

Quantum Hall states from the matrix oscillator point of view

S.Meljanac¹, A. Samsarov²

¹ Rudjer Boskovic Institute, Bijenicka c.54, HR-10002 Zagreb, Croatia

² Rudjer Boskovic Institute, Bijenicka c.54, HR-10002 Zagreb, Croatia

Abstract. *A quantum matrix oscillator model is shown to reproduce the quantum Hall states in a direct way and is intimately related to the regularized matrix model introduced by Polychronakos. By transferring the consideration to the Bargmann representation with the help of a particular similarity transformation, we show that the quantum matrix oscillator describes the quantum mechanics of electrons in the lowest Landau level with the ground state described by the Laughlin-type wave function.*

Key words: *Matrix oscillator, Bargmann representation, Laughlin states*

1. INTRODUCTION

The investigation of the quantum dynamics of nonrelativistic electrons in a uniform magnetic field has a far reaching application to the physics of the quantum Hall effect. The physics of electrons in the lowest Landau level exhibits some interesting features, the example of which is the occurrence of the incompressible fluidlike [1] states of condensed electrons whose excitations have fractional charge and obey fractional statistics [2], [3]. These states appear only when the electron densities are certain rational fractions of the density corresponding to a fully filled lowest Landau level and the gap in their excitation spectrum gives rise to the experimentally observed fractional quantum Hall effect. They are described by the Laughlin wave functions [4]. The tools for studying the exactness and universality of the Laughlin wave functions are offered in a natural way in the realm of matrix models [5].

Received: 20 August 2005

Recently, an attempt was made by Susskind [6] to describe the incompressible quantum Hall fluid in terms of the noncommutative Chern-Simons theory on the plane. As the Chern-Simons theory on the plane necessarily describes a spatially infinite quantum Hall system, it was also of interest to find a description of finite systems with a finite number of electrons and this was achieved by the model introduced by Polychronakos [7]. Such a regularized model, proposed as a theory of finite matrices with additional boundary vector fields, has provided a description of the quantum Hall droplet and its boundary excitations [8]. The quasiparticle and quasihole states were explained in terms of Schur functions within an algebraic approach [9].

The finite matrix Chern-Simons model is described by two matrices X_1 , X_2 or A , A^\dagger . It was shown [9] that both these matrices could not be diagonal simultaneously with some operators on the diagonal. This would lead to inconsistencies and to only two towers of states of the Bose and Fermi type, respectively. There was also a problem with the construction of the general Laughlin states [10]. However, the strong connection of the matrix Chern-Simons model with the Calogero model and the quantum Hall effect was pointed out in [7-10].

Recently, a quantum matrix oscillator was proposed and its equivalence to the Calogero-type models was established [11], [12]. The classical version of the matrix oscillator was introduced in [13] and the path integral quantization of this model was performed in [14].

In this work we propose a quantum matrix oscillator and establish its connection to the finite matrix Chern-Simons model introduced by Polychronakos. We use the matrix oscillator model [11] to find the physical states of electrons in the lowest Landau level. The ground states are Laughlin-type states and the analysis leading to this result, together with the construction of the excited states, relies heavily on the consideration that is carried out in the Bargmann representation. The main point here is to reduce the eigenvalue problem to a much simpler one and then to transfer the obtained results back to the original problem, with the help of a conveniently constructed similarity transformation. Although the analysis is performed for the one-dimensional case only, it can as well be straightforwardly extended to two and higher dimensions as long as identical particles are considered. As a consequence, the results obtained can be analytically continued onto the whole complex plane incorporating in such a way the wave functions of the true Laughlin form that depend on complex variables.

2. MATRIX OSCILLATOR, ACTION AND QUANTIZATION

Let us construct an action for the matrix oscillator described by $N \times N$ matrices X, \mathcal{P} with operator-valued matrix elements, $(X_{ij})^\dagger = X_{ji}$, $(\mathcal{P}_{ij})^\dagger = \mathcal{P}_{ji}$; $i, j = 1, 2, \dots, N$. We take the matrix X to be diagonal, with real elements. The Hamiltonian and commutation relations [11] are then ($\hbar = 1$)

$$H = R\left(\frac{1}{2m}\mathcal{P}^2 + \frac{1}{2}m\omega^2 X^2\right)C, \quad (1)$$

$$[X, \mathcal{P}] = i\mathcal{V}, \quad \mathcal{V} = (1 - \nu)\mathbf{1} + \nu\mathcal{J}, \quad (2)$$

