FACTA UNIVERSITATIS
Series: **Physics, Chemistry and Technology** Vol. 8, N° 1, 2010, pp. 45 - 56
DOI: 10.2298/FUPCT1001045F

DIFFERENCES BETWEEN SOURCE AND NEW FORM OF THE EULER-BERNOULLI EQUATION AND ITS SOLUTION[†]

UDC 532.513"311" : 532.513"312"

Mirjana Filipović[*]

Mihajlo Pupin Institute, Volgina 15, 11000 Belgrade, Serbia

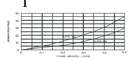
Abstract. *In construction of complex robotic and related mechanisms, lightweight materials are often used and they are characterized by flexibility segments. To achieve precise and rapid control of motion of such a mechanism, it is necessary to synthesize their high fidelity model. This would be of great importance for its analysis and synthesis. The work in this paper starts from the first research in this area, and it includes the original form of the Euler-Bernoulli equation and its solution, which, compared with current knowledge, imposes the need for the expansion of the same equation from multiple points of view. The new form of the Euler-Bernoulli equation and its solution are based on current knowledge in robotics, as well as the knowledge of classical mechanics. This is the only way of how to fully preserve the information of the complexity of the kinematics and dynamics of elastic mechanisms.*

Key words: *Euler-Bernoulli equation, elasticity of link, coupling, kinematics, dynamics*

NOMENCLATURE

DOF	degree of freedom
$t[s]$	time
$x_{i,j}$, $y_{i,j}$, $z_{i,j}$	local coordinate frame, which is set in the base of considered mode
x_j, y_j, z_j	local coordinate frame, which is set in the base of the considered link
$k_s = [x \quad y \quad z \quad \psi \quad \wp \quad \varphi]^T$	Cartesian (external) coordinates
x, y, z	basic coordinate frame, which is set in the root of the considered robotic system
$j = 1, 2, 3, \ldots, n_i$	serial number of the mode of considered link
$i = 1, 2, 3, \ldots, m$	ordinal number of the link

[†] The paper is a part of the research done within the project TR 14001. This research was supported by Ministry of Science and Technological Development of Serbia for financing the national research project "Humanoid robotics: Theory and Application".
[*] Corresponding author. E-mail: mira@robot.imp.bg.ac.rs.
Received 14th December 2010; revised 17th March 2011; accepted 18th March 2011.

$\varepsilon_{i,j} \in R^1[Nm]$	bending moment for the mode tip
$\varepsilon_1 = [\varepsilon_{1,1} \ \ldots \varepsilon_{1,n_1}]^T$	vector of bending moments dynamics
$\hat{\#}_{i,j}$	quantities that are related to an arbitrary point of the elastic line of the mode, for example: $\hat{x}_{i,j}, \hat{\varepsilon}_{i,j}$
$\#_{i,j}$	quantities that are not designated by "^" are defined for the mode tip, for example: $x_{i,j}$, $\varepsilon_{i,j}$
$\#_i$	quantities which characterize link
$\bar{\theta}_j \in R^1[rad]$	rotation angle of the motor shaft after the reducer
$\vartheta_{i,j} \in R^1[rad]$	bending angle of the considered mode
$\omega_{i,j} \in R^1[rad]$	rotation angle of the considered mode tip
$\beta_{i,j} \in R^1[Nm^2]$	flexural rigidity
$\eta_{i,j} \in R^1[s]$	factor which characterizes part of damping in whole flexural characteristics
$\hat{H}_1 \in R^{n_1 x n_1}$	matrix characterizing the inertia of the each mode
$\hat{h}_1 \in R^{n_1 x 1}$	vector characterizing the effect of centrifugal, gravitational and Coriolis forces of each mode
$F_{uk} \in R^{6x1}[N]\ or\ [Nm]$	external contact force
$j_{e1}^T \in R^{n_1 x 6}$	the Jacobian matrix serving to map the impact of the dynamic force of contact F_{uk} on the behavior of each mode
$T_{sti,j} \in R^1[m]$	stationary part of flexible deformation caused by stationary moments that vary continuously over time
$T_{toi,j} \in R^1[m]$	oscillatory part of flexible deformation
$u_j[V]$	voltage
$i_j[A]$	rotor current
$C_{Ej}[V/(rad/s)]$	proportionality constants of the electromotive force
$C_{Mj}[Nm/A]$	proportionality constants of the moment
$B_{uj}[Nm/(rad/s)]$	coefficient of viscous friction
$I_j[kgm^2]$	inertia moments of the rotor and reducer
S_j	expression defining the reducer geometry
$z_1 \in R^{n_1 x n_1}$	matrix characterizes the mutual effect of elasticity forces of the presented modes on the observed mode
	length of each mode
$f_{i,j} \in R^1[m]$	flexure

1. INTRODUCTION

Elasticity of withy, long spread links constructed of light materials, requires an elasticity analysis. When the introduction of link flexibility in the mathematical model is concerned, it is necessary to point out some essential problems in this domain.

