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Noncommutavity in the presence of the dilaton field

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Abstract. *The open bosonic string is placed in gravitational $G_{\mu\nu}$, antisymmetric $B_{\mu\nu}$ and linear dilaton field $\Phi = \Phi_0 + a_\mu x^\mu$. We investigate here the contribution of the background fields to the noncommutativity parameter in two cases: $a^2 \neq 0$ and $a^2 = 0$. We consider the boundary conditions like constraints and use the canonical method. The case $a^2 \neq 0$ is equivalent to the dilaton free case formally. In the second case a first class constraint appears and it generates a local Weyl symmetry.*

Key words: *Open string, Boundary conditions, Noncommutativity*

1. INTRODUCTION

At the beginning we define the model introducing the action. The exact solution exists in the case when $G_{\mu\nu}$ and $B_{\mu\nu}$ are constant fields and the dilaton field is linear. The crucial technical point is extension of the space time i.e. adding of the conformal part of the intrinsic metric F to the coordinate x^μ. In this way we transform the starting action into the form of the dilaton free case. The vector a_μ can be either light cone or not.

The case when the dilaton field gradient is not light cone vector is the subject of the next section. It is almost like dilaton free case. There is one commutative coordinate and that is $x = a_\mu x^\mu$.

In the second case it turns out there is a first class constraint(FCC) in the theory. According to the Dirac theory of the constrained systems, the FCC generates the local symmetry. We recognize easily the two dimensional Weyl symmetry. The action is the same as dilaton free one formally, but expressed in terms of the gauge invariant variables.

At the end we gave some concluding remarks.

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2. STRING THEORY WITH THE DILATON FIELD

2.1. Definition of the model

Let us consider the open string action given by the following expression:

$$S = \kappa \int_\Sigma d^2\xi \sqrt{-g} \left\{ \left[\frac{1}{2} g^{\alpha\beta} G_{\mu\nu} + \frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}} B_{\mu\nu} \right] \partial_\alpha x^\mu \partial_\beta x^\nu + \Phi R^{(2)} \right\} \qquad (1)$$

where $G_{\mu\nu}(x)$ is the space-time metric, $B_{\mu\nu}(x)$ is antisymmetric tensor field, $\Phi(x)$ is dilaton field and $R^{(2)}$ is scalar curvature of the two dimensional world-sheet. Here, $\xi^\alpha (\alpha = 0, 1)$ are coordinates of the world-sheet Σ and x^μ $(\mu = 0, 1, 2, ..., D-1)$ are coordinates of the space-time $\mathcal{M}$. Metric $g_{\alpha\beta}$ is *intrinsic* world-sheet metric . We use the following notation in the whole article: $\partial_\alpha = \frac{\partial}{\partial \xi^\alpha}$, $\partial_\mu = \frac{\partial}{\partial x^\mu}$, $\xi^0 = \tau$ and $\xi^1 = \sigma$ $(\sigma \in [0, \pi])$.

We use the conformal gauge $g_{\alpha\beta} = e^{2F} \eta_{\alpha\beta}$. For simplicity, we split the space-time coordinates x^μ in the Dp-brane part x^i $(i = 0, 1, ..., p)$ and the rest x^a $(a = p+1, p+2, ..., D-1)$. The background fields are chosen in the following way:

$$B_{\mu\nu} \to B_{ij} \qquad a_\mu \to a_i \qquad (2)$$

$$G_{\mu\nu} = 0 \quad \mu = i \in \{0, 1, .., p\} \quad \nu = a \in \{p+1, ..., D-1\} \qquad (3)$$

where $a_\mu = \partial_\mu \Phi$.

If we aplly these conventions we split the action (1) in the free theory action (x^a directions) and the action (4) interesting for our further analysis

$$S_2 = \kappa \int_\Sigma d^2\xi \left[\left(\frac{1}{2} \eta^{\alpha\beta} G_{ij} + \epsilon^{\alpha\beta} B_{ij} \right) \partial_\alpha x^i \partial_\beta x^j + 2\eta^{\alpha\beta} a_i \partial_\alpha x^i \partial_\beta F \right]. \qquad (4)$$

2.2. Solution of the space-time equations

The condition for preserving the Weyl symmetry on the quantum level represents the next set of conditions

$$\beta^G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_\nu{}^{\rho\sigma} + 2D_\mu a_\nu = 0 \qquad (5)$$

$$\beta^B_{\mu\nu} \equiv D_\rho B^\rho_{\mu\nu} - 2a_\rho B^\rho_{\mu\nu} = 0 \qquad (6)$$

$$\beta^\Phi \equiv 4\pi\kappa \frac{D-26}{3} - R + \frac{1}{12} B_{\mu\rho\sigma} B^{\mu\rho\sigma} - 4D_\mu a^\mu + 4a^2 = 0 \qquad (7)$$

$a_\mu = \partial_\mu \Phi$, $B_{\mu\rho\sigma}$ is field strength of the field $B_{\mu\nu}$.