where $R = (1, \dots, 1)$ is a row-vector whose all components are units, and $C = R^T$ is a transpose of R . Also, we have $RC = N$ and $CR = \mathcal{J}$, where \mathcal{J} is the $N \times N$ matrix with units at all positions. The matrix \mathcal{V} is symmetric, $\mathcal{V}^T = \mathcal{V}$, where $\nu > -\frac{1}{N}$ is a real parameter and m is the mass. Generally, \mathcal{V} is a Hermitian matrix $\mathcal{V}^\dagger = \mathcal{V}$, with $\nu_{ii} = 1$ and $\nu_{ij}^* = \nu_{ji}$, $\forall i, j$, and the effective Hamiltonian contains three-body interactions [12].

In order to describe two-dimensional systems of N charged particles with charge e in a magnetic field B , it is convenient to define the matrix $X_1 \equiv X$ and a second matrix X_2 expressed in terms of \mathcal{P} as

$$X_2 = -\frac{1}{eB}\mathcal{P} = -\frac{1}{m\omega}\mathcal{P}, \quad (3)$$

where $\omega = \frac{eB}{m}$. Note that the trace $Tr[X_1, X_2]$ is equal to $\frac{N}{ieB}$, in accordance with the relation (2).

The coordinates of the electrons can be globally parametrized in a fuzzy way by introducing two $N \times N$ Hermitian matrices X_a ; $a = 1, 2$. The action leading to the quantum matrix oscillator is then given by the regularized finite matrix Chern-Simons model introduced by Polychronakos

$$S_M = \frac{eB}{2} \int dt Tr[\varepsilon_{ab} X_a (\dot{X}_b - i[A_0, X_b]) + 2\theta A_0] - \frac{\omega eBN}{2\bar{\psi}\psi} \int dt \bar{\psi} X_a X_a \psi - \int dt \bar{\psi} (i\partial_t + A_0) \psi, \quad (4)$$

where $eB\theta = k$, A_0 is a matrix entering into the above action only linearly and ψ ($\bar{\psi} = \psi^{*T}$) is a boundary vector field. The action (4) is invariant under the transformations $X_a \rightarrow UX_a U^{-1}$, $\psi \rightarrow U\psi$, $\bar{\psi} \rightarrow \bar{\psi}U^{-1}$, $A_0 \rightarrow$

$UA_0U^{-1} + iU\partial_tU^{-1}$, where U is a unitary matrix, $U \in U(N)$. The term with ω serves as a potential box that keeps particles near the origin and also provides a Hamiltonian for the theory that chooses a unique ground state, while the last term in the action can be interpreted as a boundary term. Also, note that the minor change is made in the harmonic term in respect to the action of Ref. [7], namely $Tr(X_a)^2$ is replaced by $\bar{\psi}(X_a)^2\psi$. But, as these two parts yield the same spectrum when acting on the singlet sector of the $U(N)$ group, this replacement essentially does not make any difference.

The variation of the action S_M in the field variable A_0 gives the equation of motion for the time component A_0 of the gauge field. This equation has the form

$$ieB[X_1, X_2] + k\mathbf{1} - \psi\bar{\psi} = 0 \quad (5)$$

and can be interpreted as the Gauss law. Now we recognize the Gauss law (5) as a quantization condition imposed on the matrices X_1 and X_2 , after which their matrix elements become operators [11]. As an additional point, we require that one of the X_a matrices, say X_1 , can be diagonalized. From Eq. (5) it follows $Tr[X_1, X_2] = \frac{Tr\psi\bar{\psi} - Nk}{ieB} = \frac{N}{ieB}$, in agreement with Eqs. (2),(3). This means that $Tr\psi\bar{\psi} = N(k+1)$.

At this point it is important to note that certain quantization constraints ($k \in \mathbb{Z}$) can be imposed on the parameter k and these may be justified by some group theoretic arguments [15]. So, in further considerations, k will be an integer. The obviously redundant number of degrees of freedom is reduced to effectively $2N$ phase space variables with the help of the Gauss law constraint (5) and $U(N)$ gauge symmetry. At the beginning we had $2N^2$ degrees of freedom and $2N$ components of the boundary complex vector. After diagonalizing X_1 , and solving the Gauss constraint, we are left with $2N$ degrees of freedom, corresponding to N electrons.