With the aim to exploit the experience of previous research, Meirovitch theory was first analyzed [1]. Meirovitch proposed a "modal technique" more than 40 years ago, exactly in 1967. The author elaborated a particular application of the Euler-Bernoulli equation supposing that elastic deformation was a quantity defined in advance with respect to amplitude and frequency and, formed in this way, it was included into a dynamics model. Not finding any other solutions, many researchers in robotics [2-10], applied the solution [1] in the description of the real dynamics of the robot system elastic deformations, or they used many ways to modify the solutions from the [1].

Having not found agreement with Meirovitch and his followers, the definition of elastic deformation was made taking into account the first research studies, i.e. the original form of the Euler-Bernoulli equation.

The Euler-Bernoulli equation was written in 1750. It was written by Bernoulli, a physicist and Euler, a mathematician, his long time friend and colleague. They did not even dream about the robotics and the knowledge we have now on disposal. But, although it was made more than 250 years ago, Euler-Bernoulli equation is still usable and it can be connected logically with the contemporary knowledge from the robotics.

In this paper, the Euler-Bernoulli equation is formed but the "assumed modes technique" is not used, in contrast to contemporaries who deal with this issue as well. That means that the elastic deformation amplitude and its frequency change depending on the moments (perturbation, inertial moments, Coriolis, centrifugal moments, gravity moments as well as coupling moments between the present modes, and the play of the external forces). It, of course, depends on the mechanism configuration, weight, length of the segments of the reference trajectory choice, dynamic characteristics of the motor motions etc.

This has been elaborated in detail in the work, but it is not the only essential problem existing in the pertinent literature.

In the previous papers [2-10], the general solution of the motion of an elastic robotic system has been obtained by considering flexural deformations as transversal oscillations that can be determined by the method of particular integrals of Bernoulli.

It is taken into consideration that any elastic deformation can be presented by superimposing Bernoulli's particular solutions of the oscillatory character and stationary solution of the forced character. See papers [11-16].

The motion equation at any point of considered mode as defined in papers [17] and [18] follows directly from the Euler-Bernoulli equation for the preset boundary conditions.

Nowadays, taking into consideration significantly improved knowledge in the robotics (classical mechanics); the Euler-Bernoulli equation can not be used anymore in its original form, as a purpose of synthesis and analysis of elastic robotic systems. Therefore, with respect to Euler and Bernoulli, it is necessary to further improve the equation. It is the only way for not losing information of complexity of motion dynamics of every mode within a segment (and broader within the total robotic configuration). Thus, it is very important to connect the original Euler-Bernoulli equation and modern robotic knowledge on the principles of classical mechanics. The foundations of classical mechanics are particularly emphasized because synthesis and analysis of kinematics and dynamics of robotic configurations in stiff and elastic elements are based on them. The elasticity of segments on the principles of classical mechanics is implemented in this paper.

This research has theoretical and practical significance. The purpose is to define as realistically as possible both kinematic and dynamic model of the mechanism with stiff and elastic elements which will describe the real system very well.

In the papers [13] and [14] the general form of the mathematical model of the robotic system with elastic segments (the Euler-Bernoulli equation) is given for the first time.

The future work should be directed to the implementation of the flexibility of links on any model of a rigid robot and also on the model of a reconfigurable rigid robot as given in [19] or any other type of mechanism. The mechanism would be modeled to contain elastic elements and to generate vibrations, which are used for conveying particulate and granular materials in [20].

Section 2 analyses the source form of the Euler-Bernoulli equation and its solution. The Euler-Bernoulli equation and its solution is extended and explained in several aspects in Section 3. Section 4 gives some concluding remarks.