When $G_{\mu\nu}(x)$ and $B_{\mu\nu}(x)$ are constant fields and dilaton field is linear $\Phi(x) = \Phi_0 + a_\mu x^\mu$, there is an exact solution and that is $a^2 = \kappa\pi \frac{26-D}{3}$.

2.3. Extending of the space-time

We introduce the coordinates $y^A = (x^i, F)$ and the fields

$$G_{AB} = \begin{pmatrix} G_{ij} & 2a_i \\ 2a_j & 0 \end{pmatrix}, \qquad B_{AB} = \begin{pmatrix} B_{ij} & 0 \\ 0 & 0 \end{pmatrix}. \tag{8}$$

The action (4) gets the form of the dilaton free action:

$$S_2 = \kappa \int_\Sigma d^2\xi \left[\frac{1}{2}\eta^{\alpha\beta} G_{AB} + \varepsilon^{\alpha\beta} B_{AB}\right] \partial_\alpha y^A \partial_\beta y^B \tag{9}$$

3. CASE $a^2 \neq 0$

3.1. Basic quntities

When $a^2 \neq 0$, there is an inverse matrix of G_{AB}

$$G^{AB} = (G^{-1})^{AB} = \begin{pmatrix} P_T^{ij} & \frac{a^i}{2a^2} \\ \frac{a^j}{2a^2} & -\frac{1}{4a^2} \end{pmatrix}, \tag{10}$$

where $P^T_{ij} = G_{ij} - \frac{a_i a_j}{a^2}$.

It is usuful for practical reasons to introduce the quantity $\Pi_{\pm AB} = B_{AB} \pm \frac{1}{2} G_{AB}$. The effective open string metric is defined:

$$(\Pi_\pm \Pi_\mp)_{AB} = -\frac{1}{4} G^{eff}_{AB} = -\frac{1}{4} \begin{pmatrix} \tilde{G}_{ij} & 2a_i \\ 2a_j & 0 \end{pmatrix}, \tag{11}$$

where $\tilde{G}_{ij} = (G - 4BP^T B)_{ij}$. Inverse of the G^{eff}_{AB} is equal to

$$G^{AB}_{eff} = (G^{-1}_{eff})^{AB} = \begin{pmatrix} \tilde{P}_T^{ij} & \frac{\tilde{a}^i}{2\tilde{a}^2} \\ \frac{\tilde{a}^j}{2\tilde{a}^2} & -\frac{1}{4\tilde{a}^2} \end{pmatrix}, \tag{12}$$

where $\tilde{P}_T^{ij} = \tilde{G}^{ij} - \frac{\tilde{a}^i \tilde{a}^j}{\tilde{a}^2}$, $\tilde{a}^2 = \tilde{G}^{ij} a_i a_j$ and $\tilde{a}^i = \tilde{G}^{ij} a_j$.

3.2. Canonical analysis

The momenta are defined in a standard way

$$\pi_A = \frac{\partial \mathcal{L}}{\partial(\partial_\tau y^A)} = \kappa(G_{AB}\dot{y}^B - 2B_{AB} y^{B\prime}). \tag{13}$$

According to the definition, canonical hamiltonian is equal to

$$H_c = \int_0^\pi d\sigma \mathcal{H}_c = \int_0^\pi d\sigma (\pi_A \dot{y}^A - \mathcal{L}), \tag{14}$$

where $\mathcal{H}_c = T_- - T_+$, $T_\pm = \mp\frac{1}{4\kappa} G_{AB} j_\pm^A j_\pm^B$ and $j_{\pm A} = \pi_A + 2\kappa \Pi_{\pm AB} y^{B\prime}$.

The algebra of the currents $j_{\pm A}$ can be calculated easily ant it holds the following

$$\{j_{\pm A}, j_{\pm B}\} = \pm 2\kappa G_{AB} \delta'(\sigma - \overline{\sigma}) \,, \tag{15}$$

$$\{j_{\pm A}, j_{\mp B}\} = 0 \,. \tag{16}$$

3.3. Boundary conditions like constraints of the theory

If we vary the action(1) we obtain the Euler-Lagrange equations of motion and boundary conditions. When the equations of motion are obeyed, it holds

$$\delta S = \int_{\tau_i}^{\tau_f} \gamma_A^{(0)} \delta y^A \Big|_0^\pi \,, \tag{17}$$

where $\gamma_A^{(0)} = \frac{\partial \mathcal{L}}{\partial_\sigma y^A} = \Pi_{-AB} j_+^B + \Pi_{+AB} j_-^B$. We can obtain the same result using the hamiltonian approach. If we demand the differentiability of Hamiltonian we get the regular terms(proportianal to the δy^A and $\delta \pi_A$) and one extra term which is the same like boundary term obtained in Lagrangian approach.

