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NEW INTEGRABLE SYSTEMS RELATED TO HYDROGEN ATOMS IN COMBINED CIRCULARLY POLARIZED AND MAGNETIC FIELDS

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Abstract The Hamiltonian function $H = \frac{\vec{p}^2}{2} - \frac{1}{r} + fx + \frac{\gamma^2}{8}(\lambda_1^2 x^2 + y^2 + \lambda_2^2 z^2)$ is investigated. By direct construction of the two additional integrals of motion it is demonstrated that the cases $\lambda_1 = \lambda_2 = 1$ and $\lambda_1 = 2$, $\lambda_2 = 1$ are integrable. In the case $\lambda_1 = 1$, $\lambda_2 = 1/2$ one additinal integral of motion is constructed and the integrability is further tested numerically.

1 Introduction

In recent paper [1] we have shown that the three-dimensional dynamical system defined with the Hamiltonian function with velocity dependent potential:

$$H_\lambda = \frac{\vec{p}^2}{2} - \frac{1}{r} - \omega l_z + fx + \frac{\omega^2}{18}(x^2 + y^2 + \lambda^2 z^2) \tag{1}$$

is integrable if $\lambda = 1$ and $\lambda = 2$, i.e it possesses two additional (besides the Hamiltonian function itself) independent integrals of motion which are in involution.

This dynamical system is a rare example of three-dimensional integrable but not separable Hamiltonian system which is at the same time related to important realistic systems: hydrogen atoms in circularly polarized (CP) microwave fields and hydrogen atoms in crossed magnetic and electric fields [2].

Even more closely related to the above integrable Hamiltonian function is the system which describes the hydrogen atom in the presence of the CP field together with the constant magnetic field orthogonal to the polarization plane. The corresponding Hamiltonian function in the frame rotating together with the CP field reads (in atomic units, $m = 1, e = 1$)

$$H_{CP\gamma} = \frac{\vec{p}^2}{2} - \frac{1}{r} - (\omega - \frac{\gamma}{2})l_z + fx + \frac{\gamma^2}{8}(x^2 + y^2). \tag{2}$$

where ω and f are the frequency and intensity of the CP field while γ is intensity of the magnetic field. This system has recently been studied in connection with the nondispersive wave packets and atomic traps [3], [4], [5]. Note that in the case of the two dimensional

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motion in the z-plane ($z = 0, p_z = 0$) and in the special case when $(\omega - \gamma/2)^2/18 = \gamma^2/8$ the two Hamiltonian functions (1) and (2) coincide.

Another interesting special case of the last Hamiltonian function is when $\omega = \gamma/2$ i.e. when the Coriolis-like term is absent. It appears that the two-dimensional variant of that case with the Hamiltonian function

$$H'_{CP\gamma} = \frac{p_x^2}{2} + \frac{p_y^2}{2} - \frac{1}{(x^2+y^2)^{1/2}} + fx + \frac{\gamma^2}{8}(x^2+y^2) \tag{3}$$

is separable in eliptic coordinates. It is natural to ask if there are three-dimensional integrable generalizations of the last Hamiltonian function analogous to Eq.(1). Our task in this paper is to (at least partially) answer that question i.e. to find out if there are such values of the parameters λ_1 and λ_2 for which the Hamiltonian function

$$H_{\lambda_1,\lambda_2} = \frac{\vec{p}^2}{2} - \frac{1}{r} + fx + \frac{\gamma^2}{8}(\lambda_1^2 x^2 + y^2 + \lambda_2^2 z^2) \tag{4}$$

is integrable. In Section 2 we shall apply the generalization of the Whittaker's method developed in [1], i.e. we shall search for the additional integrals of motion which are polynomial in momenta. In this way we shall prove that the system Eq.(4) is integrable if $\lambda_1 = \lambda_2 = 1$ and $\lambda_1 = 2$, $\lambda_2 = 1$. Also in the case $\lambda_1 = 1$, $\lambda_2 = 1/2$ we shall construct one additional integral of motion quadratic in momenta. In Section 3 we shall numerically invastigate integrability of the last case using Poincare surfaces of section of the corresponding reduced two-dimensional systems. The calculations indicate that the last case is also integrable.