2. The Source Form of the Euler-Bernoulli Equation

Any elastic body, when exposed to external forces, is deformed - it elongates, shortens, bends, twists, depending on the position and direction of forces acting on it. Only transversal deformations of the prismatic beam will be analyzed in detail in this paper.

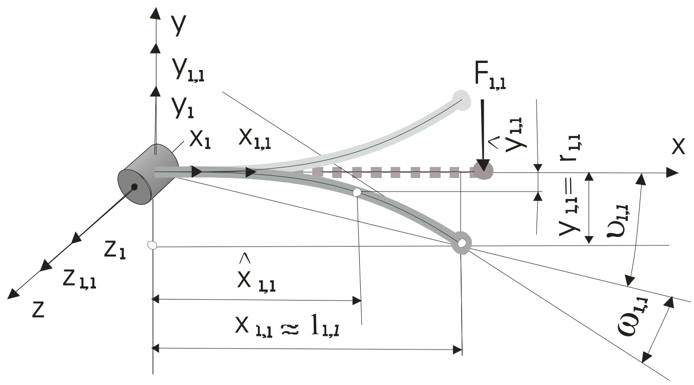


Fig. 1 Idealized motion of elastic body according to D. Bernoulli

When disturbance of the equilibrium state of an elastic body occurs, it causes motion, i.e. vibration of particles. The motion propagates through the body as a wave process whose characteristic is that in every place of the elastic medium exists the same disturbed state prevails in each place of the flexible medium, but with a delayed phase.

The Euler-Bernoulli equation of beam bending is of the following form:

$$\beta_{1,1} \cdot \frac{\partial^2 \hat{y}_{1,1}}{\partial \hat{x}_{1,1}^{\;2}} + \hat{m}_{el1,1} \cdot \frac{d^2 \hat{y}_{1,1}}{dt^2} \cdot (x_{1,1} - \hat{x}_{1,1}) = 0 \tag{1}$$

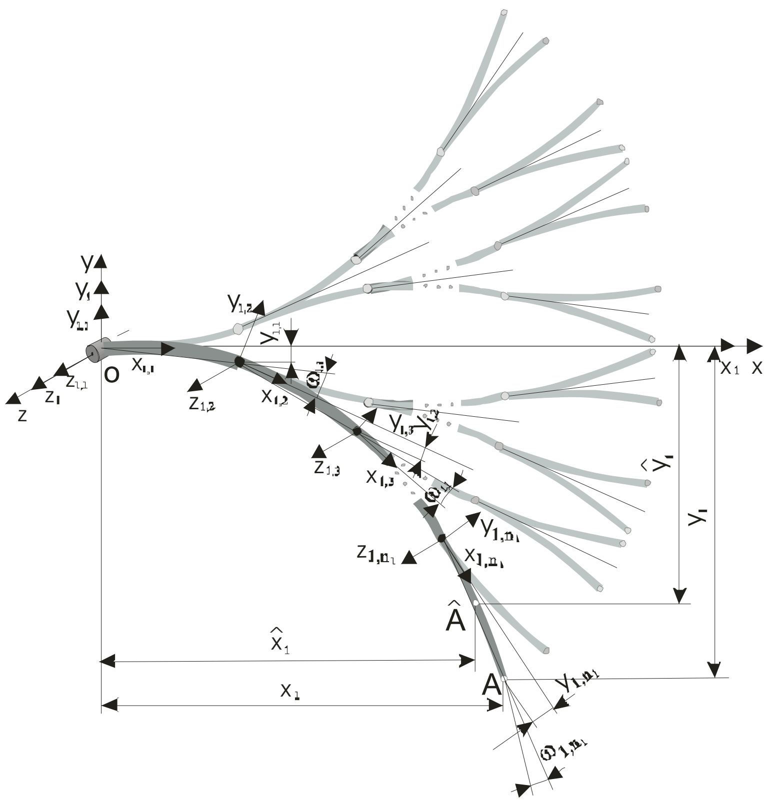


Fig. 2 Possible positions of the tip of elastic line with n_1 modes, simplified for $\overline{\theta}_1 = 0$

Where $\hat{M}_{1,1} = \hat{m}_{el1,1} \cdot \dfrac{d^2 \hat{y}_{1,1}}{dt^2} \cdot (x_{1,1} - \hat{x}_{1,1})\ [Nm]$ is the load moment, in these source equations encompassing only inertia, $\hat{\varepsilon}_{1,1} = \beta_{1,1} \cdot \dfrac{\partial^2 \hat{y}_{1,1}}{\partial \hat{x}_{1,1}^2} [Nm]$ is bending moment. $\beta_{1,1} = E_l \cdot I_{mom1,1} [Nm^2]$ is the flexural rigidity.

