=\lambda_p(a)\,|2a|_p^{-1/2}\chi_p\left(-\frac{b^2}{4a}\right),\quad a\neq 0,\ \forall p. \tag{2.20}$$

Arithmetic function $\lambda_p(a)$ has the following values:

$$\text{for } p\neq 2,\quad \lambda_p(a)=\begin{cases} 1\ , \text{ if } \gamma(a) \text{ is even in expansion (2.3)} \\ \left(\frac{a_0}{p}\right)\ , \text{ if } \gamma(a) \text{ is odd and } p\equiv 1(\mathrm{mod}\,4) \\ i\left(\frac{a_0}{p}\right)\ , \text{ if } \gamma(a) \text{ is odd and } p\equiv 3(\mathrm{mod}\,4) \end{cases}. \tag{2.21}$$

Function $\lambda_p(a)$ has the following properties:

$$|\lambda_p(a)|_\infty=1,\ \lambda_p(a)\lambda_p(-a)=1,\ \lambda_p(ac^2)=\lambda_p(a),\ a,c\neq 0, \tag{2.22}$$

$$\lambda_p(a)\lambda_p(b)=\lambda_p(a+b)\lambda_p\left(\frac{1}{a}+\frac{1}{b}\right),\ 0\neq a, 0\neq b, a+b\in\mathbb{Q}_p. \tag{2.23}$$

The Gauss's integral on B_γ $(\forall p)$ is

$$\int_{B_\gamma}\chi_p(ax^2+bx)\,dx=\begin{cases} p^\gamma\Omega(p^\gamma\,|b|_p)\ , & |a|_p p^{2\gamma}\leq 1 \\ \lambda_p(a)\,|2a|_p^{-1/2}\chi_p\left(-\frac{b^2}{4a}\right)\Omega\left(p^{-\gamma}\left|\frac{b}{2a}\right|_p\right), & |4a|_p p^{2\gamma}>1, \end{cases} \tag{2.24}$$

where $\Omega(x)$ is a characteristic function

$$\Omega(|x|_p)=\begin{cases} 1, & 0\leq |x|_p\leq 1 \\ 0, & |x|_p>1 \end{cases}. \tag{2.25}$$

The function $\Omega(x)$ is one of the simplest examples for locally constant functions with compact support. It is invariant with respect to Fourier transformation

$$F[\Omega](y)=\int_{\mathbb{Q}_p}\chi_p(xy)\Omega(|x|_p)\,dx=\int_{|x|_p\leq 1}\chi_p(xy)\,dx=\Omega(|y|_p) \tag{2.26}$$

and belongs to Hilbert spaces of quadratically integrable functions $L_2(\mathbb{Q}_p)$.

3. Classical p-Adic Harmonic Oscillator with a Time-Dependent Frequency (HOTDF)

Classical p-adic HOTDF is given by Lagrangian

$$L(\dot q,q,t)=\frac{\dot q^2}{2}-\frac{\omega^2(t)\,q^2}{2}\ ,\quad m{=}1,\dot q,q,t,\omega(t)\in\mathbb{Q}_p, \tag{3.1}$$

where $\omega(t)$ is assumed to be an analytic function of t. Using Euler-Lagrange equation one obtains the corresponding equation of motion

$$\ddot{q}+\omega^2(t)\,q=0,\ \ddot{q}\in \mathbb{Q}_p. \tag{3.2}$$

There is an analytic solution of (3.2), which can be presented in the same form as in the real case [18,19], i.e.

$$q(t)=s(t)\,[A\cos\gamma(t)+B\sin\gamma(t)],\quad s(t),\gamma(t),A,B\in\mathbb{Q}_p, \tag{3.3}$$

where $|\gamma(t)|_p\leq p^{-1}$ if $p\neq 2$, and $|\gamma(t)|_p\leq 4^{-1}$ if $p=2$. Functions $s(t)$ and $\gamma(t)$ represent amplitude and phase, respectively. A and B are constants which can be determined by initial conditions.

Substituting $q(t)$ in (3.2) in the form (3.3) one can easily obtain equations for $s(t)$ and $\gamma(t)$:

$$\ddot{s}-\dot{\gamma}^2(t)\,s+\omega^2(t)\,s=0, \tag{3.4}$$

$$\dot{\gamma}(t)\,s^2=C^2, \tag{3.5}$$

where $C^2\in\mathbb{Q}_p$ is a constant independent of initial conditions. The corresponding momentum to $q(t)$ is

$$k(t)=\dot{q}=\dot{s}(t)\,[A\cos\gamma(t)+B\sin\gamma(t)]\ +s(t)\,\dot{\gamma}(t)\Big(-A\sin\gamma(t)+B\cos\gamma(t)\Big). \tag{3.6}$$

