FACTA UNIVERSITATIS
Series: **Physics, Chemistry and Technology** Vol. 4, N^o 2, 2006, pp 371 -379

Deformed Diffeomorphisms and Gravity

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Abstract. *The concepts necessary for an algebraic construction of a gravity theory on noncommutative spaces are introduced. The θ-deformed diffeomorphisms are studied and a tensor calculus is defined. This leads to a deformed Einstein-Hilbert action which is invariant with respect to deformed diffeomorphisms. This contribution is based on joint work with P. Aschieri, C. Blohmann, M. Dimitrijević, P. Schupp and J. Wess.*

Key words: *Noncommutative Geometry, Deformed Diffeomorphisms, Deformed Gravity*

1. Noncommutative Spaces

It is expected that in order to obtain a better understanding of physics at short distances and in order to cure the problems occuring when trying to quantize gravity one has to change the nature of space-time in a fundamental way. One way to do so is to implement noncommutativity by taking coordinates which satisfy the commutation relations

$$[\hat{x}^\mu, \hat{x}^\nu] = C^{\mu\nu}(\hat{x}) \neq 0\,. \tag{1}$$

The function $C^{\mu\nu}(\hat{x})$ is unknown. For physical reasons it should be a function that vanishes at large distances where we experience the commutative world and may be determined by experiments [1]. We denote the algebra generated by noncommutative coordinates $\hat{x}^\mu$ which are subject to the relations (1) by $\hat{\mathcal{A}}$ (*algebra of noncommutative functions*). In what follows we will exclusively consider the θ-deformed case which may at very short distances provide a reasonable approximation for $C^{\mu\nu}(\hat{x})$

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} = \text{const.} \tag{2}$$

Received: 24 October 2005

but we note that the algebraic construction presented here can be generalized to more complicated noncommutative structures of the above type which possess the Poincaré-Birkhoff-Witt (PBW) property.

2. Symmetries on Deformed Spaces

In general the commutation relations (1) are not covariant with respect to undeformed symmetries. For example the canonical commutation relations (2) break Lorentz symmetry if we assume that the noncommutativity parameters $\theta^{\mu\nu}$ do not transform.

The question arises whether we can *deform* the symmetry in such a way that it acts consistently on the deformed space (i.e. leaves the deformed space invariant) and such that it reduces to the undeformed symmetry in the commutative limit. The answer is yes: Lie algebras can be deformed in the category of Hopf algebras (Hopf algebras coming from a Lie algebra are also called Quantum Groups)[2]. Quantum group symmetries lead to new features of field theories on noncommutative spaces. Because of its simplicity, θ-deformed spaces are very well-suited to study those.

In the following we will construct explicitly a θ-deformed version of diffeomorphisms which consistently act on the noncommutative space (2). Then we present a gravity theory which is invariant with respect to this deformed diffeomorphisms [2, 3, 4].

3. Diffeomorphisms

Diffeomorphisms are generated by vector-fields ξ. Acting on functions, vector-fields are represented as linear differential operators $\xi = \xi^\mu \partial_\mu$. Vector-fields form a Lie algebra Ξ with the Lie bracket given by

$$[\xi, \eta] = \xi \times \eta$$

where $\xi \times \eta$ is defined by its action on functions

$$(\xi \times \eta)(f) = (\xi^\mu(\partial_\mu \eta^\nu)\partial_\nu - \eta^\mu(\partial_\mu \xi^\nu)\partial_\nu)(f).$$

The Lie algebra of *infinitesimal diffeomorphisms* Ξ can be embedded into its universal enveloping algebra which we want to denote by $\mathcal{U}(\Xi)$. The universal enveloping algebra is an associative algebra and possesses a natural Hopf

[2]To be more precise the universal enveloping algebra of a Lie algebra can be deformed. The universal enveloping algebra of any Lie algebra is a Hopf algebra and this gives rise to deformations in the category of Hopf algebras.

algebra structure. The coproduct is defined as follows on the generators[3]:

$$\begin{aligned}\Delta : \mathcal{U}(\Xi) &\rightarrow \mathcal{U}(\Xi) \otimes \mathcal{U}(\Xi)\\ \Xi \ni \xi &\mapsto \Delta(\xi) := \xi \otimes 1 + 1 \otimes \xi.\end{aligned} \tag{3}$$

For a precise definition and more details on Hopf algebras we refer the reader to text books [5]. For our purposes it shall be sufficient to note that the coproduct implements how the Hopf algebra acts on a product in a representation algebra (Leibniz-rule). Scalar fields are defined by their transformation property with respect to infinitesimal coordinate transformations:

$$\delta_\xi \phi = -\xi\phi = -\xi^\mu(\partial_\mu \phi). \tag{4}$$

The product of two scalar fields is transformed using the Leibniz-rule

$$\delta_\xi(\phi\psi) = (\delta_\xi\phi)\psi + \phi(\delta_\xi\psi) = -\xi^\mu(\partial_\mu \phi\psi) \tag{5}$$

such that the product of two scalar fields transforms again as a scalar.