Equation (1) is defined under the assumption that the bending moment is opposed only by the proper inertial moment.

The solution of the Euler-Bernoulli equation original form (1) can be analyzed. A general solution of motion, i.e. the form of transversal oscillations of flexible beams can be found in the method of particular integrals of Bernoulli, that is:

$$\hat{y}_{to1,1}(\hat{x}_{1,1}, t) = \hat{X}_{1,1}(\hat{x}_{1,1}) \cdot \hat{T}_{to1,1}(t) \quad (2)$$

Besides, it is supposed that according to the definition, the motion in (1) is caused by an external force $F_{1,1}$, added and then removed with the solution (2) of Bernoulli and it satisfies these assumptions.

By superimposing the particular solutions (2), any transversal oscillation can be presented in the following form:

$$\hat{y}_{to1}(\hat{x}_{1,j}, t) = \sum_{j=1}^{\infty} \hat{X}_{1,j}(\hat{x}_{1,j}) \cdot \hat{T}_{to1,j}(t) \tag{3}$$

Equations (1-3) need a short explanation. The authors wrote equation (3) based on "vision", as they did not define mathematical model of a link with infinite number of modes, whose solution is equation (3). They left this task to their successors. Transversal oscillations defined by equation (3) describe the motion of elastic beam to which we assigned an infinite number of DOFs (modes), see Fig. 2, and which can be described by a mathematical model composed of an infinite number of equations, in the form:

$$\hat{M}_{1,j} + \hat{\varepsilon}_{1,j} = 0, \qquad j = 1, 2, ..., j, ...\infty \tag{4}$$

Dynamics of each mode is described by one equation. The equations in the model (4) are not of equal structure as our contemporaries, authors of numerous works presently interpret it. We think that the coupling between the modes involved leads to structural diversity among the equations in the model (4).

3. The New Form of the Euler-Bernoulli Equation

The point of application of forces displace in time, so these forces perform certain work on this path. Work is opposed by:

* the potential energy of the elastic body, which depends on the stiffness characteristic and flexure, and

* the dissipation energy of the elastic body, which depends on the damping characteristic and flexure change velocity.

The presence of dissipation energy is especially expressed in an oscillatory regime, while its presence is minimal in a stationary regime, when displacement velocity of the material particles of an elastic body with respect to an equilibrium position is minimal. To include the damping effect into analysis, source equation (1) should be extended as follows:

$$\hat{m}_{el1,1} \cdot \frac{d^2 \hat{y}_{1,1}}{dt^2} \cdot (x_{1,1} - \hat{x}_{1,1}) + \beta_{1,1} \cdot \frac{\partial^2 (\hat{y}_{1,1} + \eta_{1,1} \cdot \dot{\hat{y}}_{1,1})}{\partial \hat{x}_{1,1}^2} = 0 \tag{5}$$

$\eta_{1,1}$ is a factor characterizing the share of damping in the total elasticity characteristic.

The Bernoulli solution (1-4) describes only partially the nature of motion of real elastic beams. More precisely, it is only one component of motion. The original equations (1-4) should be expanded in order to be applicable in a broader analysis of elasticity. By supplementing these equations with the expressions that come out directly from the motion dynamics of elastic bodies, they become more complex.

The motion of the considered robotic system (or link with infinite number of modes presented in Fig. 2 is far more complex than the motion of the body presented in Fig. 1. This means that the equations that describe the robotic system (its elements) must also be more complex than the equations (1-4), formulated by Euler and Bernoulli. This fact is overlooked, and the original equations are widely used in the literature to describe the robotic system motion. This is far inadequate because valuable pieces of information about the complexity of the elastic robotic system motion are thus lost. Hence, it should be especially emphasized the necessity of expanding the source equations for the purpose of modeling robotic systems, and this should be done in the following way:

* based on the known laws of dynamics, equation (1) is to be supplemented by all the forces that participate in the formation of the bending moment of the considered mode.