There are a few choices of boundary conditions. We choose the Neuman boundary conditions i.e. $\gamma_A^{(0)} = 0$ at the string endpoints. The boundary conditions are considered like constraints here.

3.4. Consistency conditions

Boundary conditions are considered like constraints and we must examine the conistency conditions. We obtain

$$\gamma_A^{(n)} = \left\{ H_c, \gamma_A^{(n-1)} \right\} = \Pi_{-AB} (-1)^n \partial_\sigma^n j_+^B + \Pi_{+AB} \partial_\sigma^n j_-^B \,. \tag{18}$$

Using Taylor expansion, the infinite set of consistency conditions turns into one condition. At the point $\sigma = 0$, we have:

$$\Gamma_A(\sigma) = \sum_{n\geq 0} \frac{\sigma^n}{n!} \gamma_A^{(n)}(\sigma = 0) = \Pi_{+AB} j_-^B(\sigma) + \Pi_{-AB} j_+^B(-\sigma) \,. \tag{19}$$

Similarly, at the point $\sigma = \pi$:

$$\tilde{\Gamma}_A(\sigma) = \sum_{n\geq 0} \frac{(\sigma - \pi)^n}{n!} \gamma_A^{(n)}(\pi) = \Pi_{+AB} j_-^B(\sigma) + \Pi_{-AB} j_+^B(2\pi - \sigma) \,. \tag{20}$$

The Poisson bracket between the canonical hamiltonian and quantity Γ_A is equal to σ derivative of Γ_A, so there are no more constraints in the theory. From the constraints' algebra

$$\{\Gamma_A(\sigma), \Gamma_B(\overline{\sigma})\} = -\kappa G^{eff}_{AB}\delta'(\sigma - \overline{\sigma})\,, \tag{21}$$

we conclude all constraints are of the second class(assumption $\tilde{a}^2 \neq 0$).

3.5. Solution of the boundary conditions

For some field W we can define the symmetric and antisymmetric part under transformation $(\sigma \to -\sigma)$

$$w(\sigma) = \frac{1}{2}\left[W(\sigma) + W(-\sigma)\right] \qquad\qquad \bar{w}(\sigma) = \frac{1}{2}\left[W(\sigma) - W(-\sigma)\right]\,. \tag{22}$$

On this way we define the symmetric and antisymmetric parts of the coordinates $(q^A, \bar{q}^A)$ and momenta $(p_A, \bar{p}_A)$. Using the definition of the current $j_{\pm A}$ (14), boundary conditions (19) and new coordinates and momenta, we can rewrite the boundary conditions in terms of the new variables. We solve the boundary conditions equalizing separatly the symmetric and antisymmetric part with zero. The final result is

$$\pi_A = p_A\,, \quad y^A(\sigma) = q^A - 2\Theta^{AB}\int^{\sigma} d\sigma_1 p_B\,, \tag{23}$$

where $\Theta^{AB} = -\frac{1}{\kappa}(G^{-1}_{eff}BG^{-1})^{AB}$.

3.6. Noncommutativity

Using the algebra of the "old" canonical variables, we can calculate the algebra of the "new" variables:

$$\{q^A(\tau,\sigma), p_B(\tau,\overline{\sigma})\} = \delta^A{}_B\delta_s(\sigma,\overline{\sigma})\,, \tag{24}$$

where $\delta_s(\sigma,\overline{\sigma}) = \frac{1}{2}[\delta(\sigma - \overline{\sigma}) + \delta(\sigma + \overline{\sigma})]$.
From the (24) and (23), we obtain the final result:

$$\{y^A(\tau,\sigma), y^B(\tau,\overline{\sigma})\} = 2\Theta^{AB}\theta(\sigma + \overline{\sigma}) \tag{25}$$

where the function $\theta(x)$ is defined:

$$\theta(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1/2 & \text{if } 0 < x < 2\pi \\ 1 & \text{if } x = 2\pi \end{cases} \tag{26}$$