2 Integrals of motion

2.1. Integrals of motion as polynomials in momenta: General set of equations

Given the Hamiltonian function H_{λ_1,λ_2} Eq.(4), we shall assume that there exists a dynamical function $I(\vec{r},\vec{p})$ whose Poisson bracket with the Hamiltonian function vanishes:

$$\{H_{\lambda_1,\lambda_2}, I\}_{PB} = 0. \tag{5}$$

It appears that the most efficient way to solve the above equation (in I) is if one first complexifies the equations by introducing the complex coordinates and momenta [1]:

$$q = x + iy, \quad q^* = x - iy, \quad p = p_x + ip_y, \quad p^* = p_x - ip_y. \tag{6}$$

In complex variables the Hamiltonian function is:

$$H_{\lambda_1,\lambda_2} \equiv E = \frac{1}{2}(pp^* + p_z^2) + W(q, q^*, z), \tag{7}$$

whith the potential

$$W = -\frac{1}{(qq^* + z^2)^{1/2}} + \frac{1}{2}f(q+q^*) + \frac{\gamma^2}{8}\left(\frac{\lambda_1^2}{4}(q+q^*)^2 - \frac{1}{4}(q-q^*)^2 + \lambda_2^2 z^2\right), \tag{8}$$

while the Poisson bracket becomes:

$$\{H,I\}_{PB} = 2(\frac{\partial H}{\partial q}\frac{\partial I}{\partial p^*} - \frac{\partial H}{\partial p^*}\frac{\partial I}{\partial q} + \frac{\partial H}{\partial q^*}\frac{\partial I}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial I}{\partial q^*}) + \frac{\partial H}{\partial z}\frac{\partial I}{\partial p_z} - \frac{\partial H}{\partial p_z}\frac{\partial I}{\partial z}. \quad (9)$$

For the integral of motion I we assume the polynomial form:

$$I = \sum_{n=0}^{N}\sum_{k=0}^{n}(1+\delta_{k0})^{-1}(a_{n,k}(q,q^*,z;E)p^k + a^*_{n,k}(q,q^*,z;E)p^{*k})p_z^{n-k}, \quad a_{n,0} = a^*_{n,0}. \quad (10)$$

The terms $p^j p^{*k}$ for both $j,k \neq 0$ are omitted in the above expansion because from Eq.(7) one finds that $pp^* = 2(E-W) - p_z^2$ and therefore if e.g. $j-k = m \geq 0$ then $p^j p^{*k} = p^m(2(E-W)-p_z^2)^k$. Substituing the expressions (7) and (10) into Eq.(9) one obtaines the following set of coupled linear partial differential equations for the complex coefficients $a_{n,k}(q,q^*,z;E)$:

$$\frac{\partial a_{n,k}}{\partial q} + 2(E-W)\frac{\partial a_{n+2,k+2}}{\partial q^*} - \frac{\partial a_{n,k+2}}{\partial q^*} + \frac{\partial a_{n,k+1}}{\partial z}$$

$$-2(k+2)\frac{\partial W}{\partial q^*}a_{n+2,k+2} - (n-k+1)\frac{\partial W}{\partial z}a_{n+2,k+1} = 0, \quad 0 \leq k \leq n \leq N, \quad (11)$$

$$\frac{\partial a_{n-2,0}}{\partial z} + 2(E-W)(\frac{\partial a_{n,1}}{\partial q^*} + \frac{\partial a^*_{n,1}}{\partial q}) - \frac{\partial a_{n-2,1}}{\partial q^*} - \frac{\partial a^*_{n-2,1}}{\partial q}$$

$$-n\frac{\partial W}{\partial z}a_{n,0} - 2\frac{\partial W}{\partial q^*}a_{n,1} - 2\frac{\partial W}{\partial q}a^*_{n,1} = 0, \quad 1 \leq n \leq N+2, \quad (12)$$

where we have adopted the convention $a_{n,k} = 0$ if $n < k$, $N < n$ or $n < 0$.

We see that the coefficients $a_{2n,k}$ with the even order are decoupled from those of the odd order $a_{2n+1,k}$, i.e. the polynomial integrals of motions if they exist possess either only odd or even powers in momenta.

2.2 Integrals of motion of first and second order

It is easy to verify that Eqs.(11) and (12) for $N = 1$ and with W from Eq.(8) possess a nonzero solution only when $\lambda_2 = 1$. Obviously, the corresponding (linear in momenta) integral of motion is $l_x = yp_z - zp_y$.

Solving Eqs.(11) and (12) for $N = 2$ one finds out that there are three cases for which nontrivial solutions exist. The corresponding three solutions are given in Appendix.