* to define the form of elastic line of the considered robotic system it is necessary to expand the previously known solutions (3). Supplement it by adding stationary solution to the particular solution of D. Bernoulli, which is of oscillatory character. This means that the given solution depends directly on the overall system dynamics. As the link elastic line does not usually conform to the direction of the preset axes but extends in the space, we cannot define it by only one equation. General form of the elastic line is a direct outcome of the dynamics of system motion and cannot be represented by one equation but three equations are needed to define position and three equations to define orientation of each point on the elastic line. The equation of elastic line of the robotic system should also encompass the angles of motor shaft rotation $\overline{\theta}_i$ and the robot configuration.

The load moment is composed of all moment acting on the first mode of the link and these are perturbation moments, inertial moments (single and coupled moments), centrifugal, gravitational, Coriolis moments (single and coupled), coupled bending moments of the other modes, as well as the external force (which can be defined as static or dynamic force), which is *via* Jacobian matrix transferred to the motion of the first mode that come out directly from the motion dynamics of elastic bodies. They become more complex. This means that all these forces participate in generating of bending moment i.e. in forming elastic deformation as well as of the elasticity line of the first mode. In that case the model of elastic line of the first mode of the elastic link has the form of the Euler-Bernoulli equation:

$$\hat{H}_{1,j}\frac{d^2\hat{y}_{1,j}}{dt^2}+\hat{h}_{1,1}+j^T_{1,1}F_{uk}+z_{1,j}\cdot\varepsilon_1+\beta_{1,1}\cdot\frac{\partial^2(\hat{y}_{1,1}+\eta_{1,1}\cdot\dot{\hat{y}}_{1,1})}{\partial\hat{x}^2_{1,1}}=0 \tag{6}$$

The elastic line model of the first link that has n_1 modes is given in a matrix form by the following Euler-Bernoulli equation:

$$\hat{H}_1\cdot\frac{d^2\hat{y}_{1,j}}{dt^2}+\hat{h}_1+j^T_{e1}\cdot F_{uk}+z_1\cdot\varepsilon_1+\hat{\varepsilon}_1=0 \tag{7}$$

Researchers are especially interested in the motion of the first mode tip.

The motion equation of the forces involved at any point of the elastic line of first mode, including the point of the first mode tip, can be defined from the Euler-Bernoulli equation (6). The motion equation of all forces at the first mode tip for the given boundary conditions can be defined by the following equation:

$$H_{1,j}\frac{d^2 y_{1,j}}{dt^2} + h_{1,1} + j_{e1,1}{}^T \cdot F_{uk}{}^T + z_{1,j} \cdot \varepsilon_{1,j} + \varepsilon_{1,1} = 0 \Bigg| \begin{array}{l} \Sigma F = 0 (\Sigma M = 0) \\ \text{at the point of} \\ \text{first mode tip} \end{array} \tag{8}$$

Equation (8) is interesting because it allows one to calculate the position of the first mode tip. If we know the position of each mode tip we can always calculate the position of the link tip too and eventually the position of the robot tip.

Vector motion equation of all the forces at the tip of each mode of the first link can be defined from equation (7) for the preset boundary conditions:

$$H_1\frac{d^2 y_{1,j}}{dt^2} + h_1 + j_{e1}^T F_{uk} + z_1\varepsilon_1 + \varepsilon_1 = 0 \Bigg| \begin{array}{l} \Sigma F = 0 \\ \text{at the point of} \\ \text{each mode tip} \\ \text{of the first link} \end{array} \tag{9}$$

This equation should be supplemented by the mathematical model of motor written in following form:

$$\begin{array}{l} u_1 = R_1 \cdot i_1 + C_{E1} \cdot \dot{\overline{\theta}}_1 \\ C_{M1} \cdot i_1 = I_1 \cdot \ddot{\overline{\theta}}_1 + B_{u1} \cdot \dot{\overline{\theta}}_1 - S_1 \cdot (z_{m1,j} \cdot \varepsilon_1 + \varepsilon_{1,1}) \Bigg| \begin{array}{l} \Sigma M = 0 \\ \text{about the rotation axis} \\ \text{of the } \textit{first} \text{ motor} \end{array} \end{array} \tag{10}$$

Let us define it by setting for each motor the motion equation of all the moments acting about the rotation axis of the given motor. It has the form of the mathematical model of the motor of a rigid robotic system, but the difference being in that the moment of the *i*-th motor is not opposed by the mechanism moment (as with rigid robotic systems). The motor moment is opposed by the bending moment of the first elastic mode that comes after the motor, and also in part, by the bending moments of the other elastic modes that are connected in series after the given motor. All the modes after the motor, due to their position, influence the dynamics of motor motion. The overall order of the system (7) and (10) is (n_1+1).

