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Quantization and eigenvalue distribution of noncommutative scalar field theory

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Abstract.

The quantization of noncommutative scalar field theory is studied from the matrix model point of view, exhibiting the significance of the eigenvalue distribution. This provides a new framework to study renormalization, and predicts a phase transition in the noncommutative ϕ^4 model. In 4-dimensions, the corresponding critical line is found to terminate at a non-trivial point.

Key words: *Noncommutative field theory, phase transition, matrix model, renormalization*

1. INTRODUCTION: NC FIELD THEORY AND UV/IR MIXING

Noncommutative (NC) field theory has been studied intensively in recent years, see e.g. [1, 2] and references therein. Part of the motivation has been the hope that the UV divergences of Quantum Field Theory should be controlled and perhaps regularized by noncommutativity, which in turn should be related to quantum fluctuations of geometry and quantum gravity. In the simplest case of the quantum plane $\mathbb{R}^d_\theta$, the coordinate functions $x_i, i = 1, ..., d$ satisfiy the canonical commutation relations

$$[x_i, x_j] = i\theta_{ij}. \tag{1}$$

We assume for simplicity that θ_{ij} is nondegenerate and maximally-symmetric, characterized by a single parameter θ with dimension (length)^2. This implies

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uncertainty relations among the coordinates,

$$\Delta x_i \Delta x_j \geq \frac{1}{2}\theta_{ij} \approx \theta. \tag{2}$$

This leads to an energy scale

$$\Lambda_{NC} = \sqrt{\frac{1}{\theta}}, \tag{3}$$

and noncommutativity is expected to be important for energies beyond Λ_{NC}.

The most naive guess would be that Λ_{NC} plays the role of a UV cutoff in Quantum Field Theory. However, this turns out not to be the case. The quantization of NC field theory not only suffers from divergences in the UV, but also from new divergencies in the IR, which are in general not under control up to now. This type of behavior can to some extent be expected from (1). To understand this, expand a scalar field on $\mathbb{R}^d_\theta$ in a basis of plane waves, $\phi(x) = \int d^d p\, \phi(p) \exp(ipx)$, and consider the range of momenta p. Formally, $p \in \mathbb{R}^d$ has neither an IR nor a UV cutoff. As usual in field theory, it is necessary however to impose cutoffs $\Lambda_{IR} \leq p \leq \Lambda_{UV}$, with the hope that the dependence on the cutoffs can be removed in the end; this is essentially the defining property of a renormalizable field theory.

In view of (2), if Δx_i is minimized according to $\Delta x_i = \Lambda_{UV}^{-1}$, it implies that some $\Delta x_j \geq \theta \Lambda_{UV}$. This means that momenta near Λ_{UV} are inseparably linked with momenta in the IR, and suggests $\Lambda_{UV}\Lambda_{IR} = \Lambda_{NC}^2$. A consistent way to impose IR and UV cutoffs in NC field theory is to replace $\mathbb{R}^n_\theta$ by some compact NC spaces such as fuzzy tori, fuzzy spheres etc. The algebra of functions is then given by a finite matrix algebra $Mat(\mathcal{N} \times \mathcal{N}, \mathbb{C})$. One then finds quite generally

$$\Lambda_{IR} = \sqrt{\frac{1}{N\theta}} \quad \ll \quad \Lambda_{NC} = \sqrt{\frac{1}{\theta}} \quad \ll \quad \Lambda_{UV} = \sqrt{\frac{N}{\theta}} \tag{4}$$

where $\mathcal{N} \approx N^{d/2}$ (depending on the details of the compact NC space) is the dimension of the representation space. This shows that the scales in the UV and IR are intimately related, which is the origin of UV/IR mixing in NC field theory. It is then not too surprising that a straightforward Wilsonian renormalization scheme involving only Λ_{UV} will generically fail [3].

The remainder of this paper is a qualitative discussion of a new approach to quantization of euclidean NC scalar field theory, which was proposed in [4] with the goal to take into account more properly the specific features of NC field theory.

2. NONCOMMUTATIVE SCALAR FIELD THEORY

Consider the scalar ϕ^4 model on a suitable d-dimensional NC space,

$$S = \int d^d x \left(\frac{1}{2}\phi\Delta\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4 \right) = S_{kin} + \int d^d x\, V(\phi). \tag{5}$$

We assume that the algebra of functions on the NC space is represented as operator algebra on some *finite-dimensional* Hilbert space (this is the defining property of the so-called fuzzy spaces), so that $\phi \in Mat(\mathcal{N} \times \mathcal{N}, \mathbb{C})$ is a Hermitian matrix. The integral is then replaced by the appropriately normalized trace, $\int d^d x = (2\pi\theta)^{n/2} Tr()$.