From equations (3.3) and (3.6) one obtains

$$A=q_0\Big(\frac{\cos\gamma_0}{s_0}+\frac{\dot{s}_0\sin\gamma_0}{C^2}\Big)-k_0\Big(\frac{s_0\sin\gamma_0}{C^2}\Big), \tag{3.7}$$

$$B=q_0\Big(\frac{\sin\gamma_0}{s_0}-\frac{\dot{s}_0\cos\gamma_0}{C^2}\Big)+k_0\Big(\frac{s_0\cos\gamma_0}{C^2}\Big), \tag{3.8}$$

where $q_0=q(t_0)$, $k_0=k(t_0)$, $s_0=s(t_0)$ and $\gamma_0=\gamma(t_0)$.

Classical evolution can be described as a motion in the phase space:

$$z_t=\begin{pmatrix}q(t)\\ k(t)\end{pmatrix}=T(t,t_0)\begin{pmatrix}q(t_0)\\ k(t_0)\end{pmatrix}. \tag{3.9}$$

Replacing A and B in (3.3) and (3.6) by expressions (3.7) and (3.8), we obtain

$$T(t,t_0)=$$

$$\begin{bmatrix} \frac{s_t}{s_0}\cos(\gamma_t-\gamma_0)-\frac{s_t\dot{s}_0}{C^2}\sin(\gamma_t-\gamma_0) & \frac{s_ts_0}{C^2}\sin(\gamma_t-\gamma_0)\\ \Big(\frac{\dot{s}_t}{s_0}-\frac{\dot{s}_0}{s_t}\Big)\cos(\gamma_t-\gamma_0)-\Big(\frac{\dot{s}_t\dot{s}_0}{C^2}+\frac{C^2}{s_ts_0}\Big)\sin(\gamma_t-\gamma_0) & \frac{s_0}{s_t}\cos(\gamma_t-\gamma_0)+\frac{\dot{s}_ts_0}{C^2}\sin(\gamma_t-\gamma_0)\end{bmatrix}. \tag{3.10}$$

One can show that $\det T(t,t_0)=1$ and that $T(t,t)$ is a unit matrix. On the phase space $V=\mathbb{Q}_p\times\mathbb{Q}_p$ one can introduce a bilinear symplectic form

$$B(z,z')=k'q-kq',\ z,z'\in V, \tag{3.11}$$

which is invariant under transformations given by (3.10), i.e.

$$B(z_1,z_1') = B(T(t_1,t_0)z, T(t_1,t_0)z') = B(z_0,z_0'). \tag{3.12}$$

Note that matrices (3.10) do not make a group. Namely, $T(t_3,t_2)\,T(t_1,t_0) \neq T(t_3,t_0)$ if $t_1 \neq t_2$. However, the product

$$T(t_2,t_1)\,T(t_1,t_0) = T(t_2,t_0). \tag{3.13}$$

is well defined.

4. p-Adic Quantum Mechanics

We will consider p-adic quantum mechanics in which canonical variables are p-adic numbers and states are described by complex-valued wave functions $\psi \in L_2(\mathbb{Q}_p)$. In VV formulation of quantum mechanics over p-adic fields there is neither way nor need to define p-adic "momentum" and "Hamiltonian" operators. In the the real case they are infinitesimal generators of space and time translations, but, since $\mathbb{Q}_p$ is disconnected field, these infinitesimal transformations become meaningless. The usual (canonical) quantization does not work here, because wave functions and their variables belong to different valued number fields. However, there is a possibility to use Weyl representation [20]. Finite transformations remain meaningfull and the corresponding Weyl and evolution operators are p-adically well defined.

Standard quantum mechanics starts with a representation of the Heisenberg commutation relation

$$[q,k] = i, \tag{4.1}$$

in the space $L_2(\mathbb{R})$. In the Weyl representation a pair of unitary operators is considered

$$e^{i\hat{k}q}\,\psi(x) = \psi(x+q),\ e^{i\hat{q}k}\,\psi(x) = e^{ixk}\,\psi(x). \tag{4.2}$$

Commutation relation takes the form:

$$V_k U_q = e^{ikq}\, U_q V_k. \tag{4.3}$$

In the p-adic case, the coresponding operators act in space $L_2(\mathbb{Q}_p)$.