Hence, it should be especially emphasized the necessity of expanding the source solution (3) with the stationary character of the elastic deformation caused by the forces involved.

By superposing the particular solution of oscillatory nature, and the stationary solution of forced nature, any flexible deformation of a considered mode may be presented in the following general form:

$$\hat{y}_{1,1} = \hat{X}_{1,1}(\hat{x}_{1,1}) \cdot (\hat{T}_{st1,1}(t) + \hat{T}_{to1,1}(t)) \tag{11}$$

$\hat{T}_{st1,1}$ is the stationary part of flexible deformation caused by stationary forces that vary continuously over time. $\hat{T}_{to1,1}$ is the oscillatory part of flexible deformation as in (2).

Component $\hat{X}_{1,1}(\hat{x}_{1,1})$ describes a possible geometrical relation between $\hat{y}_{1,1}$ and $\hat{x}_{1,1}$. Component $\hat{T}_{st1,1}+\hat{T}_{to1,1}$ describes the dependence of flexure $\hat{y}_{1,1}$ on flexibility force, which is the only time-varying quantity in expression (11). By superposing solutions (11), any flexible deformations of a flexible link with an infinite number of degrees of freedom may be presented in the following form:

$$\hat{y}_1(\hat{x}_{1,j},t) = R_1(\overline{\theta}_1,t) + \sum_{j=1}^{\infty} \hat{X}_{1,j}(\hat{x}_{1,j}) \cdot (\hat{T}_{st1,j}(t) + \hat{T}_{to1,j}(t)) \tag{12}$$

The equation of Bernoulli (12) (see Fig. 2) defines a geometrical position of any spot on the elastic body line $\hat{y}_1$ in direction y_1 - axis, and in a direction of x_1 - axis it would be a $\hat{x}_1$ coordinate which is also a geometrical size and it can be presented in an analogue way as well as the size $\hat{y}_1$.

$$\hat{x}_1(\hat{y}_{1,j},t) = N_1(\overline{\theta}_1,t) + \sum_{j=1}^{\infty} \hat{Y}_{1,j}(\hat{y}_{1,j}) \cdot (\hat{T}_{st1,j}(t) + \hat{T}_{to1,j}(t)) \tag{13}$$

Any form of elastic line and the pertinent transversal oscillations, as well as the motor motion, can be presented by equations (12) and (13). To this equation one should add also the equation defining the orientation of each point on the elastic line of the link.

$$\hat{\psi}_1(\hat{x}_{1,j},\hat{y}_{1,j},t) = K_1(\overline{\theta}_1,t) + \sum_{j=1}^{\infty} \Re_{1,j}(\hat{x}_{1,j},\hat{y}_{1,j}) \cdot (\hat{T}_{st1,j}(t) + \hat{T}_{to1,j}(t)) \tag{14}$$

The position and orientation of a tip of a presented body with indefinite number of modes is defined by coordinates x_1, y_1 and ψ_1 in x_1, y_1 level. It is supposed that all motions are made in $x_1 - y_1$ level, and a coordinate is $z_1 = 0$ in this case. Equations (12-14) are actually the solution of dynamics of the presented body's motion during the time. However, in order to calculate the coordinates $\hat{x}_1$, $\hat{y}_1$, $\hat{\psi}_1$ in some specific moment of time (as is seen from Fig. 2) , it is necessary (except from angles $\omega_{1,1}$, $\omega_{1,2}$, $\omega_{1,3}$, ... $\omega_{1,j}$) to know sizes of elastic deformations of all modes $y_{1,1}$, $y_{1,2}$, $y_{1,3}$, ... $y_{1,j}$ and $x_{1,1}$, $x_{1,2}$, $x_{1,3}$, $x_{1,j}$ defined in a space of local coordination system $x_{i,j}$, $y_{i,j}$, $z_{i,j}$. Generally, coordinates $\hat{x}_1$, $\hat{y}_1$ and $\hat{\psi}_1$ are the total of elastic deformations, but precisely, in geometrical terms, $\hat{x}_1$ and $\hat{y}_1$ are the total of projected elastic deformations on axes x_1, y_1 respectively.

Equation (3) has significance as elastic deformation for each mode for Meirovitch [1] and his followers [2-10], and in this way defined is entered in the total dynamic model.

