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Noncommutativity and B-field in p-string theory

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Abstract: *We explore the consequence of a noncommutative deformation on the effective field theory of the tachyon of the p-adic string. Exact soliton solutions that interpolate between the noncommutative soliton and that of the p-adic tachyon are obtained. A worldsheet origin of the noncommutativity is sought in the antisymmetric Neveu-Schwarz B-field.*

Key words: *p-adic string, noncommutative field theory, soliton*

The p-adic string theory (or simply *p-string theory* for short), introduced and developed in Refs.[1, 2, 3], is an intriguing theory. Apparently it describes open string theory in an exotic discretisation of the worldsheet[4] in the form of a Bethe lattice or an infinite (Bruhat-Tits) tree, whose boundary is (isomorphic to) the field of p-adic numbers[6]. This helps to compute any tree-level correlator of its lowest excitation, the p-tachyon field. As a consequence of this, the low energy effective field theory of the p-tachyon is known *exactly.* (See [5] for a review.) It is possible, therefore, to check[7] that the p-tachyon field behaves according to the conjectures by Sen[8] concerning the open string tachyons on D-branes.

One might hope that the p-adic string theory will provide us a useful guide to difficult questions in (usual) string theory. One is further encouraged by the fact that in the $p \to 1$ limit, p-string theory seems to reduce to (an approximation of) the usual string theory[9]. To realise this in practice, however, we require a better understanding of the p-adic string itself. As it is, we only know some properties of its D-branes in flat spacetime. Among the nontrivial backgrounds, a particularly simple one a is that of a constant rank two Neveu-Schwarz field B. In the usual case, its effect is to provide a *noncommutative deformation* of the effective field theory[10].

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With this objective let us consider a noncommutative deformation of the spacetime effective field theory of the p-tachyon on a D-brane of p-string theory[11]:

$$\mathcal{L}_{NC}^{(p)} = -\varphi \star p^{-\frac{1}{2}\Box}\varphi + \frac{1}{p+1}\,(\star\varphi)^{p+1}, \tag{1}$$

where, for simplicity, we assume noncommutativity along two spatial directions and the Moyal product

$$f \star g = f(x^1,x^2)\exp\left(\frac{i}{2}\theta\epsilon^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right) g(x^1,x^2), \tag{2}$$

is used instead of ordinary mutiplication along these directions. Let us recall that the $\theta = 0$ version of Eq.(1) was derived in [2]. Using this action, Ref.[7] checked the Sen conjectures for the p-tachyon. To be more precise, the maximum of the potential at $\varphi = 1$ is the unstable vacuum corresponding to the D-brane, but it also has a local minimum at $\varphi = 0$. The various soliton solutions of [2] are naturally identified to the lower dimensional D-branes[7].

It is trivial to check that the constant configurations $\varphi = 0$ and 1 still satisfy the deformed equation of motion:

$$p^{-\frac{1}{2}\Box}\varphi = (\star\varphi)^p. \tag{3}$$

It is only a little more work to verify that the gaussian soliton solutions with nontrivial space dependence generalise in the noncommutative case. Since the $\star$-product of gaussians is again a gaussian:

$$Ae^{-a|z|^2} \star Be^{-b|z|^2} = \frac{AB}{1+ab\,\theta^2}\,\exp\left(\frac{a+b}{1+ab\,\theta^2}|z|^2\right), \tag{4}$$

the ansatz

$$\varphi(x^1,x^2) = A^2\exp\left(-a\left[(x^1)^2+(x^2)^2\right]\right) \tag{5}$$

can be used in the equation of motion (3) to determine the width a as a polynomial of degree p

$$\sum_{i=0}^{\lfloor p/2\rfloor}\binom{p}{2i}(a\theta)^{2i} - (1-2a\ln p)\sum_{i=0}^{\lfloor (p-1)/2\rfloor}\binom{p}{2i+1}(a\theta)^{2i} = 0. \tag{6}$$

Although we cannot find an analytic expression for the root $a(\theta)$ as a function of θ in general, it is straightforward to see that in the limiting cases $\theta = 0$

and $\theta \to \infty$, we recover the width of the gaussian lump of [2] $a = \frac{p-1}{2p \ln p}$ and the 'noncommutative soliton' of [12] $a = 1/\theta$ respectively. The amplitude A is determined in terms of a as

$$A^{2p-2} = \sum_{i=0}^{\lfloor (p-1)/2 \rfloor} \binom{p}{2i+1} (a\theta)^{2i}\,, \tag{7}$$

which, again, interpolates between the commutative limit ($A = p^{1/2(p-1)}$ for $\theta = 0$) and the limit of large noncommutativity ($A = \sqrt{2}$ for $\theta \to \infty$). Thus we see that the *noncommutative p-soliton* given by Eqs.(5), (6) and (7), interpolates smoothly between the commutative p-soliton of [2] the noncommutative soliton of [12].