$$U_q\psi(x) = \psi(x+q),\ V_k\psi(x) = \chi(kx)\psi(x), \tag{4.4}$$

where $q,k,x \in \mathbb{Q}_p$, and χ is additive character on $\mathbb{Q}_p$. A family of unitary operators:

$$W(z) = \chi_p(-\tfrac{1}{2}qk)\,U_q V_k,\ z = \begin{pmatrix} q \\ k \end{pmatrix} \in \mathbb{Q}_p \times \mathbb{Q}_p, \tag{4.5}$$

satisfies the Weyl relation:

$$W(z)\,W(z') = \chi(\tfrac{1}{2}B(z,z'))\,W(z+z'). \tag{4.6}$$

The operator $W(z)$ acts on the wave function by the following way

$$W(z)\,\psi(x) = \chi_p(kx + \frac{kq}{2})\psi(x+q). \tag{4.7}$$

$W(z)$ is a unitary representation of the Heisenberg-Weyl group on $L_2(\mathbb{Q}_p)$. Recall that the Heisenberg-Weyl group consists of the elemets (z,α) with the group product

$$(z,\alpha)\cdot(z',\alpha')=(z+z',\alpha+\alpha'+\tfrac{1}{2}B(z,z')), \tag{4.8}$$

where α is a p-adic parameter.

How can one describe dynamics in p-adic quantum mechanics? As mentioned above, because of the impossibility of defining momentum and quantum hamiltonian as generators of infinitesimal space and time translations, respectively, introduction of evolution operator for quantum systems in a usual way is impossible. The unitary operators $U(t)$ for autonomous systems form one-parameter commutative multiplicative group [21]. The kernel of $U(t)$ is expressed as a Feynman functional integral

$$K_t(x,y)=\int\chi\Big(\int_0^t L(q,\dot q)\Big)\prod_t dq(t), \tag{4.9}$$

where integration is performed over classical p-adic trajectories with the boundary conditions $q(0)=y$ and $q(t)=x$. The integration $\int L\,dt=S(t)$ in the formula (4.9) is understood as an operation which is inverse to differentiation.

Assumption we have for quadratic p-adic lagrangians [9,11] that

$$K_t(x,y)\sim\chi(S_{\mathrm{kl}}(t)), \tag{4.10}$$

which is valid in a real case, turnes out to be quite justified for the harmonic oscillator as well [22]. The evolution operator is defined by

$$[U\psi](x)=\int_{\mathbf{Q}_p}K_t(x,y)\,\psi(y)\,dy,\qquad K_0(x,y)=\delta(x-y). \tag{4.11}$$

The operator $U(t)$ satisfies the important group relation

$$U(t)\,U(t')=U(t+t') \tag{4.12}$$

and consequently one has for the kernel

$$\int_{\mathbf{Q}_p}K_t(x,y)\,K_{t'}(y,x')\,dy=K_{t+t'}(x,x'). \tag{4.12'}$$

The operators $W(z)$ and $U(t)$ are connected by the relation

$$U(t)\,W(z)\,U(t)^{-1}=W(T_t z), \tag{4.13}$$

where T_t is the matrix of the clasical evolution in phase space. In this way, the determination of p-adic quantum systems dynamics reduces to the eigenvalue problem of the evolution operator. This problem is formulated by

$$U(t)\psi_{\alpha\beta}(x)=\chi_p(E_\alpha t)\psi_{\alpha\beta}(x), \tag{4.14}$$

where E_α corresponds to p-adic energy, and indices α and β denote energy levels and degeneration, respectively.

5. Quantum p-Adic Harmonic Oscillator with a Time-dependent Frequency

The kernel K_p of evolution operator for p-adic HOTDF will be taken in the natural form (for the real case see [18,19])

$$K_p(x_1,t_1;x_0,t_0)=\lambda_p(t_1,t_0)N_p(t_1,t_0)\,\chi_p\left[\frac{1}{2}\left(\frac{\dot{s}_0\,x_0^2}{s_0}-\frac{\dot{s}_1x_1^2}{s_1}\right)\right]$$

$$\cdot\,\chi_p\left[-\frac{\dot{\gamma}_1x_1^2+\dot{\gamma}_0x_0^2}{2\tan(\gamma_1-\gamma_0)}+\frac{x_1x_0\sqrt{\dot{\gamma}_1\dot{\gamma}_0}}{\sin(\gamma_1-\gamma_0)}\right], \tag{5.1}$$

where the evolution operator $U(t_1,t_0)$ itself is

$$[U_p\psi](x_1,t_1)=\int K_p(x_1,t_1;x_0,t_0)\psi(x_0,t_0)\,dx_0=\psi(x_1,t_1). \tag{5.2}$$

In the above expressions:

1) $\lambda_p(t_1,t_0)$ – Legendre symbol, which depends on the initial (t_0) and final (t_1) times of the system evolution;
2) $N_p(t_1,t_0)$ – normalyzing factor in a propagator K_p, which will be determineted from relation analogous to (4.12) and (4.12′);
3) t_1 and t_0 have to be such that $|\gamma_1-\gamma_0|_p\le\frac{1}{p}$ ($t_1,t_0\in G_p'$), and $\dot{\gamma}_1,\dot{\gamma}_0,\gamma_1,\gamma_0,s_1$ and s_0 are analytical functions;
4) Oscillator mass m and Planck constant h are taken to be equal to unity ($m=1,h=1$).