The elastic deformation cannot be defined in advance (with both amplitude and frequency) and put in the system but completely inversely. The elastic deformation is a dynamic value which depends on the total dynamics of the robot system motions.

With new knowledge collected through generations, the intensive development of the new technical areas such as robotics especially strengthen by the development of the data computing process, demanded and enabled that elastic deformation was considered really as the dynamic value which depended on the system parameters. The elastic deformation is a dynamic value by both amplitude and frequency and it is the result of the total system motion's i.e. outer and inner, dynamic and static forces. Such elastic deformation should exist in the dynamics of the robot system motions. The synthesis of the robot system dynamics should be processed on the basis of the completely new different principles comparing to [1], with models based on the known, classic dynamics, the elasticity theory and the oscillation theory, where the elastic deformations are described as dynamic values of the inner and outer load which influence the total dynamics of the robot system motions.

That means that the elastic deformation amplitude and its frequency change depending on the forces (inertial forces, Coriolis, centrifugal forces, gravity forces as well as coupling forces between the present modes, and the play of the environment forces). It, of course, depends on the mechanism configuration, weight, length of the segments of the reference trajectory choice, dynamic characteristics of the motor motions etc.

In this paper, as explained above, equations (12-14) have completely new meaning. Equations (12-14) are solution of dynamic models (7) and (10), i.e. form of link elastic lines in space of Cartesian coordinates. It should be pointed out that the form of elastic line comes out directly from the dynamics of the system motion.

The robot motion is based on motor rotating angles, elastic deformation values and all other kinematics and dynamic robot mechanism characteristics (such as its geometry, configuration, weight disposal, motor characteristics, reference trajectory choice, as well as, many other important characteristics that influence the robot system motion dynamics). In robotics, this procedure is called the solution of "direct kinematics".

4. CONCLUSIONS

The paper describes the methods of expanding the Euler-Bernoulli equation from multiple points of view. Elastic deformation (moment of load) not only builds a perturbation and inertial moments, but there is the influence of gravitational, centrifugal, Coriolis torques (single and coupled), bending moments of other modes (which due to the coupling affect the motion of the considered mode), and moments that are caused by the action of external forces. Due to the strong coupling, there is a diversity in the structure of the extended form of the Euler-Bernoulli equation of each mode. Damping is an integral part of the characteristics of elasticity of real systems and is naturally included in the Euler-Bernoulli equation. All of these features and this whole discussion is not just related to the Euler-Bernoulli equation but also to motion equation for any point (and the top point) of the elastic line. This is the case because the motion equation follows directly from the Euler-Bernoulli equation defining boundary conditions.

It is concluded that the definition of kinematic models is of particular importance. A special attention is paid to a new interpretation of equation (3). In this paper, this is a solution to the Euler-Bernoulli equation that defines the form of the elastic line (position and orientation of every point in the space of the Cartesian coordinates) of the considered mode, segment and finally the robot.

It is pointed out that the elastic deformation is the consequence of the total robot system dynamics which is essentially different from the widely used method that implies the adaptation of the "assumed modes technique".

The dynamics of mechanism just over the sizes of elastic deformation is included into its definition, resulting from the dynamics of the motion of the mechanism.

Acknowledgment: *I am grateful to Prof. Dr Katica R. (Stevanović) Hedrih, manager director of the Department of Mechanics, from the Mathematical Institute SANU, Belgrade, Serbia, for helpful consultations during the implementation of this paper.*