In component form we have:

$$\{x^i(\sigma), x^j(\bar{\sigma})\} = 2\Theta^{ij}\theta(\sigma+\bar{\sigma})\,, \tag{27}$$

$$\{x^i(\sigma), F(\bar{\sigma})\} = 2\Theta^{i}\theta(\sigma+\bar{\sigma})\,, \tag{28}$$

$$\{F(\sigma), F(\bar{\sigma})\} = 0\,. \tag{29}$$

where $\Theta^{ij} = -\frac{1}{\kappa}(\tilde{P}^T B P^T)^{ij}$ and $\Theta^i = \frac{(aB\tilde{G}^{-1})^i}{2\kappa a^2} = \frac{(\tilde{a}B)^i}{2\kappa\tilde{a}^2}$.
It is not so difficult to see that:

$$a_i\Theta^{ij} = 0 \quad , \quad a_i\Theta^i = 0 \tag{30}$$

Because of that, we have one commutative coordinate $x = a_i x^i$.

4. CASE $a^2 = 0$

Because of $detG_{AB} = -4a^2 detG_{ij}$, metric of the extended space-time is singular. If we use the definition of the momenta $\pi_A = (\pi_i, \pi_F)$ (13) and currents, we obtain that, for $a^2 = 0$, the following identity holds

$$j = a^i\pi_i - \frac{1}{2}\pi_F + 2\kappa a^i B_{ij}x'^j = a^i j_{\pm i} - \frac{1}{2}j_{\pm F} = 0\,. \tag{31}$$

This is a primary constraint. We can prove by definition that the components of the canonical hamiltonian are $T_\pm = \mp\frac{1}{4\kappa}G^{ij}j_{\pm i}j_{\pm j}$. The total hamiltonian is

$$\mathcal{H}_T = \mathcal{H}_c + \lambda j\,, \tag{32}$$

where λ is an Lagrange multiplier. Using the currents' algebra (15) and the total hamiltonian (32), we can conclude easily that j is a first class constraint. As a consequence of that, it generates a simmetry. The local symmetry generator and its action on variables are defined like

$$\delta_\eta X = \{X, G\}\,, \qquad G \equiv \int d\sigma\eta(\sigma)j(\sigma)\,, \tag{33}$$

where $\eta(\sigma)$ is a parameter of the gauge transformation. If we act with G on $z^i = x^i + 2a^iF$ and $P_i = \pi_i + 4\kappa F' a^j B_{ji}$, we obtain they are gauge invariant. Poisson bracket of these variables tells us they are canonically conjugated.

The current $j_{\pm i}$ can be rewritten in the form of the current $j_{\pm A}$ but in terms of the gauge invariant variables:

$$j_{\pm i} = \pi_i + 2\kappa\Pi_{\pm ij}x'^j \pm 2\kappa a_i F' = P_i + 2\kappa\Pi_{\pm ij}z'^j \,. \tag{34}$$

We know $T_\pm = \mp\frac{1}{4\kappa}G^{ij}j_{\pm i}j_{\pm j}$. So, hamiltonian is gauge invariant. On the other side, if we implement the transformation on the action (4), we obtain it is gauge invariant too. Formally, the noncommutativity relation is the same as in dilaton free case

$$\left\{z^i(\tau,\sigma), z^j(\tau,\overline{\sigma})\right\} = 2\Theta^{ij}\theta(\sigma+\overline{\sigma})\,, \tag{35}$$

where $\Theta^{ij} = -\frac{1}{\kappa}(G^{-1}_{eff}BG^{-1})^{ij}$ and the function $\theta(x)$ is defined in Eq.(26). The number of the noncommutative coordinates remains the same as in free dilaton case because the constraint removed one degree of freedom.

5. CONCLUDING REMARKS

We have considerd here the contribution of the linear dilaton field to the noncommutativity parameter. In the specific background, extending the space-time, we transformed the starting action into the form of the dilaton free action. But, two cases appeared- $a^2 \neq 0$ and $a^2 = 0$.

The first case, in extended space-time, is the same as dilaton free case. But, when we split the extended space-time in x and F part, we conclude that *number of noncommutative coordinates are the same as before.* We have one commutative Dp-brane coordinate in the direction of the dilaton gradient a_i.

The second case is much more interesting. Metric of the extended space-time is not invertible. That is a sign that we have a symmetry in the theory cancelling one degree of freedom. After short canonical analysis we found that theory has one first class constraint. It is a generator of the symmetry and responsible to "put things in order". Action and Hamiltonian are expressed in terms of the new, gauge invariant variables and have the form as in dilaton free case. Consequently, the noncommutativity relation remains the same but it holds for the gauge invariant variables and the number of the noncommutative coordinates is unchanged.

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