It was shown in [9] that in the $p \to 1$ limit p-string theory reduces to the *boundary string field theory (BSFT)* description of ordinary bosonic string (truncated to two derivatives). In this limit, the lagrangian (1) reduces to:

$$\mathcal{L}_{NC}^{(p\to 1)} = -\frac{1}{2}\varphi \star \Box \varphi - \frac{1}{2}\varphi \star \varphi \left(\ln_\star(\varphi \star \varphi) - 1\right). \tag{8}$$

This noncommutative deformation was considered in [14]. Amazingly, the noncommutative p-solitons discussed above also make sense in the $p \to 1$ limit[11]. To get the $\star$-deformed logarithm of an ordinary exponential, we extrapolate the n-fold $\star$-product of the gaussian following from (4) to fractional powers. The width a and the amplitude A of the gaussian lump are now given by the transcendental equations:

$$\begin{aligned} 2a &= \frac{1-(a\theta)^2}{2a\theta} \ln\left(\frac{1+a\theta}{1-a\theta}\right), \\ 2\ln A &= \frac{1+a\theta}{2a\theta}\ln(1+a\theta) - \frac{1-a\theta}{2a\theta}\ln(1-a\theta). \end{aligned}$$

These too interpolate smoothly between the BSFT solution $a = \frac{1}{2}$, $A = \sqrt{e}$ at $\theta = 0$ [13] to the GMS solution $a = \frac{1}{\theta}$, $A = \sqrt{2}$ [12, 14]. Further details may be found in [11].

We will now consider the effect of coupling a B-field to the worlsheet of the p-string. It is not obvious what this means as the B-field arises in the closed string sector which is not known in p-string theory. Our objective, however, is modest: we are interested in a constant B-field background. This is known to be gauge equivalent (after an integration by parts)

$$\frac{1}{2}\int_{\Sigma} B_{\mu\nu} dX^{\mu}\wedge dX^{\nu} = \int_{\partial\Sigma} d\xi\ B_{\mu\nu}X^{\mu}\,\partial_{\xi}X^{\nu}, \tag{9}$$

to a background gauge field $A_{\nu}\sim B_{\mu\nu}X^{\mu}$, which couples to the boundary of the worldsheet. It is not that we know about the gauge field excitation of the open p-string (although existence of the translation zero modes for the transverse scalars on the solitonic D-branes is a strong hint[7]). Nevertheless, the second form above is a better starting point for a p-adic generalisation: the p-adic worldsheet being a tree lacks an obvious 2-form[4]. However, starting with the discrete Polyakov action on the p-adic 'worldsheet' and integrating over the bulk degrees of freedon, one can write an effective non-local action on the boundary

$$\mathcal{S}_p = \frac{p(p-1)\beta_p}{4(p+1)}\int_{\mathbf{Q}_p} d\xi\, d\xi'\,\frac{\left(X^{\mu}(\xi)-X^{\mu}(\xi')\right)^2}{|\xi-\xi'|_p^2}. \tag{10}$$

where, $\mathbf{Q}_p$ is the field of p-adic numbers and $|\cdot|_p$ the norm in it. Historically this action was proposed in [15] even before the local action and indeed before the p-string 'worldsheet' was understood.

There remains one hurdle still. There is no natural notion of a (tangential) derivative for the complex/real valued functions $X^{\mu}(\xi)$ of a p-adic variable. Fortunately, there is a way to write this as a kernel[6, 16]

$$\partial_{\xi}X^{\mu} = \int_{\mathbf{Q}_p} d\xi'\,\frac{\mathrm{sgn}_{\tau}(\xi-\xi')}{|\xi-\xi'|_p^2}\,X^{\mu}(\xi'), \tag{11}$$

where $\mathrm{sgn}_{\tau}(\xi)$ is a p-adic "sign function" depending on the choice of $\tau\in\mathbf{Q}_p$:

$$\mathrm{sgn}_{\tau}(\xi) = \begin{cases} +1, & \text{if } \xi=\zeta_1^2-\tau\zeta_2^2 \quad \text{for some } \zeta_1,\zeta_2\in\mathbf{Q}_p, \\ -1 & \text{otherwise.}\end{cases}$$

Of the three inequivalent choices of τ, $\mathrm{sgn}_{\tau}(\xi)$ is antisymmetric only when $\tau = p, \omega p$ ($\omega^{p-1}=1$) and that too for $p = 3$ (mod 4)[6]. In order to follow the usual string case closely, we shall restrict to these cases.