The first two cases are defined with $\lambda_1 = \lambda_2 = 1$ and $\lambda_1 = 2$, $\lambda_2 = 1$. Substituting the solutions Eq.(28) and Eq.(29) into the expansion Eq.(10) and returning to the real variables one obtaines the corresponding integrals of motion:

$$I_{\lambda_1=1,\lambda_2=1} = I_{1,1} = fA_x + \frac{f^2}{2}(y^2+z^2) - \frac{\gamma^2}{8}(l_x+l_y+l_z)^2, \quad (13)$$

and

$$I_{2,1} = A_x + \frac{f}{2}(y^2+z^2) + \frac{\gamma^2}{4}x(y^2+z^2) \quad (14)$$

where A_x is the x component of the Runge-Lenz vector:

$$A_x = p_z l_y - p_y l_z + \frac{x}{r}. \tag{15}$$

In both above cases $\lambda_2 = 1$, i.e l_x is also integral of motion since it commutes with the Hamiltonian function. Moreover l_x commutes with and is independent of both invariants $I_{1,1}$ and $I_{2,1}$. This proves that the dynamical system with Hamiltonian function Eq.(4) is integrable when $\lambda_1 = \lambda_2 = 1$ and $\lambda_1 = 2$, $\lambda_2 = 1$.

Interestingly enough, these two cases are not only integrable but also separable. It can easily be verified that when $\lambda_1 = 2$, $\lambda_2 = 1$ the corresponding Hamilton-Jacoby equation separates in semiparabolic coordinates u, v, ϕ:

$$x = \frac{u^2 - v^2}{2}, \quad \rho = (y^2 + z^2)^{1/2} = uv, \quad \tan\phi = \frac{y}{z},$$

where ρ, ϕ, x are the standard cylindrical coordinates (with respect to x-axes). On the other hand, when $\lambda_1 = \lambda_2 = 1$ the Hamilton-Jacoby equation separates in eliptic coordinates μ, ν, ϕ where

$$x = \frac{4f}{\gamma^2}(1 + \cosh\mu\cos\nu), \quad \rho = \frac{4f}{\gamma^2}\sinh\mu\sin\nu.$$

The third case for which Eqs.(11) and (12) (with $N = 2$) allow a nonzero solution is when $\lambda_1 = 1$, $\lambda_2 = 1/2$. For future convenience we write down explicitely the Hamiltonian function in this case:

$$H_{1,1/2} = \frac{\vec{p}^2}{2} - \frac{1}{r} + fx + \frac{\gamma^2}{8}(x^2 + y^2 + \frac{1}{4}z^2). \tag{16}$$

The corresponding integral of motion obtained from the solution Eq.(30) reads

$$I_{1,1/2} = fA_x + \frac{f^2}{2}(y^2 + z^2) - \frac{\gamma^2}{8}l_z^2 + \frac{f\gamma^2}{16}xz^2. \tag{17}$$

In contrast with the previous two cases there is no linear (in momenta) integral of motion and the question of integrability of the Hamiltonian function Eq.(16) remaines open. To prove that it is integrable one might try to construct the second integral of motion by solving Eqs.(11) and (12) for $N \geq 4$. The second integral of motion if it exists should be at least the fourth order polynomial in momenta (if it is polynomial at all). This follows from the well known fact [6] that the special case when $f = 0$ of the Hamiltonian function Eq.(16) is integrable and possesses (besides l_z) an integral of motion which is quartic in momenta. However the search for the general solution of Eqs.(11) and (12) for $N \geq 4$ and W from Eq.(8) was beyond our technical capabilities. Therefore, in the following section, we have tested numerically the integrability of H with the surfaces of section of the corresponding reduced two-dimensional system.

3 The case $\lambda_1 = 1$, $\lambda_2 = 1/2$: Numerical calculations

In the case (like ours) when one has one additional integral of motion besides energy and the number of degrees of freedom n is greater than two (and if the motion generated by the integral of motion treated as a Hamiltonian function satisfies certain conditions [7]) one can define equivalent family of reduced $n-1$-dimensional systems, i.e. one can effectively lower the number of degrees of freedom by one. Then, the system is integrable if and only if the same is true for these $n-1$-dimensional systems. The reduced systems are defined in the following way [7]. First one fixes the value of the additional integral of motion; the corresponding level set in the phase space is $2n-1$-dimensional hypersurface. Then one identifies the points on the hypersurface which lie on the same trajectory generated by the integral of motion. In other words the points in the phase space of the reduced system are trajectories of the additional integral of motion which lie on the same level set (of the additonal invariant). The phase space is therefore $2n-2$-dimensional, i.e. it is of even dimension. The motion in such quotient phase space is then generated by the Hamiltonian function. It is well defined becuse the motion of the Hamiltonian function in the original phase space commutes with that of the additional integral of motion (under above mentioned conditions the quotient phase space is differential manifold and the reduced motion is Hamiltonian [7]).

