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A NEW CONTINUOUS ENTROPY FUNCTION AND ITS INVERSE

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Abstract. *A new continuous entropy function of the form h(x)=-k∫p'(x)*ln(p(x)) is constructed based on Lebesgue integral. Solution of the integral is of the form h(x)=-k*p(x)*ln(e^{-1}*p(x)). Inverse solution of this function in the form of p(x)=h(x)*W^{-1}(h(x)* e^{-1}) has been obtained and this is a novelty. The fact that the integral of a logarithmic function is also a logarithmic function has been exploited and used in a more general ansatz for a deliberate function. The solution of the differential equation Λ(x)= g(x)'*ln(g(x)) for a deliberate function g(x) includes the Lambert W function. The solution is a parametric function g(x)=(s(x)+C)/W(e^{-1}*(s(x)+C)). Parameter C has a minimal value and, consequently, an infimal function g(x) exists. This transform is of general use, and is particularly suitable for use in neural networks, the measurement of complex systems and economic modeling, since it can transform multivariate variables to exponential form.*

Key words: *sigmoid functions, logarithmic probability density and logarithmic cumulative probability functions, information entropy, solution of the inverse entropy problem fatigans*

1. INTRODUCTION

Entropy is a notion that started with statistical mechanics and Boltzmann equation in the 19th century. In the 20th century, with the emergence of information science, a similarity with the information communication was discovered by Shannon [1] and it showed a similarity with Boltzmann equation [2-4]. That is why Shannon used the term information (discrete) entropy that he expressed as:

$$H(x) = -\sum_{i=1}^{n} p(x_i)\log_2 p(x_i) \tag{1}$$

where $H(x)$ denotes entropy, $p(x_i)$ the probability density function of discrete values x_i and $\log_2$ a binary logarithmic function.

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Similarly, the continuous entropy of the continuous probability function was defined as:

$$H(p(x)) = -\int_a^b p(x)\log_2 p(x)dx \tag{2}$$

where $H(p(x))$ stands for continuous entropy, $p(x)$ for continuous probability function, while a and b are the limits of the interval. Here it should be clearly stated that the continuous entropy is not a limit of discrete entropy since it differs from the limit by ∞.

For this reason the search for other continuous entropy formulations has taken place, among which most notably the Kullback–Leibler [5] divergence was designed. Basically, this entropy is a measure of similarity of two continuous probability density functions and is established as:

$$D_{KL}(P,Q) = \int_{-\infty}^{+\infty} p(x)\ln(\frac{p(x)}{q(x)})dx \tag{3}$$

where D_{KL} denotes Kullback–Leibler relative entropy from continuous cumulative probability function P to Q with continuous probability density functions $p(x)$ and q(x), respectively. This entropy is not symmetric, which can easily be observed from the equation (3) if $p(x)$ and $q(x)$ are interchanged.

Jaynes proposed a similar function

$$H(p(x)) = -\int_{-\infty}^{+\infty} p(x)\ln(\frac{p(x)}{m(x)})dx \tag{4}$$

where $m(x)$ represents a general function.

2. NEW APPROACH FOR A NEW CONTINUOUS ENTROPY FUNCTION

We propose a new approach for a continuous entropy function based on Lebesgue integral. It is well known that Lebesgue integral has the following form [6]:

$$F(x) = \int f(x)d\mu(x) \tag{5}$$

or written differently

$$F(x) = \int f(x)\frac{d\mu(x)}{dx}dx \tag{6}$$

where $F(x)$ designates the Lebesgue integral, f(x) is a non-negative function and μ(x) is a measure.

There have been some attempts [7,8] approaching the entropy with Lebesgue integral. They have chosen the function -ln(p(x)) for f(x) and p(x) as a derivative measure thus yielding for the continuous entropy:

$$H(p(x)) = -k\int p(x)\ln(p(x))dx \tag{7}$$

Now our approach is different, taking for function f(x) = - ln(p(x)) as well but for μ(x) = p(x) thus yielding

$$h(p(x)) = -k\int \frac{dp(x)}{dx}\ln(p(x))dx \tag{8}$$

where *h*(*p*(*x*)) denotes our new entropy.

In order to obtain a solution let us consider the logarithmic equation of the general form

$$u(x) = A + B\ln(f(x)) \tag{9}$$

where *f*(*x*) is a positive function. Then the derivative

$$\frac{du(x)}{dx} = B\frac{df(x)}{dx}\frac{1}{f(x)} \tag{10}$$

is, in general, not a logarithmic function. *A* an*d* B are constants.

Let us consider if a function of an explicit logarithmic form could be constructed.
Let us recall that if

$$v(x) = \ln(x) \tag{11}$$

its integral

$$\int v(x)dx = x\ln(xe^{-1}) - C\ . \tag{12}$$

3. THE DERIVATION OF A LOGARITHMIC FORM FOR A FUNCTION s(x) AND OF A LOGARITHMIC FORM FOR ITS DERIVATIVE Λ(x)

Let us now choose a similar ansatz for a general function g(x)

$$\Lambda(x) = \frac{dg(x)}{dx}\ln(g(x)) \tag{13}$$

Integrating (13)

$$\int_{-\infty}^{x}\Lambda(p)dp = \int_{-\infty}^{x}\frac{dg(p)}{dp}\ln(g(p))dp \tag{14}$$

yields the result

$$s(x) = g(x)\ln(g(x)e^{-1}) - C \tag{15}$$

It can be observed that both Λ(x) and s(x) are logarithmic functions of g(x). Thus the objective has been met with this ansatz.