REFERENCES

1. L. Meirovitch, *Analytical methods in vibrations*, Macmillan, New York, 1967.
2. J. Cheong, W.K. Chung and Y. Youm, Bandwidth modulation of rigid subsystem for the class of flexible robots, *Proc. Int. Conf. on Robotics and Automation, San Francisco, USA*, 2000, pp. 1478-1483.
3. A. De Luca and B. Siciliano, Closed-form dynamic model of planar multilink lightweight robots, *IEEE Transactions on Systems, Man, and Cybernetics*, **21** (4), 826-839 (1991).
4. S.E. Khadem and A.A. Pirmohammadi, Analytical development of dynamic equations of motion for a three-dimensional flexible link manipulator with revolute and prismatic joints, *IEEE Transactions on Systems, Man and Cybernetics, part B: Cybernetics*, **33** (2), 237-249 (2003).
5. H.K. Low, A systematic formulation of dynamic equations for robot manipulators with elastic links, *Journal of Robotic Systems*, **4** (3), 435-456 (1987).
6. H.K. Low and M. Vidyasagar, A lagrangian formulation of the dynamic model for flexible manipulator systems, *Transactions of ASME-Journal of Dynamic Systems, Measurement, and Control*, **110** (2), 175-181 (1988).
7. H.K. Low, Solution schemes for the system equations of flexible robots, *Journal of Robotics Systems*, **6** (4), 383-405 (1989).
8. F. Matsuno and T. Kanzawa, Robust control of coupled bending and torsional vibrations and contact force of a constrained flexible arm, *Proc. Int. Conf. on Robotics and Automation, Minneapolis, Minnesota*, 1996, pp. 2444-2449.
9. M. Moallem, K. Khorasani and V.R. Patel, Tip position tracking of flexible multi-link manipulators: an integral manifold approach, *Proc. Int. Conf. on Robotics and Automation*, *Minneapolis, Minnesota*, 1996, pp. 2432-2436.
10. D. Surdilović and M. Vukobratović, One method for efficient dynamic modeling of flexible manipulators, *Mechanism and Machine Theory,* **31** (3), 297-315 (1996).
11. M. Filipović and M. Vukobratović, Modeling of flexible robotic systems, *Proc. The Int. Conf. on "Computer as a Tool" (EUROCON 2005), Belgrade, Serbia and Montenegro,* 2005, pp. 1196-1199.
12. M. Filipović and M. Vukobratović, Contribution to modeling of elastic robotic systems, *Engineering and Automation Problems*, **5** (1), 22-35 (2006).
13. M. Filipović and M. Vukobratović, Complement of source equation of elastic line, *Journal of Intelligent and Robotic Systems*, **52** (2), 233-261 (2008).
14. M. Filipović and M. Vukobratović, Expansion of source equation of elastic line, *Robotica*, **26** (6), 739-751 (2008).
15. M. Filipović, New form of the Euler-Bernoulli rod equation applied to robotic systems, *Theoretical and Applied Mechanics*, **35** (4), 381-406 (2008).
16. M. Filipović and V. Kokotović, Contribution to new interpretation of the Euler-Bernoulli equation, *Facta Universitatis, Series: Automatic Control and Robotics*, **8** (1), 137-147 (2009).
17. W.J. Book, Recursive Lagrangian dynamics of flexible manipulator arms, *International Journal of Robotics Research,* **3** (3), 87-101 (1984).
18. W.J. Book, Analysis of massless elastic chains with servo controlled joints, *Transactions of ASME-Journal of Dynamic Systems, Measurement, and Control*, **101** (3), 187-192 (1979).
19. A.M. Djurić and W.H. ElMaraghy, Unified reconfigurable robots Jacobian, Proc. of the 2nd Int. Conf. on Changeable, Agile, Reconfigurable and Virtual Production, Toronto, Canada, 2007, pp. 811-823.
20. Ž. Despotović and Z. Stojiljković, Power converter control circuits for two-mass vibratory conveying system with electromagnetic drive: Simulations and experimental results, *IEEE Transactions on Industrial Electronics,* **54**(1), 453-466 (2007).

RAZLIKE IZMEĐU IZVORNE I NOVE FORME EULER-BERNOULLI JEDNAČINE KAO I NJENOG REŠENJA

Mirjana Filipović

Pri konstrukciji složenih robotskih i sličnih mehanizama, često se koriste laki materijali koje karakteriše elastičnost segmenata. Radi postizanja precizne i brze kontrole kretanja takvih mehanizama potrebno je da se sintetizuje njihov visoko verodostojni model. To bi bilo od velike važnosti za njegovu analizu i sintezu. Polazi se od prvih istraživanja u ovoj oblasti, a to su originalna forma Euler-Bernoulli jednačine kao i njeno rešenje, što pri poređenju sa savremenim znanjima nameće potrebu proširenja istih jednačina sa više stanovišta. Nova forma Euler-Bernoulli jednačine, a takođe i njeno rešenje, se bazira na savremenim znanjima iz robotike, odnosno na znanjima klasične mehanike. To je jedini način da se kompletno sačuva informacija o složenosti kinematike i dinamike kretanja elastičnog mehanizma.

Ključne reči: *Euler-Bernoulli jednačina***,** *elastičnost linka, sprezanje, kinematka, dinamika*