In this finite setup, the quantization is defined by an integral over all Hermitian matrices, e.g.

$$\begin{aligned} Z &= \int [\mathcal{D}\phi] e^{-S}, & (6) \\ \langle \phi_{i_1 j_1} \cdots \phi_{i_n j_n} \rangle &= \frac{1}{Z} \int [\mathcal{D}\phi] e^{-S}\, \phi_{i_1 j_1} \cdots \phi_{i_n j_n} & (7) \end{aligned}$$

As a warm-up, we recall the analogous but simpler case of

Pure matrix models. These are defined by an action of the form

$$S^{M.M.} = \mathcal{N} Tr(V(\phi)) \tag{8}$$

for some polynomial $V(\phi)$, where again $\phi \in Mat(\mathcal{N} \times \mathcal{N}, \mathbb{C})$ is a Hermitian matrix. These pure matrix models can be solved exactly by the following change of coordinates in matrix space:

$$\phi = U^{-1} D U \tag{9}$$

where D is a diagonal matrix with real eigenvalues ϕ_i, and $U \in U(\mathcal{N})$. Using $[\mathcal{D}\phi] = \Delta^2(\phi_i) dU d\phi_1 ... d\phi_N$, where dU denotes the Haar measure for $U(\mathcal{N})$ and $\Delta(\phi_i) = \prod_{i<j}(\phi_i - \phi_j)$ is the Vandermonde-determinant. The integral over dU can be carried out trivially, reducing the path integral to an integral over eigenvalues,

$$Z = \int d\phi_i \Delta^2(\phi_i - \phi_j) \exp(-\mathcal{N} \sum_i V(\phi_i)) = \int d\phi_i \exp(-S_{eff}^{M.M}(\vec{\phi})) \tag{10}$$

with an "effective action" of the form

$$\exp(-S_{eff}^{M.M.}(\phi_i)) = \exp(\sum_{i \neq j} \log|\phi_i - \phi_j| - \mathcal{N} \sum_i V(\phi_i)). \tag{11}$$

We can assume that the eigenvalues are ordered, $\phi_1 \leq \phi_2 \leq ... \leq \phi_{\mathcal{N}}$, and denote with $\mathcal{E}_{\mathcal{N}}$ the space of ordered $\mathcal{N}$-tupels $\vec{\phi} = (\phi_1, ..., \phi_{\mathcal{N}})$. They will be interpreted as coordinates of a point in the space of eigenvalues $\mathcal{E}_{\mathcal{N}}$.

The remaining integral over ϕ_i can be evaluated with a variety of techniques such as orthogonal polynomials, Dyson- Schwinger resp. loop equations, or the saddle-point method. It turns out that the large $\mathcal{N}$ limit is correctly reproduced by the saddle-point method, because the Vandermone-determinant $\Delta^2(\phi_i) = \exp(\sum_{i\neq j} \ln|\phi_i - \phi_j|)$ corresponds to a strongly repulsive potential between the eigenvalues, which leads to a strong localization in $\mathcal{E}_{\mathcal{N}}$. This means that the effective action $\exp(-S_{eff}^{M.M.}(\vec{\phi}))$ is essentially a delta-function in $\mathcal{E}_{\mathcal{N}}$,

$$\exp(-S_{eff}^{M.M.}(\vec{\phi})) \approx \exp(-\mathcal{N} Tr(V(\vec{\phi}_0)))\, \delta(\vec{\phi} - \vec{\phi}_0). \tag{12}$$

One now proceeds to determine the localization of the maximum $\vec{\phi}_0 \in \mathcal{E}_{\mathcal{N}}$ by solving $\frac{\partial}{\partial \phi_i} S_{eff} = 0$, which can be written as an integral equation in the large $\mathcal{N}$ limit. Expectation values can then be computed as

$$\langle f(\vec{\phi})\rangle = f(\vec{\phi}_0) = f(\langle\vec{\phi}\rangle) \tag{13}$$

which is indeed correct in the large $\mathcal{N}$ limit of pure matrix models, consistent with (12). This "factorization" of expectation values is characteristic for a delta-function integral density, and can be used to test the localization hypothesis (12) resp. the sharpness of the (approximate) delta function.