4. THE GENERAL FUNCTION g(x) AS A SOLUTION FOR THE LOGARITHMIC FORM OF FUNCTION s(x) AND THE LOGARITHMIC FORM FOR ITS DERIVATIVE Λ(x)

The solution of differential equation (13) has the general form

$$g(x) = e^{W(e^{-1}(\int \Lambda(x)dx + C)) + 1} \tag{16}$$

and its equivalent for obtaining $g(x)$ from (14)

$$g(x) = \frac{s(x) + C}{W((s(x) + C)e^{-1})}. \tag{17}$$

$g(x)$ is expressed in terms of another function, the Lambert W function that is the solution of the equation

$$x = ye^{y} \tag{18}$$

$W(x)$ can be expressed explicitly as

$$y = W(x) \tag{19}$$

It is known [9] that the Lambert W function has a real value for the interval $x \in [0, -e^{-1}]$ composed of two branches denoted W for $x \in [0, -e^{-1})$&$W(x)$>=-1 and W_{-1} for $x \in [-e^{-1}, 0)$&W(x)<=-1.

Based on that the estimation of a general constant C can be obtained from (17), the argument of the function W, $e^{-1}(s(x) + C) \geq -e^{-1}$, from which we obtain

$$C \geq -(1 + s(x)) = -(1 + \min(s(x)) = -1 \tag{20}$$

Solutions with $C = 0$ are thus possible and simplify the function g(x).

It can be observed that (16) expresses $g(x)$ as a function of $\Lambda(x)$ and (17) as a function of $s(x)$. Function $g(x)$ generates both $\Lambda(x)$ and $s(x)$ as logarithmic functions.

5. Derivatives of $g(x)$

Differentiating (16) and (17) yields respectively

$$\frac{dg(x)}{dx} = \frac{\Lambda(x)}{1 + W(e^{-1}(\int \Lambda(x)dx + C))} \tag{21}$$

and

$$\frac{dg(x)}{dx} = \frac{\frac{ds(x)}{dx}}{1 + W(e^{-1}(s(x) + C))}. \tag{22}$$

6. Applicability of Theoretical Results to Entropy

Now let us return to our entropy. Comparing equations (8) and (13) we obtain from (15):

$$h(x) = -kp(x)\ln(p(x)e^{-1}) \tag{23}$$

and for its derivative

$$\frac{dh(x)}{dx} = -k\frac{dp(x)}{dx}\ln(p(x)) \tag{24}$$

getting as well

$$\frac{dh(x)}{dp(x)} = -k\ln(p(x)) \tag{25}$$

With this approach we have yielded quite an interesting characteristic, namely the entropy related function and its derivative are both logarithmic functions. This enables simple adding of derivatives and consequently the functions of multiple probability density functions itself. This can be effectively used in multivariate probability functions in economics, management, statistics etc.

Now let us suppose that the entropy related function is known and we ask ourselves for which probability one yields that entropy related value. This is the problem of the inverse entropy related function and can be obtained implementing equation (17) at C=0. The answer is though

$$p(x) = \frac{h(x)}{W(e^{-1}h(x))} \tag{26}$$

or equivalently

$$p(x) = e^{W(e^{-1}h(x))+1} \tag{27}$$

where W represents Lambert W function.

Based on this equation one can determine the probability density function knowing the entropy related function.

The inverse of the equation (23) gives

$$p(x) = -\frac{h(x)}{kW(\frac{-h(x)e^{-1}}{k})} \tag{28}$$

7. Conclusion and Future Work

We have developed a new entropy related function and its inverse. This is, according to the knowledge of the author of this paper, unique. Our ansatz was to use the logarithmic form of a general function and logarithmic form of its derivative. This allows for a more compact formulation of sums of functions in the form of the product.

However, since this method is applicable to statistical methods in general, further research on logarithmic forms of probability density and cumulative probability density functions may be obtained for known statistical distributions such as normal Gauss, student, logistic, lognormal and other. Relations between h(p(x)) and Fisher information function should be explored as well. It seems that this transform might be appropriate for economic modeling because it may be used in the summary formulation of multivariate problems.

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NOVA KONTINUALNA ENTROPIJSKA FUNKCIJA I NJEN INVERZ

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Nova funkcija neprekidne entropije oblika $h(x)=-k\int p'(x)*\ln(p(x))$ *definisana je pomoću Lebesgue-ovog integrala. Rešenje ovog integrala je oblika* $h(x)=-k*p(x)*\ln(e^{-1}*p(x))$. *Novina je da je pronađeno inverzno rešenje pomenute funkcije oblika* $p(x)=h(x)*W^{-1}(h(x)*e^{-1}$. *Činjenica da je integral logaritamske funkcije, takođe, logaritamska funkcija korišćena je u opštijem kontekstu vezanom za deliberate funkciju. Rešenje diferencijalne jednačine* $\Lambda(x)= g(x)'*\ln(g(x))$ *za funkciju* $g(x)$ *uključuje i Lambert-ovu* W *funkciju. Rešenje je parametarska funkcija* $g(x)=(s(x)+C)/W(e^{-1}*(s(x)+C))$. *Kako parametar* C *ima minimalnu vrednost, postoji infimalna funkcija* $g(x)$.

Ovakav pristup ima široku primenu i može se koristiti kod neuronskih mreža, merenja kompleksnih sistema i u ekonomskom modelovanju, s obzirom na to da se njime multivarijantne promenljive mogu transformisati u eksponencijalni oblik.

Ključne reči: *sigmoidna funkcija, logaritamska funkcija gustine i logaritamska funkcija zbirne verovatnoće, informaciona entropija, rešenje inverznog problema entropije*