In the action (4) we have introduced a quadratic potential $\frac{N}{2i\psi\bar{\psi}}m\omega^2\bar{\psi}(X_a)^2\psi$ which, after the diagonalization of the matrix X_1 , becomes equal to the quantum matrix oscillator Hamiltonian (1). More explicitly, the unitary transformation U which diagonalizes the matrix X_1 , $UX_1U^\dagger = X'_1 = \text{diag}(x_1, \dots, x_N)$, will change the vector ψ into $\phi = U\psi$ and the matrix X_2 into X'_2 , so that, after solving the quantization constraint (5), it can be represented [11] with the following operator-valued elements:

$$-ieB(X'_2)_{ij} = \left(\frac{\partial}{\partial x_i} + \sum_{k \neq i} \frac{\lambda_{ik}}{x_i - x_k} \right) \delta_{ij} - \frac{1 - \delta_{ij}}{x_i - x_j} \phi_i \bar{\phi}_j, \quad (6)$$

where the eigenvalues of X_1 can be interpreted as the particle coordinates in the x_1 direction. The parameters λ_{ik} , $i, k = 1, \dots, N$, are gauge parameters and ϕ is the vector we end up with, after the vector ψ is rotated by the transformation U , $\phi = U\psi$. In the following, we work in the gauge where all gauge parameters λ_{ik} are equal to zero. The Gauss law (5) is now just a deformed quantization condition (2) that can be rewritten in the form

$$-eB[X'_1, X'_2] = i\mathcal{V}', \quad \mathcal{V}' = -k\mathbf{1} + \phi\bar{\phi}. \tag{7}$$

If one of the matrices X'_1, X'_2 in the relation (7) is diagonal, which is the case here, then the consistency of the solution of the commutation relation (7) necessarily requires that the matrix \mathcal{V}' should be of the form $\mathcal{V}' = -k\mathbf{1} + (k+1)\mathcal{J}$, where the matrix \mathcal{J} has already been defined after Eq. (2). Namely, a more detailed analysis shows that $|\phi_i| \equiv |(U\psi)_i| = \sqrt{k+1}$, $k \geq -1$ and the residual $U(1)^N$ gauge freedom can be used to choose the phase factors of ϕ_i so that $\phi_i = \sqrt{k+1}$. The matrix \mathcal{V}' is equal to the matrix $\mathcal{V} = (1-\nu)\mathbf{1} + \nu\mathcal{J}$, where $\nu = k+1$. In the classical limit $\hbar \rightarrow 0$ or equivalently $\nu \rightarrow \infty$, we have $Tr[X_1, X_2] = 0$ and the diagonal elements in \mathcal{V}' are equal to zero. In regard to the parameter k , the two specially interesting cases are when $k = -1(\nu = 0)$ and $k = 0(\nu = 1)$. The former corresponds to the Bose system and the latter corresponds to the Fermi system.

3. BARGMANN REPRESENTATION

Let us introduce matrix operators

$$\mathcal{A}^\pm = \sqrt{\frac{m\omega}{2}}(X'_1 \pm iX'_2) \tag{8}$$

such that the following commutation relation holds:

$$[\mathcal{A}^-, \mathcal{A}^+] = -k\mathbf{1} + \phi\bar{\phi} = (1-\nu)\mathbf{1} + \nu\mathcal{J}. \tag{9}$$

Owing to the fact that the fields $\phi, \bar{\phi}$ are proportional to R, C matrices, i.e. $\phi = \sqrt{k+1}C$, $\bar{\phi} = \sqrt{k+1}R$, the Hamiltonian can now be written in a way as Eq. (1),

$$H = \frac{\omega}{2(k+1)}\bar{\phi}\{\mathcal{A}^-, \mathcal{A}^+\}\phi = \frac{\omega}{2}R\{\mathcal{A}^-, \mathcal{A}^+\}C. \tag{10}$$

The ground state is a column vector $\|0\rangle_\nu$ that is annihilated by the operator \mathcal{A}^- ,

$$\mathcal{A}^- \mathcal{J} \|0\rangle_\nu = \mathcal{A}^- \|0\rangle_\nu = 0, \quad \|0\rangle_\nu \sim C \prod_{i < j} (x_i - x_j)^\nu e^{-\frac{m\omega}{2} \sum_i x_i^2} \quad (11)$$

and the full Fock space [11] is given by the states

$$\prod_n (Tr(\mathcal{A}^{+n} \mathcal{J}))^{m_n} \|0\rangle_\nu = \prod_n \left(\sum_i (a_i^+)^n \right)^{m_n} |0\rangle_\nu, \quad (12)$$

where $(a_i^+)^n = (R\mathcal{A}^{+n})_i$, $(a_i^-)^n = (\mathcal{A}^{+n}C)_i$, $i = 1, \dots, N$, with a_i^+ , a_i^- being the one-particle creation and annihilation operators [11] for the Hamiltonian H . The corresponding energies are

$$E_{\{m\}} = E_0 + \omega \sum_n nm_n, \quad (13)$$

where

$$E_0 = \omega \left(\frac{N}{2} + \nu \frac{N(N-1)}{2} \right), \quad \nu \geq 0. \quad (14)$$