We shall determine now $\lambda_p(t_1,t_0)$ and $N_p(t_1,t_0)$ so that the relation (4.12) is fulfilled $\forall t_0,t_1,t_2\in G_p'$:

$$\begin{gathered}U(t_2,t_1)\,U(t_1,t_0)=U(t_2,t_0)\\ \int_{\mathbf{Q}_p}K(x_2,t_2;x_1,t_1)K(x_1,t_1;x_0,t_0)\,dx_1=K(x_2,t_2;x_0,t_0).\end{gathered} \tag{5.3}$$

To obtain (5.3) we can write down

$$\int_{\mathbf{Q}_p}K(x_2,t_2;x_1,t_1)K(x_1,t_1;x_0,t_0)\,dx_1=\lambda_p(2,1)\lambda_p(1,0)N_p(2,1)N_p(1,0)$$

$$\cdot\int_{\mathbf{Q}_p}\chi_p\left[\frac{1}{2}\left(\frac{\dot{s}_1x_1^2}{s_1}-\frac{\dot{s}_2\,x_2^2}{s_2}\right)\right]\chi_p\left[\frac{1}{2}\left(\frac{\dot{s}_0x_0^2}{s_0}-\frac{\dot{s}_1x_1^2}{s_1}\right)\right]$$

$$\cdot\,\chi_p\left[-\frac{\dot{\gamma}_2{x_2}^2+\dot{\gamma}_1{x_1}^2}{2\tan(\gamma_2-\gamma_1)}+\frac{x_2x_1\sqrt{\dot{\gamma}_2\dot{\gamma}_1}}{\sin(\gamma_2-\gamma_1)}\right]\chi_p\left[-\frac{\dot{\gamma}_1x_1^2+\dot{\gamma}_0x_0^2}{2\tan(\gamma_1-\gamma_0)}+\frac{x_1x_0\sqrt{\dot{\gamma}_1\dot{\gamma}_0}}{\sin(\gamma_1-\gamma_0)}\right]dx_1$$

$$=\lambda_p(2,1)\lambda_p(1,0)N_p(2,1)N_p(1,0)\chi_p\left[\frac{1}{2}\left(\frac{\dot{s}_0x_0^2}{s_0}-\frac{\dot{s}_2x_2^2}{s_2}\right)\right]\chi_p\left[-\frac{\dot{\gamma}_2{x_2}^2}{2\tan(\gamma_2-\gamma_1)}-\frac{\dot{\gamma}_0x_0^2}{2\tan(\gamma_1-\gamma_0)}\right]$$

$$\cdot\int_{\mathbf{Q}_p}\chi_p\left[-\frac{\dot{\gamma}_1x_1^2}{2}\left(\frac{1}{\tan(\gamma_2-\gamma_1)}+\frac{1}{\tan(\gamma_1-\gamma_0)}\right)+x_1\left(\frac{\sqrt{\dot{\gamma}_2\dot{\gamma}_1}}{\sin(\gamma_2-\gamma_1)}x_2+\frac{\sqrt{\dot{\gamma}_1\dot{\gamma}_0}}{\sin(\gamma_1-\gamma_0)}x_0\right)\right]dx_1.$$

Taking into account (2.20), introducing notation $\tan(\gamma_2-\gamma_1)=\tan(2,1),\ldots$, using the properties (2.15), (2.22), equation (3.5) and some elementary trigonometry, we come to

$$\lambda_p(2,1)=\lambda_p(t_2,t_1)=\lambda_p(2(\gamma_2-\gamma_1))=\lambda_p(2\sin(\gamma(t_2)-\gamma(t_1)),$$

or, in a general case

$$\lambda_p(t_j,t_i)=\lambda_p\Big(2\sin(\gamma(t_j)-\gamma(t_i))\Big), \tag{5.4}$$

$$N_p(t_i,t_j)=\left|\frac{\sqrt{\dot\gamma_i\dot\gamma_j}}{(\gamma_i-\gamma_j)}\right|_p^{1/2}=\left|\frac{\sqrt{\dot\gamma_i\dot\gamma_j}}{\sin(\gamma_i-\gamma_j)}\right|_p^{1/2}. \tag{5.5}$$

On the basis of the above results, the following important conclusions can be deduced:

1) the kernel of the evolution operator (propagator) of p-adic HOTDF, for two space-time points $(x_1,t_1;x_0,t_0)$, has the form

$$K_p(x_1,t_1;x_0,t_0)=\lambda_p(2\sin(\gamma(t_1)-\gamma(t_0))\left|\frac{\sqrt{\dot\gamma_1\dot\gamma_0}}{\sin(\gamma_1-\gamma_0)}\right|_p^{1/2}\chi_p\left[\frac{1}{2}\left(\frac{\dot s_0x_0^2}{s_0}-\frac{\dot s_1x_1^2}{s_1}\right)\right]$$

$$\cdot\chi_p\left[-\frac{\dot\gamma_1x_1^2+\dot\gamma_0x_0^2}{2\tan(\gamma_1-\gamma_0)}+\frac{x_1x_0\sqrt{\dot\gamma_1\dot\gamma_0}}{\sin(\gamma_1-\gamma_0)}\right], \tag{5.6}$$

2) for $t_1=t_0 \Rightarrow$ $\quad K_p(x_1,t_0;x_0,t_0)=\delta(x_1-x_0),$ (5.7)

3) the relations (5.3) are satisfied,

4) like to classical mechanics, the operators of quantum evolution do not form a group with respect to multiplication neither in standard nor in p-adic cases.

One can show that the operator $U(t_1,t_0)$ satisfies the relation of the form (4.13)

$$U(t_1,t_0)\,W(z)\,U^{-1}(t_1,t_0)=W(T(t_1,t_0)z). \tag{5.8}$$

Thus, the evolution of quantum p-adic HOTDF within VV formalism is complitely formulated.

Eigenvalue problem of the evolution operator $U(t_1,t_0)$ for HOTDF has the form

$$U(t_1,t_0)\,\psi(x_0,t_0)=\chi(\alpha(\gamma_1-\gamma_0))\,\psi(x_0,t_0), \tag{5.9}$$

where α corresponds to p-adic energy. For the $\alpha=0$ case we have ground (vacuum) state of the oscillator.

6. Vacuum State

In p-adic quantum theory, it is of major importance to enlighten the question of the existance of vacuum states, i.e. such functions $\psi_0\in L_2(\mathbb{Q}_p)$, for which the condition

$$U_p(t)\psi_0(x)=\int K_t(x,y)\psi_0(y)\,dy=\psi_0(x) \tag{6.1}$$

is valid. These states are invariant with respect to action of the evolution operator. Our task now is to answer whether characteristic function $\Omega(x)$ over $\mathbb{Z}_p$ satisfies (6.1). Note that we have

$$[U(t_1,t_0)\,\Omega]\,(x_1) = \int\limits_{\mathbf{Q}_p} \lambda_p(2\sin(\gamma_1-\gamma_0))\left|\frac{\sqrt{\dot\gamma_1\dot\gamma_0}}{\sin(\gamma_1-\gamma_0)}\right|_p^{1/2} \chi_p\left[\frac{1}{2}\left(\frac{\dot s_0 x_0^2}{s_0}-\frac{\dot s_1 x_1^2}{s_1}\right)\right]$$

$$\cdot\chi_p\left[-\frac{\dot\gamma_1 x_1^2+\dot\gamma_0 x_0^2}{2\tan(\gamma_1-\gamma_0)}+\frac{x_1x_0\sqrt{\dot\gamma_1\dot\gamma_0}}{\sin(\gamma_1-\gamma_0)}\right]\Omega(x_0)\,dx_0$$

$$= \lambda_p(2\sin(\gamma_1-\gamma_0))\left|\frac{\sqrt{\dot\gamma_1\dot\gamma_0}}{\sin(\gamma_1-\gamma_0)}\right|_p^{1/2} \chi_p\left[-\frac{x_1^2}{2}\left(\frac{\dot\gamma_1}{\tan(\gamma_1-\gamma_0)}+\frac{\dot s_1}{s_1}\right)\right]$$

$$\cdot\int\limits_{|x_0|_p\le 1} \chi_p\left[\frac{x_0^2}{2}\left(\frac{\dot s_0}{s_0}-\frac{\dot\gamma_0}{\tan(\gamma_1-\gamma_0)}\right)+\frac{x_1x_0\sqrt{\dot\gamma_1\dot\gamma_0}}{\sin(\gamma_1-\gamma_0)}\right]dx_0. \tag{6.2}$$

Let us introduce some notations:

$$a(t_1,t_0) = \frac{\dot s_0}{2s_0}-\frac{\dot\gamma_0}{2\tan(\gamma_1-\gamma_0)} \tag{6.3}$$

$$b(t_1,t_0) = \frac{\sqrt{\dot\gamma_1\dot\gamma_0}}{\sin(\gamma_1-\gamma_0)}, \tag{6.3'}$$

$$c(t_1,t_0) = \frac{\dot\gamma_1}{2\tan(\gamma_1-\gamma_0)}+\frac{\dot s_1}{2s_1}. \tag{6.3''}$$