Similarly one studies tensor representations of $\mathcal{U}(\Xi)$. For example vector fields are introduced by the transformation property

$$\begin{aligned}\delta_\xi V_\alpha &= -\xi^\mu(\partial_\mu V_\alpha) - (\partial_\alpha \xi^\mu)V_\mu\\ \delta_\xi V^\alpha &= -\xi^\mu(\partial_\mu V^\alpha) + (\partial_\mu \xi^\alpha)V^\mu.\end{aligned}$$

The generalization to arbitrary tensor fields is straight forward:

$$\begin{aligned}\delta_\xi T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n} &= -\xi^\mu(\partial_\mu T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}) + (\partial_\mu \xi^{\mu_1})T^{\mu\cdots\mu_n}_{\nu_1\cdots\nu_n} + \cdots + (\partial_\mu\xi^{\mu_n})T^{\mu_1\cdots\mu}_{\nu_1\cdots\nu_n}\\ &\quad -(\partial_{\nu_1}\xi^\nu)T^{\mu_1\cdots\mu_n}_{\nu\cdots\nu_n} - \cdots - (\partial_{\nu_n}\xi^\nu)T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu}.\end{aligned}$$

As for scalar fields, we also find that the product of two tensors transforms like a tensor. Summarizing, we have seen that scalar fields, vector fields and tensor fields are representations of the Hopf algebra $\mathcal{U}(\Xi)$, the universal enveloping algebra of infinitesimal diffeomorphisms. The Hopf algebra $\mathcal{U}(\Xi)$ acts via *infinitesimal coordinate transformations* δ_ξ which are subject to the relations:

$$[\delta_\xi, \delta_\eta] = \delta_{\xi\times\eta} \tag{6}$$

$$\Delta\delta_\xi = \delta_\xi \otimes 1 + 1 \otimes \delta_\xi \quad . \tag{7}$$

[3]The structure maps are defined on the generators $\xi \in \Xi$ and the universal property of the universal enveloping algebra $\mathcal{U}(\Xi)$ assures that they can be uniquely extended as algebra homomorphisms (respectively anti-algebra homomorphism in case of the antipode S) to the whole algebra $\mathcal{U}(\Xi)$.

The transformation operator δ_ξ is explicitly given by differential operators which depend on the representation under consideration. In case of scalar fields this differential operator is given by $-\xi^\mu\partial_\mu$.

4. Deformed Diffeomorphisms

The concepts introduced in the previous subsection can be deformed in order to establish a consistent tensor calculus on the noncommutative space-time algebra (2). In this context it is necessary to account the full Hopf algebra structure of the universal enveloping algebra $\mathcal{U}(\Xi)$.

In our setting the algebra $\hat{\mathcal{A}}$ possesses a noncommutative product defined by

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}. \tag{8}$$

We want to deform the structure maps (7) of the Hopf algebra $\mathcal{U}(\Xi)$ in such a way that the resulting deformed Hopf algebra which we denote by $\mathcal{U}(\hat{\Xi})$ consistently acts on $\hat{\mathcal{A}}$. Let $\mathcal{U}(\hat{\Xi})$ be generated as algebra by elements $\hat{\delta}_\xi$, $\xi \in \Xi$. We leave the algebra relation undeformed

$$[\hat{\delta}_\xi, \hat{\delta}_\eta] = \hat{\delta}_{\xi\times\eta} \tag{9}$$

but we deform the co-sector

$$\Delta\hat{\delta}_\xi = e^{-\frac{i}{2}h\theta^{\rho\sigma}\hat{\partial}_\rho\otimes\hat{\partial}_\sigma}(\hat{\delta}_\xi\otimes 1 + 1\otimes\hat{\delta}_\xi)e^{\frac{i}{2}h\theta^{\rho\sigma}\hat{\partial}_\rho\otimes\hat{\partial}_\sigma}, \tag{10}$$