We shall now describe how in the case of our system Eqs.(16) and (17) the above procedure is effectivelly performed. First one fixes the level set of $I_{1,1/2}$ by setting

$$I_{1,1/2}(\vec{r},\vec{p}) = C. \tag{18}$$

Then one fixes the value of one of the coordinates e.g. y:

$$y = y_0. \tag{19}$$

This is equivalent of choosing the representatives for the trajectories generated by $I_{1,1/2}$, i.e. with the above two equations the four-dimensional phase space is defined. The effective two-dimensional Hamiltonian function $H^{C,y_0}_{1,1/2}(x,z,p_x,p_z)$ is then obtained simply by solving Eq.(18) for p_y:

$$p_{y_0}(x,z,p_x,p_z;C) = \frac{-b \pm (b^2-4ac)^{1/2}}{2a},$$

$$a = fx + \frac{\gamma^2}{8}x^2, \quad b = -(f + \frac{\gamma^2}{4}x)p_x y_0,$$

$$c = fp_z^2 x - f\frac{x}{r} + \frac{\gamma^2}{8}p_x^2 y_0^2 - fp_x p_z z - \frac{f\gamma^2}{16}xz^2 - \frac{f^2}{2}(y_0^2+z^2) + C \tag{20}$$

and substituing the solution in the Hamiltonian function $H_{1,1/2}$, Eq.(16):

$$H^{C,y_0}_{1,1/2} = \frac{p_x^2+p_z^2}{2} + \frac{p_{y_0}^2(x,z,p_x,p_z;C)}{2} - \frac{1}{(x^2+z^2+y_0^2)^{1/2}} + fx + \frac{\gamma^2}{8}(x^2+y_0^2+\frac{1}{4}z^2). \tag{21}$$

Now, the Hamiltonian function $H_{1,1/2}$ is integrable if and only if for each pair C, y_0 the Hamiltonin function $H^{C,y_0}_{1,1/2}$ is also integrable. The integrability of any two-dimensional system can be tested numericaly by construction of surfaces of section (SOS).

In Figure 1 we have shown the typical SOS corresponding to the motion of $H^{C,y_0}_{1,1/2}$. The semiparabolic coordinates $x = (u^2-v^2)/2$, $z = uv$ which regularize the Coulomb

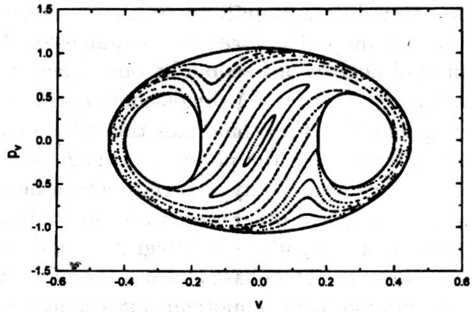

Fig.1. SOS of the trajectories corresponding to $H_{1,1/2}^{C,y_0}$, Eq.(21), with $f = 2.2$, $\gamma = 1.9$, $C = 0.964$, $y_0 = 0$. The effective energy is $H_{1,1/2}^{C,y_0} \equiv E = -3$. SOS is defined with $u_0 = 0.1$, $\dot{u} > 0$. The energy hypersurface defined with $H_{1,1/2}^{C,y_0}(u, v, p_u, p_v) = E$ is compact and the solid lines in the plot are the boundaries of its intersection with the $u = u_0$ plane.

singularity are used for the construction of the SOS. We have calculated many SOS's for various values of C, y_0, f, γ and E and the picture was always analogous to Figure 1, i.e. there were no traces of the chaotic motion and the phase space apeared to be foliated entirely by the two-dimensional tori. This strongly suggests that the system with the Hamiltonian function $H_{1,1/2}$, Eq.(16), is integrable. However, the actaul proof of integrability (i.e. the construction of the second additional integral of motion) remaines an open problem.