Now we analyze the structure of energy eigenstates in the Bargmann representation, as was done for the generalized Calogero model in arbitrary dimension [16]. Starting from the matrices X_1, X_2 , we define the combinations $\mathcal{A}_B^\pm = \sqrt{\frac{m\omega}{2}}(X_1 \pm iX_2)$, where the label B indicates that we are working in the Bargmann representation, the transfer to which is realized by the similarity transformation

$$\mathcal{A}_B^+ = S^{-1} \mathcal{A}^+ S, \quad \mathcal{A}_B^- = S^{-1} \mathcal{A}^- S, \quad (15)$$

where S is the following operator:

$$S = e^{-\omega T_+} e^{-\frac{1}{2\omega} T_-}. \quad (16)$$

The operators T_+, T_- , together with the operator T_0 , are the generators [11] of the $SU(1,1)$ algebra and are given as follows:

$$T_+ = \frac{m}{2} \bar{\psi} X_1^2 \psi = \frac{m}{2} R X_1'^2 C,$$

$$T_- = -\frac{m\omega^2}{2} \bar{\psi} X_2^2 \psi = -\frac{m\omega^2}{2} R X_2'^2 C,$$

$$T_0 = -\frac{m\omega}{4}\bar{\psi}(X_1X_2 + X_2X_1)\psi = -\frac{m\omega}{4}R(X'_1X'_2 + X'_2X'_1)C. \quad (17)$$

The frequency ω is assumed to be different from zero. Note that the same transformation connects the Hamiltonians $H = \frac{\omega}{2}R\{\mathcal{A}^-, \mathcal{A}^+\}C$ and $H_B = \frac{\omega}{2}R\{\mathcal{A}_B^-, \mathcal{A}_B^+\}C$, together with their corresponding ground states, namely

$$H = SH_B S^{-1} = 2\omega ST_0 S^{-1}, \quad |0\rangle_\nu = S|0\rangle_\nu^B \quad (18)$$

and that the commutation relation satisfied by $\mathcal{A}_B^-, \mathcal{A}_B^+$ is still unchanged

$$[\mathcal{A}_B^-, \mathcal{A}_B^+] = -k\mathbf{1} + \phi\bar{\phi} = (1 - \nu)\mathbf{1} + \nu\mathcal{J}. \quad (19)$$

As the same relation (18), up to the factor 2ω , is also satisfied by the operator T_0 , which, when rewritten in an explicit form, is equal to $\frac{1}{2}(\sum_i x_i \frac{\partial}{\partial x_i} + \frac{N}{2})$, we conclude that the Hamiltonian in the Bargmann representation H_B is exactly the operator $2\omega T_0$. This does not mean that the operators $\mathcal{A}_B^-, \mathcal{A}_B^+$ can be identified with the matrices X'_1, X'_2 . In other words, \mathcal{A}_B^+ is not a diagonal matrix, but rather it satisfies the relation $R\mathcal{A}_B^{+n}C = \sum_i (a_i^+)_B^n$, where $(a_i^+)_B^n = (R\mathcal{A}_B^{+n})_i$. An analogous relation holds for \mathcal{A}_B^- , namely $R\mathcal{A}_B^{-n}C = \sum_i (a_i^-)_B^n$, $(a_i^-)_B^n = (\mathcal{A}_B^{-n}C)_i$. These totally symmetric combinations of operators $(a_i^\pm)_B^n$ act as creation operators for the Hamiltonian $H_B = 2\omega T_0$, so that the whole Fock space for H_B can be constructed by applying them to the vacuum state

$$|0\rangle_\nu^B = \prod_{i < j} (x_i - x_j)^\nu, \quad (20)$$

which is annihilated by the covariant derivative d_i

$$d_i|0\rangle_\nu^B \equiv \left(\frac{\partial}{\partial x_i} - \nu \sum_{l \neq i} \frac{1}{x_i - x_l}\right) \prod_{j < k} (x_j - x_k)^\nu = 0. \quad (21)$$