where $[\hat{\partial}_\rho, \hat{\delta}_\xi] = \hat{\delta}_{(\partial_\rho\xi)}$. The deformed coproduct (10) reduces to the undeformed one (7) in the limit $\theta \to 0$. We have to check whether the above deformation is a good one in the sense that it leads to a consistent action on $\hat{\mathcal{A}}$. First we need a differential operator acting on fields in $\hat{\mathcal{A}}$ which represents the algebra (9). Let us consider the differential operator

$$\hat{X}_\xi := \sum_{n=0}^{\infty}\frac{1}{n!}(-\frac{i}{2})^n\theta^{\rho_1\sigma_1}\cdots\theta^{\rho_n\sigma_n}(\hat{\partial}_{\rho_1}\cdots\hat{\partial}_{\rho_n}\hat{\xi}^\mu)\hat{\partial}_\mu\hat{\partial}_{\sigma_1}\cdots\hat{\partial}_{\sigma_n}. \tag{11}$$

Then indeed we have

$$[\hat{X}_\xi, \hat{X}_\eta] = \hat{X}_{\xi\times\eta}. \tag{12}$$

It is therefore reasonable to introduce scalar fields $\hat{\phi} \in \hat{\mathcal{A}}$ by the transformation property

$$\hat{\delta}_\xi\hat{\phi} = -(\hat{X}_\xi\hat{\phi}).$$

The next step is to work out the action of the differential operators $\hat{X}_\xi$ on the product of two fields. A calculation [2] shows that

$$(\hat{X}_\xi(\hat{\phi}\hat{\psi})) = \mu \circ (e^{-\frac{i}{2}h\theta^{\rho\sigma}\hat{\partial}_\rho \otimes \hat{\partial}_\sigma}(\hat{X}_\xi \otimes 1 + 1 \otimes \hat{X}_\xi)e^{\frac{i}{2}h\theta^{\rho\sigma}\hat{\partial}_\rho \otimes \hat{\partial}_\sigma}\hat{\phi} \otimes \hat{\psi}).$$

This means that the differential operators $\hat{X}_\xi$ act via a *deformed Leibniz rule* on the product of two fields. Comparing with (10) we see that the deformed Leibniz rule of the differential operator $\hat{X}_\xi$ is exactly the one induced by the deformed coproduct (10):

$$\hat{\delta}_\xi(\hat{\phi}\hat{\psi}) = e^{-\frac{i}{2}h\theta^{\rho\sigma}\hat{\partial}_\rho \otimes \hat{\partial}_\sigma}(\hat{\delta}_\xi \otimes 1 + 1 \otimes \hat{\delta}_\xi)e^{\frac{i}{2}h\theta^{\rho\sigma}\hat{\partial}_\rho \otimes \hat{\partial}_\sigma}(\hat{\phi}\hat{\psi}) = -\hat{X}_\xi \triangleright (\hat{\phi}\hat{\psi}).$$

Hence, the deformed Hopf algebra $\mathcal{U}(\hat{\Xi})$ is indeed represented on scalar fields $\hat{\phi} \in \hat{\mathcal{A}}$ by the differential operator $\hat{X}_\xi$. The scalar fields form a $\mathcal{U}(\hat{\Xi})$-module algebra.

In analogy to the previous section we can introduce vector and tensor fields as representations of the Hopf algebra $\mathcal{U}(\hat{\Xi})$. The transformation property for an arbitrary tensor reads

$$\begin{aligned}\hat{\delta}_\xi \hat{T}^{\mu_1\cdots\mu_r}_{\nu_1\cdots\nu_s} \quad = \quad & -(\hat{X}_\xi \hat{T}^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}) + (\hat{X}_{(\partial_\mu \xi^{\mu_1})}\hat{T}^{\mu\cdots\mu_n}_{\nu_1\cdots\nu_n}) + \cdots + (\hat{X}_{(\partial_\mu \xi^{\mu_n})}\hat{T}^{\mu_1\cdots\mu}_{\nu_1\cdots\nu_n}) \\ & -(\hat{X}_{(\partial_{\nu_1}\xi^\nu)}\hat{T}^{\mu_1\cdots\mu_n}_{\nu\cdots\nu_n}) - \cdots - (\hat{X}_{(\partial_{\nu_n}\xi^\nu)}\hat{T}^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu}).\end{aligned}$$

Up to now we have seen the following:

- Diffeomorphisms are generated by vector-fields $\xi \in \Xi$ and the universal enveloping algebra $\mathcal{U}(\Xi)$ of the Lie algebra Ξ of vector-fields possesses a natural Hopf algebra structure defined by (7).
- The algebra of scalar fields $\phi \in \mathcal{A}$ is a $\mathcal{U}(\Xi)$-module algebra.
- The universal enveloping algebra $\mathcal{U}(\Xi)$ can be deformed to a Hopf algebra $\mathcal{U}(\hat{\Xi})$ defined in (9,10).
- $\mathcal{U}(\hat{\Xi})$ consistently acts on the algebra of noncommutative functions $\hat{\mathcal{A}}$, i.e. the algebra of noncommutative functions is a $\mathcal{U}(\hat{\Xi})$-module algebra.
- Regarding $\mathcal{U}(\hat{\Xi})$ as the underlying "symmetry" of the gravity theory to be built on the noncommutative space $\hat{\mathcal{A}}$, we established a full tensor calculus as representations of the Hopf algebra $\mathcal{U}(\hat{\Xi})$.