By means of (2.24), the expression (6.2) can be written in a more suitable form for discussing the existence of vacuum state

$$[U(t_1,t_0)\,\Omega]\,(x_1) = \lambda_p(2\sin(\gamma_1-\gamma_0))\,|\,b(t_1,t_0)\,|_p^{1/2}\chi_p\big(-x_1^2\,c(t_1,t_0)\big)$$

$$\cdot\begin{cases}\Omega\big(|\,b(t_1,t_0)\,x_1\,|_p\big), \text{ if } |\,a(t_1,t_0)\,|_p\le 1,\\[2ex] \lambda_p(a(t_1,t_0))\,|\,2a(t_1,t_0)\,|_p^{-1/2}\chi_p\left(-\dfrac{b^2(t_1,t_0)}{4a(t_1,t_0)}x_1^2\right)\Omega\left(\left|\dfrac{b(t_1,t_0)}{2a(t_1,t_0)}\,x_1\right|_p\right), \text{ if } |\,a(t_1,t_0)\,|_p>4.\end{cases}$$

This means that $\Omega(\,|\,x_0\,|_p)$ is vacuum state for HOTDF under the following conditions:

1) if $|\,a(t_1,t_0)\,|_p\le 1$ then

i) $\lambda_p(2\sin(\gamma_1-\gamma_0)) = 1,$

ii) $|\,b(t_1,t_0)\,|_p^{1/2} = 1,$ (6.4)

iii) $|\,c(t_1,t_0)\,x_1^2\,|_p = |\,a(t_0,t_1)\,|_p\le 1;$

2) if $|\,a(t_1,t_0)\,|_p>4$ then

$$i)\ \lambda_p(2\sin(\gamma_1-\gamma_0))\,\lambda_p(a(t_1,t_0))=1,$$

$$ii)\ \left|\frac{b(t_1,t_0)}{2a(t_1,t_0)}\right|_p^{1/2}=1, \tag{6.5}$$

$$iii)\ \left|c(t_1,t_0)\,x_1^2+\frac{b^2(t_1,t_0)}{4a(t_1,t_0)}x_1^2\right|_p\le 1.$$

We have in mind that: $|\gamma_1-\gamma_0|_p=|\sin(\gamma_1-\gamma_0)|_p\le\frac{1}{p}$ for $p\neq 2$, and $|x_1|_p\le 1$.

It is worth noting that the ground state of p-adic HOTDF, unlike the real one, can be degenerated. Namely, the state $\Omega_{0\nu}(x)$, for $p\equiv 1(\bmod 4)$,

$$\Omega_{0\nu}(x)=p^{-\nu/2}(1-p^{-1})^{-1/2}\chi_p(\tau x^2)\,\delta(p^\nu-|x|_p),\ \nu\in\mathbb{N}\ \text{and}\ \tau^2=-1 \tag{6.6}$$

is a vacuum state provided that

$$1)\ \lambda_p(2\sin(\gamma_1-\gamma_0))\,\lambda_p(\tau+a(t_1,t_0))=1,$$

$$2)\ \left|\frac{b(t_1,t_0)}{\tau+a(t_1,t_0)}\right|_p^{1/2}=1, \tag{6.7}$$

$$3)\ -\frac{b^2(t_1,t_0)}{4(\tau+a(t_1,t_0))}-c(t_1,t_0)=\tau.$$

7. Examples

In the simplest case $\omega(t)=\omega=1$ $(m=1)$, Lagrangian has the form

$$L(\dot q,q)=\frac{\dot q^2}{2}-\frac{q^2}{2},\quad \dot q,q\in\mathbb{Q}_p. \tag{7.1}$$

The matrix of classical evolution of the phase point (3.10) obtains very simple form [21]

$$T(t)=\begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}. \tag{7.2}$$

Now, action of the evolution operator can be represented as

$$U_p(t)\psi(x)=\int_{\mathbb{Q}_p}\lambda_p(2\sin t)\,|\sin t|_p^{-1/2}\chi_p\Big(\frac{xy}{\sin t}-\frac{x^2+y^2}{2\tan t}\Big)\psi(y)\,dy. \tag{7.3}$$

Since, the harmonic oscillator belongs to an autonomous system, operator U and its kernel K satisfy properties (4.12) and (4.12′) [23]. In accordance with Weyl quantization, the operators U and W satisfy relation (4.13), as a special case of relation (5.3). $\Omega(x)$ is a vacuum state of the harmonic oscillator for all times for wich the kernel K is defined. The harmonic oscillator problem was studied in detail in literature, so it will not be of further interest here.