The operators x_i, d_i , $i, j = 1, \dots, N$ satisfy the commutation relations $[d_i, x_j] = \delta_{ij}$; $[d_i, d_j] = 0$ and the Hamiltonian H_B can be expressed in terms of them in the following way:

$$H_B = E_0 + \omega \sum_i x_i d_i. \quad (22)$$

As a consequence, we have the following set of relations:

$$[H_B, x_i] = x_i,$$

$$[H_B, d_i] = -d_i, \tag{23}$$

which allows us to interpret x_i, d_i as a pair of creation and annihilation operators for the Hamiltonian H_B . However, only totally symmetric combinations of these operators have the physical meaning, so the true Fock space for H_B is constructed by applying the operators

$$\begin{aligned} B_n^+ &\equiv \sum_i x_i^n, \\ B_n^- &\equiv \sum_i d_i^n \end{aligned} \tag{24}$$

to the vacuum (20). In view of the arguments just stated, the sums of powers of the operators $(a_i^+)_B, (a_i^-)_B$ in the Bargmann representation are in fact reduced to $\sum_i (a_i^+)_B^n = \sum_i x_i^n$, i.e. $\sum_i (a_i^-)_B^n = \sum_i d_i^n$.

The $SU(N)$ invariant ground-state vacuum in the Bargmann representation, for a fixed ν , is

$$|0\rangle_\nu^B \sim (\varepsilon_{i_0 \dots i_{N-1}} \prod_{k=0}^{N-1} (R\mathcal{A}_B^{+k})_{i_k})^\nu |0\rangle_0^B \equiv (\varepsilon_{i_0 \dots i_{N-1}} \prod_{k=0}^{N-1} (a_{i_k}^+)_B^k)^\nu |0\rangle_0^B, \tag{25}$$

where $(a_{i_k}^+)_B^k = (R\mathcal{A}_B^{+k})_{i_k}$, $(a_{i_k}^-)_B^k = (\mathcal{A}_B^{+k}C)_{i_k}$, $i_k = 1, \dots, N$, with $(a_{i_k}^+)_B, (a_{i_k}^-)_B$ being the one-particle creation and annihilation operators [11] for the Hamiltonian H_B . Owing to the fact that we know the transformation from H_B to H , we also know the transformation between the corresponding ladder operators

$$S \sum_i (a_i^\pm)_B^n S^{-1} = \sum_i (a_i^\pm)^n. \tag{26}$$

As a consequence, in the Bargmann representation the expression for the ground-state takes on the form

$$\|0\rangle_\nu^B = S^{-1} \|0\rangle_\nu \sim C \prod_{i < j} (x_i - x_j)^\nu \equiv C |0\rangle_\nu^B, \quad \nu \geq 0, \tag{27}$$

with $\mathcal{A}_B^- \|0\rangle_\nu^B = 0$. Then all states, Eq. (12), in the Bargmann representation, with the covariant matrix derivative, Eq. (21), can be represented as

$$C \prod_n (\sum_i x_i^n)^{m_n} \prod_{i < j} (x_i - x_j)^\nu. \tag{28}$$

For example, the quasihole state in the Bargmann representation is $\prod_{i=1}^N (z - x_i) \prod_{i < j} (x_i - x_j)^\nu$, where z is a complex number. Note that this result is

the same as the one we would get if we assumed the diagonal form for the operator \mathcal{A}_B^+ , i.e., $\mathcal{A}_B^+ \equiv X'_1$, $\mathcal{A}_B^- \equiv X'_2$ with x_i as real variables.

The lowest state in a given tower with fixed ν is just a Laughlin wave function. For $\nu = 0$ ($k = -1$), both operators \mathcal{A}^\pm are diagonal and the system is equivalent to N ordinary one-dimensional harmonic oscillators. Generally, the Laughlin wave function exponent $\nu = k + 1$ is an integer number and, particularly, if ν is even, the system behaves as the Bose system, and for ν odd, it behaves as the Fermi system. Therefore, we have shown that the transition to the Bargmann representation allows us to eliminate the gauge degrees of freedom and reduce the finite Chern-Simons matrix model to the quantum mechanics of N variables with ground state given by the Laughlin wave function.