5. Noncommutative Geometry

The *covariant derivative* $\hat{D}_\mu$ can algebraically be defined by demanding that acting on a vector-field it produces a tensor-field

$$\hat{\delta}_\xi \hat{D}_\mu \hat{V}_\nu \stackrel{!}{=} -(\hat{X}_\xi \hat{D}_\mu \hat{V}_\nu) - (\hat{X}_{(\partial_\mu \xi^\alpha)} \hat{D}_\alpha \hat{V}_\nu) - (\hat{X}_{(\partial_\nu \xi^\alpha)} \hat{D}_\mu \hat{V}_\alpha) \tag{13}$$

The covariant derivative is given by a *connection* $\hat{\Gamma}_{\mu\nu}{}^\rho$

$$\hat{D}_\mu \hat{V}_\nu = \hat{\partial}_\mu \hat{V}_\nu - \hat{\Gamma}_{\mu\nu}{}^\rho \hat{V}_\rho.$$

From (13) it is possible to deduce the transformation property of $\hat{\Gamma}_{\mu\nu}{}^\rho$

$$\hat{\delta}_\xi \hat{\Gamma}_{\mu\nu}{}^\rho = (\hat{X}_\xi \hat{\Gamma}_{\mu\nu}{}^\rho) - (\hat{X}_{(\partial_\mu \xi^\alpha)} \hat{\Gamma}_{\alpha\nu}{}^\rho) - (\hat{X}_{(\partial_\nu \xi^\alpha)} \hat{\Gamma}_{\mu\alpha}{}^\rho) + (\hat{X}_{(\partial_\alpha \xi^\rho)} \hat{\Gamma}_{\mu\nu}{}^\alpha) - (\hat{\partial}_\mu \hat{\partial}_\nu \hat{\xi}^\rho).$$

The *metric* $\hat{G}_{\mu\nu}$ is defined as a symmetric tensor of rank two. It can be obtained for example by a set of vector-fields $\hat{E}_\mu{}^a$, $a = 0, \ldots, 3$, where a is to be understood as a mere label. These vector-fields are called *vierbeins.* Then the symmetrized product of those vector-fields is indeed a symmetric tensor of rank two

$$\hat{G}_{\mu\nu} := \frac{1}{2}(\hat{E}_\mu{}^a \hat{E}_\nu{}^b + \hat{E}_\nu{}^b \hat{E}_\mu{}^a)\eta_{ab}.$$

Here η_{ab} stands for the usual flat Minkowski space metric. Let us assume that we can choose the vierbeins $\hat{E}_\mu{}^a$ such that they reduce in the commutative limit to the usual vierbeins $e_\mu{}^a$. Then also the metric $\hat{G}_{\mu\nu}$ reduces to the usual, undeformed metric $g_{\mu\nu}$.

The inverse metric tensor we denote by upper indices

$$\hat{G}_{\mu\nu} \hat{G}^{\nu\rho} = \delta_\mu^\rho.$$

We use $\hat{G}_{\mu\nu}$ respectively $\hat{G}^{\mu\nu}$ to raise and lower indices.

The curvature and torsion tensors are obtained by taking the commutator of two covariant derivatives[4]

$$[\hat{D}_\mu, \hat{D}_\nu]\hat{V}_\rho = \hat{R}_{\mu\nu\rho}{}^\alpha \hat{V}_\alpha + \hat{T}_{\mu\nu}{}^\alpha \hat{D}_\alpha \hat{V}_\rho$$

which leads to the expressions

$$\begin{aligned} \hat{R}_{\mu\nu\rho}{}^\sigma &= \hat{\partial}_\nu \hat{\Gamma}_{\mu\rho}{}^\sigma - \hat{\partial}_\mu \hat{\Gamma}_{\nu\rho}{}^\sigma + \hat{\Gamma}_{\nu\rho}{}^\beta \hat{\Gamma}_{\mu\beta}{}^\sigma - \hat{\Gamma}_{\mu\rho}{}^\beta \hat{\Gamma}_{\nu\beta}{}^\sigma \\ \hat{T}_{\mu\nu}{}^\alpha &= \hat{\Gamma}_{\nu\mu}{}^\alpha - \hat{\Gamma}_{\nu\mu}{}^\alpha. \end{aligned}$$

[4]The generalization of covariant derivatives acting on tensors is straight forward [2].