We are finally in a position to propose our action for open p-string coupled to constant B-field[17]:

$$\int_{\mathbf{Q}_p}\frac{d\xi\, d\xi'}{|\xi-\xi'|_p^2}\Big[\eta_{\mu\nu}\left(X^{\mu}(\xi)-X^{\mu}(\xi')\right)\left(X^{\nu}(\xi)-X^{\nu}(\xi')\right)$$

$$+\frac{i(p+1)}{p^2\Gamma_\tau(-1)}\,\mathrm{sgn}_\tau(\xi-\xi')\,B_{\mu\nu}X^\mu(\xi)X^\nu(\xi')\Bigg], \tag{12}$$

where $\Gamma_\tau(-1)$ is a (finite) number depending on p. It is straightforward to solve for the Green's function in the presence of the B-field[17]

$$\begin{aligned}\mathcal{G}^{\mu\nu}(\xi-\xi') &= -G^{\mu\nu}\ln|\xi-\xi'|_p + \frac{i}{2}\theta^{\mu\nu}\mathrm{sgn}_\tau(\xi-\xi'),\\ \text{where, }\left(\frac{1}{\eta-iB}\right)^{\mu\nu} &= G^{\mu\nu}+\frac{i}{2}\,\frac{p-1}{\alpha' p^2\ln p\,\Gamma_\tau(-1)}\,\theta^{\mu\nu}\end{aligned} \tag{13}$$

The above exactly correspond to the analogous results for the usual bosonic string[18].

We seem to be at the end of the road, since formally

$$[X^\mu,X^\nu](0)=\lim_{\substack{\mathrm{sgn}_\tau(\xi)=+1\\ |\xi|_p\to 0}}\Big(\langle X^\mu(\xi)X^\nu(0)\rangle-\langle X^\mu(0)X^\nu(-\xi)\rangle\Big)=i\,\theta^{\mu\nu}, \tag{14}$$

but this conclusion would be hasty as there are caveats. The problem is due to the fact that $\mathbf{Q}_p$ does not have a natural sense of an *order*. It is possible to define an order by, say, ordering the coefficients in the power series expansion of a p-adic number. We can also distinguish between 'positive' and 'negative' p-adic numbers. Unfortunately this notion is not GL(2,$\mathbf{Q}_p$) covariant and thus of little use.

As a result the four p-tachyon Veneziano amplitude with minimally coupled B-field as above

$$\begin{aligned}\mathcal{A}^{(4)}_{pB} &= \frac{p-1}{p}\left(\frac{c_{12}c_{34}}{p^{\alpha(s)}-1}+\frac{c_{13}c_{24}}{p^{\alpha(t)}-1}+\frac{c_{14}c_{23}}{p^{\alpha(u)}-1}\right)\\ &\quad -\frac{1}{p}\left(c_{12}c_{34}+c_{13}c_{24}+c_{14}c_{23}\right)+\frac{p+1}{p}c_{12}c_{13}c_{23},\end{aligned} \tag{15}$$

(where, $c_{ij}=\cos\frac{1}{2}k^i\theta k^j$) does not quite match the same calculated from the field theory action (1):

$$\frac{p-1}{p}\left(\frac{c_{12}c_{34}}{p^{\alpha(s)}-1}+\frac{c_{13}c_{24}}{p^{\alpha(t)}-1}+\frac{c_{14}c_{23}}{p^{\alpha(u)}-1}\right)+\frac{p-2}{p}\left(c_{12}c_{34}+c_{13}c_{24}+c_{14}c_{23}\right).$$

The problems here are analogous to those encountered in incorporating Chan-Paton factors in open p-string theory[19, 5]. For details, the reader may refer to [17].

In summary, we studied a noncommutative deformation of the exact effective action of the open string tachyon on the D-brane of p-adic string theory. A family of gaussian lump solution for *all* values of the noncommutativity parameter θ was obtained. It smoothly interpolates between the soliton of the (commutative) p-adic string theory and the noncommutative soliton. Further, it was shown that in the $p \to 1$ limit one finds smoothly interpolating solitons in the boundary string field theory description of the usual bosonic string. Finally, a minimal coupling of the constant B-field to the nonlocal action on the boundary of the p-string 'worldsheet' was analysed. Its effect was found to be qualitatively similar to the deformed effective spacetime theory, although the four-tachyon amplitude did not match precisely. Perhaps in the p-adic case, one needs to go beyond the minimally coupled B-field. Much remains to be done in understanding p-adic strings in nontrivial backgrounds.

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