Strategy for the full model. This analogy suggests to apply similar techniques also for the field theory (6). At first sight, this may appear impossible, because the action is no longer invariant under $U(\mathcal{N})$. However, let us assume that we could somehow evaluate the "angular integrals"

$$\int dU \exp(-S_{kin}(U^{-1}(\phi)U)) =: e^{-\mathcal{F}(\vec{\phi})}, \tag{14}$$

Note that the resulting $\mathcal{F}(\phi_i)$ is a totally symmetric, analytic function of the eigenvalues ϕ_i of the field ϕ, due to the integration over the unitary matrices U. Using this definition, the partition function can indeed be cast into the same form as (10),

$$Z = \int d\phi_i \Delta^2(\phi_i) \exp(-\mathcal{F}(\vec{\phi}) - (2\pi\theta)^{d/2}\, TrV(\vec{\phi})) = \int d\phi_i \exp(-S_{eff}(\vec{\phi})) \tag{15}$$

(with the obvious interpretation of $TrV(\vec{\phi})$), defining

$$\exp(-S_{eff}(\vec{\phi})) = \Delta^2(\phi_i) \exp(-\mathcal{F}(\vec{\phi}) - (2\pi\theta)^{d/2}\, TrV(\vec{\phi})). \tag{16}$$

Of course we are only allowed to determine observables depending on the eigenvalues with this effective action (the extension to other observables is discussed in [4]). These are determined by $S_{eff}(\vec{\phi})$ in the same way as for pure matrix models, since the degrees of freedom related to U are integrated out. This demonstrates that all thermodynamic properties of the quantum field theory, in particular the phase transitions, are determined by $S_{eff}(\vec{\phi})$ and the resulting eigenvalue distribution in the large $\mathcal{N}$ limit. The advantage of this formulation is that it is very well suited to include interactions, and naturally extends to the non-perturbative domain.

So far, all this is exact but difficult to apply, since we will not be able to evaluate $\mathcal{F}(\vec{\phi})$ exactly. However, the crucial point in (16) is the presence of the Vandermonde-determinant $\Delta^2(\phi_i) = \exp(\sum_{i\neq j} \ln|\phi_i - \phi_j|)$, which prevents the eigenvalues from coinciding and strongly localized them. This effect cannot be canceled by $\mathcal{F}(\vec{\phi})$, because it ts analytic. This and the analogous exact analysis of the pure matrix models suggests the following *central hypothesis* of our approach: (16) is strongly localized, and the essential features of the full QFT are reproduced by the following approximation

$$\exp(-S_{eff}(\vec{\phi})) \approx \exp(-\mathcal{F}(\vec{\phi}_0) - (2\pi\theta)^{d/2}\, TrV(\vec{\phi}_0))\, \delta(\vec{\phi} - \vec{\phi}_0), \tag{17}$$

where $\vec{\phi}_0$ denotes the maximum (saddle-point) of $S_{eff}(\vec{\phi})$. The remaining integral over the eigenvalues ϕ_i can hence be evaluated by the saddle-point method. This will be justified to some extent and made more precise below, by studying $\mathcal{F}(\vec{\phi})$ using the free case. If $\mathcal{F}(\vec{\phi})$ is known, the free energy can be found by determining the minimum ϕ_0 of $S_{eff}(\vec{\phi})$. However, (17) strongly suggests that also correlation functions can be computed by restricting the full matrix integral to the dominant eigenvalue distribution $\vec{\phi}_0$. This would provide nonperturbative control over the full NC QFT.

It is important to keep in mind that a saddle-point approximation *before* integrating out $U(\mathcal{N})$ would be complete nonsense. It is only appropriate for the effective action (17). The possibility of separating the path integral into these 2 steps is only available for NC field theory.

3. THE CASE OF FREE FIELDS

For free fields, we can verify the validity of the basic hypothesis (17), and compute the unknown function $\mathcal{F}(\vec{\phi})$ explicitly at least in some domain. This will then be applied in the interacting case.