If we assume the *torsion-free* case, i.e.

$$\hat{\Gamma}_{\mu\nu}{}^{\sigma} = \hat{\Gamma}_{\nu\mu}{}^{\sigma},$$

we find an unique expression for the metric connection (Christoffel symbol) defined by

$$\hat{D}_\alpha \hat{G}_{\beta\gamma} \stackrel{!}{=} 0$$

in terms of the metric and its inverse[5]

$$\hat{\Gamma}_{\alpha\beta}{}^{\sigma} = \frac{1}{2}(\hat{\partial}_\alpha \hat{G}_{\beta\gamma} + \hat{\partial}_\beta \hat{G}_{\alpha\gamma} - \hat{\partial}_\gamma \hat{G}_{\alpha\beta})\hat{G}^{\gamma\sigma}.$$

From the curvature tensor $\hat{R}_{\mu\nu\rho}{}^{\sigma}$ we get the curvature scalar by contracting the indices

$$\hat{R} := \hat{G}^{\mu\nu}\hat{R}_{\nu\mu\rho}{}^{\rho}.$$

$\hat{R}$ indeed transforms as a scalar which may be checked explicitly by taking the deformed coproduct (10) into account.

To obtain an integral which is invariant with respect to the Hopf algebra of deformed infinitesimal diffeomorphisms we need a measure function $\hat{E}$. We demand the transformation property

$$\hat{\delta}_\xi \hat{E} = -\hat{X}_\xi \hat{E} - \hat{X}_{(\partial_\mu \xi^\mu)}\hat{E}. \tag{14}$$

Then it follows with the deformed coproduct (10) that for any scalar field $\hat{S}$

$$\hat{\delta}_\xi \hat{E}\hat{S} = -\hat{\partial}_\mu(\hat{X}_{\xi^\mu}(\hat{E}\hat{S})).$$

Hence, transforming the product of an arbitrary scalar field with a measure function $\hat{E}$ we obtain a total derivative which vanishes under the integral. A suitable measure function with the desired transformation property (14) is for instance given by the determinant of the vierbein $\hat{E}_\mu{}^a$

$$\hat{E} = \det(\hat{\mathrm{E}}_\mu{}^{\mathrm{a}}) := \frac{1}{4!}\varepsilon^{\mu_1\cdots\mu_4}\varepsilon_{\mathrm{a}_1\cdots\mathrm{a}_4}\hat{\mathrm{E}}_{\mu_1}{}^{\mathrm{a}_1}\hat{\mathrm{E}}_{\mu_2}{}^{\mathrm{a}_2}\hat{\mathrm{E}}_{\mu_3}{}^{\mathrm{a}_3}\hat{\mathrm{E}}_{\mu_4}{}^{\mathrm{a}_4}.$$

Now we have all ingredients to write down the *Einstein-Hilbert action* on $\hat{\mathcal{A}}$ as

$$\hat{S}_{\mathrm{EH}} := \int \det(\hat{\mathrm{E}}_\mu{}^{\mathrm{a}})\hat{\mathrm{R}} + \text{complex conj.}.$$

[5]We don't introduce a new symbol for the metric connection.

It is by construction invariant with respect to deformed diffeomorphisms meaning that

$$\hat{\delta}_\xi \hat{S}_{\mathrm{EH}} = 0.$$

In this section we have presented the fundamentals of a noncommutative geometry on the algebra $\hat{\mathcal{A}}$ and defined an invariant Einstein-Hilbert action. It is a deformation of the usual Einstein-Hilbert action. Using the star-product formalism it is possible to map the algebraic quantities to functions depending on commutative variables. Then it is possible to study explicitly deviations of the undeformed theory in orders of a deformation parameter [4, 2]. Very interesting is also to study a generalization of the above concepts to a more general class of noncommutative structures given by a twist [3]. This class contains in particular lattice-like spacetime algebras which may indeed provide a regularization of the field theory under consideration.

Acknowledgments The results presented in this talk were obtained together with P. Aschieri, C. Blohmann, M. Dimitrijević, P. Schupp and J. Wess and I would like to thank them for an excellent and fruitful collaboration.

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