4. CONCLUSION

In this presentation it is shown that the quantum matrix oscillator introduced in [11] leads to Laughlin ground states in the Bargmann representation. This has provided us with the possibility of reducing the problem stemming from the action (4) to a simpler one which has made the underlying structure more obvious and has provided an immediate physical interpretation of the results obtained. Due to the fact that the above procedure can be straightforwardly extended to higher dimensions, the results obtained can be analytically continued onto the whole complex plane, so as to incorporate the wave functions that depend on the complex variables. In our approach the Gauss condition is interpreted as the deformed matrix quantization condition, and the Laughlin wave function exponent $\nu = k + 1$ is an integer number. There is a complete equivalence to the one-dimensional Calogero N -body system [11], [12], with the ground state of the Laughlin type. Therefore, the Bargmann space analysis allowed us to reduce the regularized finite Chern-Simons matrix model to the 1-dimensional quantum N -body problem with ground state given by the Laughlin wave function. This analysis in terms of the quantum matrix oscillator may be relevant to the application to the quantum Hall physics, particularly if extended and applied to higher dimensions [17], [18].

Acknowledgement One of us (A.S.) would like to thank the Organising Committee of the BW 2005 for invitation to the conference and for giving a possibility to make this short report.

REFERENCES

- [1] R. A. Prange, S. M. Girvin, *The quantum Hall effect*, Springer, Berlin(1990);
S. Das Sarma, A. Pinczuk, *Perspectives in Quantum Hall effects*, Wiley, New York(1997).
- [2] F. Wilczek ed., *Fractional Statistics and Anyon Superconductivity*, World Scientific, Singapore (1990).
- [3] A. P. Polychronakos, *Generalized statistics in one dimension, in Topological Aspects of Low-dimensional Systems*, Les Houches Summer School 1998, hep-th/9902157.
- [4] R. B. Laughlin, Phys.Rev.Lett. **50** (1983) 1395.
- [5] R. Teodorescu, E. Bettelheim, O. Agam, A. Zabrodin, P. Wiegmann, Nucl.Phys.B **700** (2004) 521;
O. Agam, E. Bettelheim, P. Wiegmann, A. Zabrodin, Phys.Rev.Lett. **88** (2002) 236802.
- [6] L. Susskind, *The quantum Hall fluid and non-commutative Chern-Simons theory*, hep-th/0101029.
- [7] A. P. Polychronakos, JHEP **0104** (2001) 011.
- [8] S. Hellerman, M. Van Raamsdonk, JHEP **0110** (2001) 039.
- [9] L. Jonke, S. Meljanac, JHEP **0201** (2002) 008;
L. Jonke, S. Meljanac, Phys.Rev.B **66** (2002) 205313.
- [10] D. Karabali, B. Sakita, Phys.Rev.B **64** (2001) 245316;
D. Karabali, B. Sakita, Phys.Rev.B **65** (2002) 075304.
- [11] S. Meljanac, A.Samsarov, Phys.Lett.B **600** (2004) 179, hep-th/0408241.

- [12] S. Meljanac, M. Mileković , A.Samsarov, Phys.Lett.B **573**(2003) 202;
S. Meljanac, M. Mileković , A.Samsarov, M. Stojić , Mod.Phys.Lett.B,
Vol.**18** (2004) 603, hep-th/0405132.
- [13] D. Kazhdan, B.Kostant, S.Sternberg, Comm.Pure Appl.Math.vol.31
(1978) 481.
- [14] A. Gorsky, N.Nekrasov, Nucl.Phys. B414 (1994) 213, hep-th/9304047.
- [15] V. P. Nair, A. P. Polychronakos, Phys.Rev.Lett. **87** (2001) 030403.
- [16] S. Meljanac, M. Mileković , A.Samsarov, Phys.Lett.B **594** (2004) 241,
hep-th/0405131.
- [17] S. Zhang, J. Hu, Science **294** (2001) 823;
S. Zhang, J. Hu, Phys.Rev.B **41** (1990) 12838;
B. A. Bernevig, C. H. Chern, J. P. Hu, N. Toumbas, S. C. Zhang, Annals
Phys. **300** (2002) 185.
- [18] D. Karabali, V. P. Nair, Nucl.Phys.B **641** (2002) 533,hep-th/0203264;
D. Karabali, V. P. Nair, Nucl.Phys.B **679** (2004) 427, hep-th/0307281;
D. Karabali, V. P. Nair, Nucl.Phys.B **697** (2004) 513, hep-th/0403111.