To check localization in the sense of (17), we need to show that the expectation values of all observables which depend only on the eigenvalues $\vec{\phi}$ factorize, and can be computed by evaluating the observable at a specific point $\vec{\phi}_0$. A complete set of such observables is given by the product of traces of various powers of ϕ, which in field theory language is $\langle (\int d^dx\phi(x)^{2n_1})...(\int d^dx\phi(x)^{2n_k})\rangle$. It is shown in [4] using Wicks theorem that in the large N resp. Λ_{UV} limit,

$$\begin{aligned}\frac{\langle (\frac{1}{V}\int d^dx\phi(x)^{2n_1})(\frac{1}{V}\int d^dx\phi(x)^{2n_k})\rangle}{\langle \frac{1}{V}\int d^dx\phi(x)^2\rangle^{n_1}\langle \frac{1}{V}\int d^dx\phi(x)^2\rangle^{n_k}} &= \frac{\langle \frac{1}{V}\int d^dx\phi(x)^{2n_1}\rangle}{\langle \frac{1}{V}\int d^dx\phi(x)^2\rangle^{n_1}}\ \frac{\langle \frac{1}{V}\int d^dx\phi(x)^{2n_k}\rangle}{\langle \frac{1}{V}\int d^dx\phi(x)^2\rangle^{n_k}} \\ &= N_{Planar}(2n_1)\ ...\ N_{Planar}(2n_k). \quad (18)\end{aligned}$$

Here $N_{Planar}(2n)$ is the number of planar contractions of a vertex with $2n$ legs. The non-planar contributions always involve oscillatory integrals, and do not contribute to the above ratio in the large N limit.

This is exactly the desired factorization property, which implies that the effective action of the eigenvalue sector localizes as in (17), after suitable rescaling. This is only true for noncommutative field theories, where non-planar (completely contracted) diagrams are suppressed by their oscillating behavior, so that only the planar diagrams contribute[2].

To complete the analysis of the free case, we need to find the dominant eigenvalue distribution $\vec{\phi}_0$. That is an easy task using the known techniques from pure matrix models described above. Writing $\varphi(s) = (\phi_0)_j, \quad s = \frac{j}{N}$ for $s \in [0,1]$ in the continuum limit, the saddle-point $\vec{\phi}_0$ then corresponds to a density of eigenvalues $\rho(\varphi) = \frac{ds}{d\varphi}$ with $\int_{-\infty}^{\infty}\rho(p)dp = 1$. This turns out to be the famous Wigner semi-circle law

$$\rho(p) = \left\{ \begin{array}{ll} \frac{2}{\pi}\sqrt{1-p^2} & p^2 < 1 \\ 0, & \text{otherwise.} \end{array} \right. \quad (19)$$

This is the same as for the pure Gaussian matrix model, and it follows that the effective action for the eigenvalue sector is given by

$$S_{eff}^{free}(\vec{\phi}) = f_0(m) - \sum_{i\neq j}\ln|\phi_i - \phi_j| + \frac{2\mathcal{N}}{\alpha_0^2(m)}(\sum \phi_i^2). \quad (20)$$

[2] this is only true for observables of the above type, not e.g. for propagators

Here $f_0(m)$ is some numerical function of m which is not important here, and

$$\alpha_0^2(m) = \begin{cases} \frac{1}{4\pi^2}\,\Lambda_{UV}^2 \left(1 - \frac{m^2}{\Lambda_{UV}^2}\ln(1+\frac{\Lambda_{UV}^2}{m^2})\right), & d=4 \\ \frac{1}{\pi}\ln(1+\frac{\Lambda_{UV}^2}{m^2}), & d=2 \end{cases} \tag{21}$$

depending on the dimension of the field theory.

3. APPLICATION: THE ϕ^4 MODEL

We can now apply the result (20) to the interacting case. Consider for example the ϕ^4 model (5), which is obtained from the free case by adding a potential of the form

$$S_{int}(\phi) = \frac{\lambda}{4}\int d^dx\phi^4(x) = \frac{\lambda}{4}(2\pi\theta)^{d/2}Tr\phi^4,$$

The goal is to derive some properties (in particular thermodynamical, but also others) of the interacting model. The basic hypothesis is again (17), where the eigenvalue distribution $\vec{\phi}_0$ in the interacting case is to be determined.

It is quite easy to understand the main effect of the interaction term using the above results (20): since S_{int} only depends on the eigenvalues, it seems very natural to simply add S_{int} to (20), suggesting

$$Z_{int} = \int d\phi_i e^{-S_{eff}^{free}(\vec{\phi})-S_{int}(\vec{\phi})}. \tag{22}$$

To justify this, we can expand the interaction term into a power series in λ. Using the factorization property (18), we can immediately compute any expectation value of observables which depend only on the eigenvalues, and sum up the expansion in λ. As in the case of pure matrix models [6], the effect of this perturbative computation is reproduced for $\mathcal{N}\to\infty$ by the second (nonperturbative) point of view, which is that (22) is localized at a new eigenvalue distribution $\vec{\phi}_0^\lambda$, which now depends on the interaction. It can again be found by the saddle-point approximation.