In the case $\omega(t)=\omega_0/(1-at)^2$, the system will be described by the Lagrangian

$$L(\dot{q},q,t)=\frac{\dot{q}^2}{2}-\frac{\omega_0^2}{(1-at)^4}\frac{q^2}{2}. \tag{7.4}$$

The corresponding equation of motion is

$$\ddot{q}+\frac{\omega_0^2}{(1-at)^4}q=0. \tag{7.5}$$

This differential equation can be solved by means of the replacements:

$$x=1-at,\ q=ux,\ x=\frac{1}{v}, \tag{7.6}$$

by which eq. (7.5) is converted into

$$\frac{d^2u}{dv^2}+\frac{\omega_0^2}{a^2}u=0, \tag{7.7}$$

i.e. it becomes differential equation with constant coefficients. The general solution of the equation (7.5) is

$$q=(1-at)\left[A\cos\left(\frac{\omega_0}{a(1-at)}\right)+B\sin\left(\frac{\omega_0}{a(1-at)}\right)\right], \tag{7.8}$$

where A and B are constants wich depend on the initial conditions. The comparison of (7.8) to (3.3), leads to the conclusion that functions $s(t)$ and $\gamma(t)$ (amplitude and phase of HOTDF) have the following form:

$$s(t)=(1-at), \tag{7.9}$$

$$\gamma(t)=\frac{\omega_0}{a(1-at)}. \tag{7.10}$$

The evolution matrix of the classical phase point from $t_0=0$ to the moment t $(t\neq\frac{1}{a})$ becomes now

$$T(t,0)=\begin{bmatrix} (1-at)\left[\cos\left(\frac{\omega_0 t}{1-at}\right)+\frac{a}{\omega_0}\sin\frac{\omega_0 t}{1-at}\right] & \frac{(1-at)}{\omega_0}\sin\frac{\omega_0 t}{1-at} \\ \frac{a^2 t}{(1-at)}\cos\left(\frac{\omega_0 t}{1-at}\right)-\frac{a^2(1-at)+\omega_0}{\omega_0(1-at)}\sin\frac{\omega_0 t}{1-at} & \frac{1}{(1-at)}\cos\left(\frac{\omega_0 t}{1-at}\right)-\frac{a}{\omega_0}\sin\frac{\omega_0 t}{1-at} \end{bmatrix} \tag{7.11}$$

If one replaces $a=0$ (and $\omega_0=1$) in the above matrix, we obtain (7.2), i.e. matrix of the evolution for the time independent oscillator. In addition to the general constraint $(t\neq 1/a)$, in the p-adic case the conditions of defining sine and cosine functions become:

$$\left|\frac{\omega_0 t}{1-at}\right|_p\leq\frac{1}{p}\ \text{ if }\ p\neq 2,\ \text{and}\ \left|\frac{\omega_0 t}{1-at}\right|_2\leq\frac{1}{4}\ \text{ if } p=2. \tag{7.12}$$

It is the general constraint to the p-adic time.

The kernel $K_p(x,t;x_0,0)$, in accordance with (5.6), (7.9) and (7.10), has the following form:

$$K_p(x,t;x_0,0)=\lambda_p\left(2\sin\frac{\omega_0 t}{1-at}\right)\left|\frac{\omega_0}{(1-at)\sin\frac{\omega_0 t}{1-at}}\right|_p^{1/2}\chi_p\left[\frac{a}{2}\left(\frac{x_1^2}{1-at}-x_0^2\right)\right]$$

$$\cdot\chi_p\left[-\frac{\frac{\omega_0}{(1-at)^2}x_1^2+\omega_0 x_0^2}{2\tan\frac{\omega_0 t}{1-at}}+\frac{x_1x_0\frac{\omega_0}{1-at}}{\sin\frac{\omega_0 t}{1-at}}\right]. \tag{7.13}$$

The kernel (7.13) satisfies the relation (5.3). Due to the feature $|\sin\alpha|_p=|\alpha|_p$ and $\lambda_p(\sin\alpha)=\lambda_p(\alpha)$, the first two terms in kernel (7.13) can be simplified to produce

$$\lambda_p\left(2\sin\frac{\omega_0 t}{1-at}\right)=\lambda_p\left(2\frac{\omega_0 t}{1-at}\right), \tag{7.14}$$

$$\left|\frac{\omega_0}{(1-at)\sin\frac{\omega_0 t}{1-at}}\right|_p^{1/2}=|t|_p^{-1/2},\ \forall a,\omega_0. \tag{7.15}$$