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Appendix: Solutions of Eqs.(11) and (12) for N=2

For N=2 Eqs.(11) and (12) which couple only even order coefficients $a_{2,k}$ and $a_{0,0}$ are given explicitely by:

$$\frac{\partial a_{2,2}}{\partial q} = 0, \tag{22}$$

$$\frac{\partial a_{2,1}}{\partial q} + \frac{\partial a_{2,2}}{\partial z} = 0, \tag{23}$$

$$\frac{\partial a_{2,0}}{\partial q} + \frac{\partial a_{2,1}}{\partial z} - \frac{\partial a_{2,2}}{\partial q^*} = 0, \tag{24}$$

$$\frac{\partial a_{2,0}}{\partial z} - \frac{\partial a_{2,1}}{\partial q^*} - \frac{\partial a_{2,1}^*}{\partial q} = 0, \tag{25}$$

$$\frac{\partial a_{0,0}}{\partial q} + 2(E - W)\frac{\partial a_{2,2}}{\partial q^*} - 4\frac{\partial W}{\partial q^*}a_{2,2} - \frac{\partial W}{\partial z}a_{2,1} = 0, \tag{26}$$

$$\frac{\partial a_{0,0}}{\partial z} + 2(E - W)(\frac{\partial a_{2,1}}{\partial q^*} + \frac{\partial a_{2,1}^*}{\partial q}) - 2\frac{\partial W}{\partial z}a_{2,0} - 2\frac{\partial W}{\partial q^*}a_{2,1} - 2\frac{\partial W}{\partial q}a_{2,1}^* = 0. \tag{27}$$

The three cases with non-zero solutions are (W is given in Eq.(8)):
i) $\lambda_1 = 2,\ \lambda_2 = 1$

$$a_{2,2} = q^*/4, \quad a_{2,1} = z/2, \quad a_{2,0} = -(q + q^*)/4,$$

$$a_{0,0} = -E(q+q^*)/2 + 3fqq^*/4 + f(q^2+q^{*2})/8 + 15\gamma^2 qq^*(q+q^*)/64$$
$$+\gamma^2(q^3+q^{*3})/64 + fz^2/2 + 3\gamma^2 z^2(q+q^*)/16. \qquad (28)$$

ii) $\lambda_1 = \lambda_2 = 1$

$$a_{2,2} = fq^*/4+\gamma^2 q^{*2}/32,\ a_{2,1} = (f/2+\gamma^2 q^*/8)z,\ a_{2,0} = -(f(q+q^*)/4+\gamma^2 qq^*/16-\gamma^2 z^2/8),$$

$$a_{0,0} = -\gamma^2(qq^*+2z^2)/(8r) + f^2(4z^2-(q-q^*)^2)/8$$
$$-[E - f(q+q^*)/2 - \gamma^2(qq^*+z^2)/8][f(q+q^*)/2 + \gamma^2(qq^*+2z^2)/8], \qquad (29)$$

where

$$r = (qq^*+z^2)^{1/2}.$$

iii) $\lambda_1 = 1,\ \lambda_2 = 1/2$

$$a_{2,2} = fq^*/4 + \gamma^2 q^{*2}/32, \quad a_{2,1} = fz/2, \quad a_{2,0} = -f(q+q^*)/4 + \gamma^2 qq^*/16,$$

$$a_{0,0} = -\gamma^2 qq^*/(8r) + f^2(4z^2-(q-q^*)^2)/8 + f\gamma^2(q+q^*)z^2/32$$
$$-[E - f(q+q^*)/2 - \gamma^2(4qq^*+z^2)/32][f(q+q^*)/2 + \gamma^2 qq^*/8]. \qquad (30)$$

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NOVI INTEGRABILNI SISTEMI POVEZANI SA VODONIKOVIM ATOMIMA U KOMBINOVANIM CIRKULARNO POLARIZOVANIM I MAGNETNIM POLJIMA

M. Raković

Sadržaj Ispitivana je Hamiltonova funkcija $H = \frac{\vec{p}^2}{2} - \frac{1}{r} + fx + \frac{\gamma^2}{8}(\lambda_1^2 x^2 + y^2 + \lambda_2^2 z^2)$. Direktno su konstruisana dva dodatna integrala kretanja za koje je pokazano da u slučajevima $\lambda_1 = \lambda_2 = 1$ i $\lambda_1 = 2$, $\lambda_2 = 1$ su integrabilni. U slučaju $\lambda_1 = 1$, $\lambda_2 = 1/2$ jedan dodatni integral kretanja je konstruisan čija integrabilnost je dalje numerički testirana.