We conclude with some remarks and results of this analysis [4]:

1. A caveat: the validity of (20) has been established only "locally", i.e. the function $\mathcal{F}(\vec{\phi})$ is really known only for $\vec{\phi}\approx\vec{\phi}_0$. The only free parameter available is the mass[3], which enters in (20) through α_0 and

[3]this can be enhanced by adding other terms such as the one in [5], which will be pursued elsewhere

allows to test a one-dimensional submanifold of $\mathcal{E}_{\mathcal{N}}$. Therefore we can trust (20) only if $\vec{\phi}_0^\lambda \approx \vec{\phi}_0$. This leads to the second observation:

2. The physical properties of the interacting model will be close to those of the free case only if $\vec{\phi}_0^\lambda \approx \vec{\phi}_0$. In particular, the correlation length (i.e. physical mass) of the interacting model will be reproduced best by the free action whose EV distribution ϕ_0 is closest to $\vec{\phi}_0^\lambda$. Working this out [4] leads to a very simple understanding of mass renormalization. Namely, it turns out that the bare mass must be adjusted to

   $$m^2 = m_{phys}^2 - \frac{3}{16\pi^2}\,\Lambda_{UV}^2 \left(1 - \frac{m_{phys}^2}{\Lambda_{UV}^2}\,\ln(\frac{\Lambda_{UV}^2}{m_{phys}^2})\right)\,\lambda. \tag{23}$$

   in the 4-dimensional case. This reproduces precisely the result of a conventional one-loop computation, however it is now based on a non-perturbative analysis and not just a formal expansion.

3. Stretching somewhat the range where (20) has been tested, one can study the thermodynamical properties and phase transitions of the interacting model. Indeed a phase transition is found at the point where the eigenvalue distribution $\vec{\phi}_0^\lambda$ breaks up into 2 disjoint pieces ("2 cuts"), due to the interaction term. This is expected to be the transition between the "striped" and disordered phase found in NC scalar field theory [7, 8]. In the most interesting case of 4 dimension, a critical line is found which ends at a nontrivial critical point $\lambda_c > 0$ [4]. This is strongly suggestive for a *nontrivial* NC ϕ^4 model in 4 dimensions, since the endpoint of the critical line $\lambda_c > 0$ should correspond to a fixed-point under a suitable RG flow.

   Since the mechanism of the phase transition is very generic and does not depend on the details of the potential, the qualitative features of this result are expected to be correct.

Full details can be found in [4].

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References

[1] M. R. Douglas and N. A. Nekrasov, "Noncommutative field theory," *Rev.Mod.Phys.* **73** (2001) 977; [hep-th/0106048]

[2] R. Szabo, "Quantum Field Theory on Noncommutative Spaces". *Phys.Rept.* **378** (2003) 207-299; [hep-th/0109162]

[3] S. Minwalla, M. Van Raamsdonk, N. Seiberg

"Noncommutative Perturbative Dynamics".

JHEP **0002** (2000) 020; [hep-th/9912072]

[4] H. Steinacker, "A non-perturbative approach to non-commutative scalar field theory". *JHEP***0503** (2005) 075; [hep-th/0501174]

[5] H. Grosse, R. Wulkenhaar, "Renormalisation of ϕ^4-theory on noncommutative R^4 in the matrix base". [hep-th/0401128]

[6] E. Brezin, C. Itzykson, G. Parisi and J. B. Zuber, "Planar Diagrams," *Commun. Math. Phys.* **59** (1978) 35.

[7] S. Gubser, S. Sondhi, "Phase structure of non-commutative scalar field theories". *Nucl.Phys.***B605** (2001) 395-424; [hep-th/0006119]

[8] W. Bietenholz, F. Hofheinz, J. Nishimura, "Phase diagram and dispersion relation of the non-commutative $\lambda\phi^4$ model in d=3". *JHEP* **0406** (2004) 042; X. Martin, "A matrix phase for the ϕ^4 scalar field on the fuzzy sphere". *JHEP* **0404** (2004) 077