According to the eigenvalue problem which we defined by the relation (5.9) one can write

$$U_p(t,0)\psi_p(x_0,0)=\chi_p\left(\frac{\alpha_p\omega_0}{(1-at)}t\right)\psi_p(x_0,0), \tag{7.16}$$

and consequently, energy levels of the HOTDF under consideration will be of the form

$$E_p=\frac{\alpha_p\omega_0}{(1-at)}\,. \tag{7.17}$$

What conditions should be satisfied for p-adic vacuum state to exist? Let us consider first the function $\Omega(|x|_p)$, namely conditions (6.4) and (6.5) that determine a possibility of its existence. At the beginning we rewrite (6.3), (6.3′) and (6,3″) in an explicit form:

$$a(t,0)=-\frac{a}{2}-\frac{\omega_0}{2\tan\frac{\omega_0 t}{1-at}}, \tag{7.18}$$

$$b(t,0)=\frac{\omega_0}{(1-at)\sin\frac{\omega_0 t}{1-at}}, \tag{7.18′}$$

$$c(t,0)=\frac{\omega_0}{2(1-at)^2\tan\frac{\omega_0 t}{1-at}}-\frac{a}{2(1-at)}, \tag{7.18″}$$

and the conditions (6.4) for $|a(t,0)|_p\le 1$:

$$\begin{aligned}&i)\ \lambda_p\left(2\frac{\omega_0 t}{1-at}\right)=1,\\ &ii)\ |t|_p^{-1/2}=1,\\ &iii)\ |c(t,0)|_p\le 1.\end{aligned} \tag{7.19}$$

Note that $|a(t,0)|_p=|c(t,0)|_p=|1/2t|_p$. This imposes a necessary condition for the vacuum state existence $|t|_p=1$. For determining the "phase" factor λ_p, it is necessary to know values of a,t and ω_0. For $|a(t,0)|_p>4$ we have

$$i)\ \lambda_p\left(2\frac{\omega_0 t}{1-at}\right)\lambda_p(a(t,0))=1,$$

ii) $|t^{-1}|_p = |2\,a(t,0)|_p,$ (7.20)

iii) $\left| c(t,0) + \frac{b^2(t,0)}{4a(t,0)} \right|_p \leq 1.$

In this case the condition *ii)* in (7.20) is always satisfied, so that the conditions of the vacuum state existence, neglecting at this moment the condition *i)* in (7.20), practically reduces to a general constraint (7.12) for p-adic time.

It is worth to note that by putting $\omega_0 = 1$ and $a = 0$, all conditions of the vacuum state reduce to the already known condition $|t|_p \leq \frac{1}{p}$, for $p \neq 2$ and $|t|_2 \leq \frac{1}{4}$, for $p = 2$, which we met in ordinary (time independent) oscillator.

For $p \equiv 1(\text{mod}\,4)$, the state (6.6) will be vacuum state, if $|a(t,0)|_p \neq 1$ and

$$1)\ \lambda_p\left(2\frac{\omega_0 t}{1-at}\right)\lambda_p(\tau + a(t,0)) = 1,$$

$$2)\ |t^{-1}|_p = |\tau + a(t,0)|_p, \qquad (7.21)$$

$$3)\ -\frac{b^2(t,0)}{4(\tau + a(t,0))} - c(t,0) = \tau.$$

Finally, note that all listed conditions represent constraints on a possible choice of p-adic parameters a, ω_0 and the duration of time interval t.

8. Concluding Remarks

In this paper we have applied p-adic quantum mechanics to the harmonic oscillator with time-dependent frequency. This is the first example of systems with time-dependent Lagrangians treated p-adically.

Two main results obtained are: (*i*) the kernel of quantum evolution operator and (*ii*) $\Omega(|x_p|_p)$ vacuum state. It is important to note that an eigenstate $\Omega(|x_p|_p)$ exists under definite conditions. It means that some real models cannot be generalized to the p-adic ones. Existence of this simplest vacuum state is a necessary condition to make the corresponding adelic model. One can argue that models with the simplest p-adic vacuum state $\Omega(|x_p|_p)$ are more fundamental than those without it.

Adelic approach [23] gives a unified treatment of real and a p-adic aspects, and for the harmonic oscillator with time-dependent frequency [24] it will be published elsewhere.

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p-ADIČNI HARMONIJSKI OSCILATOR SA VREMENSKI ZAVISNOM FREKVENCIJOM

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Sadržaj U radu je razmatran problem klasičnog i kvantnog harmonijskog oscilatora sa vremenski zavisnom frekvencijom nad poljem p-adičnih brojeva. Razmatrana je vremenska evolucija ovog sistema u skladu sa Vladimirov-Volovićevim formalizmom. Nadjen je operator vremenske evolucije i njegovo jezgro, vakuumsko stanje, i ispitivane su mogućnosti njegove egzistencije i degeneracije. Posebno su razmatrani slučajevi $\omega = \omega_0$ i $\omega(t) = \omega_0/(1 - at)^2$.