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REAL HAMILTONIAN FORMS FOR AFFINE TODA MODELS

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Abstract. *The construction of a family of real Hamiltonian forms (RHF) for the special class of affine* $1+1$*-dimensional Toda field theories (ATFT) is reported. Thus the method, proposed in [1] for systems with finite number of degrees of freedom is generalized to infinite-dimensional Hamiltonian systems. The construction method is ilustrated on an explicit nontrilial example RHF of* $\mathbf{E}_6^{(1)}$ *ATFT.*

Key words: *Solitons, Affine Toda Field Theories, Hamiltonian systems*

1. INTRODUCTION

To each simple Lie algebra $\mathfrak{g}$ one can relate Toda field theory (TFT) in $1+1$ dimensions. It allows Lax representation: $[L, M] = 0$, where L and M are first order ordinary differential operators, see e.g. [2, 3, 4, 5, 6, 7]:

$$L\psi \equiv \left(i\frac{d}{dx} - iq_x(x,t) - \lambda J_0 \right) \psi(x,t,\lambda) = 0, \qquad (1)$$

$$M\psi \equiv \left(i\frac{d}{dt} - \frac{1}{\lambda} I(x,t) \right) \psi(x,t,\lambda) = 0. \qquad (2)$$

whose potentials take values in $\mathfrak{g}$. Here $q(x,t) \in \mathfrak{h}$ - the Cartan subalgebra of $\mathfrak{g}$, $\vec{q}(x,t) = (q_1, \dots, q_r)$ is its dual r-component vector, $r = \operatorname{rank} \mathfrak{g}$, and

$$J_0 = \sum_{\alpha\in\pi} E_\alpha, \qquad I(x,t) = \sum_{\alpha\in\pi} e^{-(\alpha,\vec{q}(x,t))} E_{-\alpha}. \qquad (3)$$

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By $\pi_{\mathfrak{g}}$ we denote the set of admissible roots of $\mathfrak{g}$, i.e. $\pi_{\mathfrak{g}} = \{\alpha_0, \alpha_1, \dots, \alpha_r\}$ where $\alpha_1, \dots, \alpha_r$ are the simple roots of $\mathfrak{g}$ and α_0 is the minimal root of $\mathfrak{g}$. The corresponding TFT is known as the affine TFT. The Dynkin graph that corresponds to the set of admissible roots of $\mathfrak{g}$ is called extended Dynkin diagrams (EDD). The equations of motion are of the form:

$$\frac{\partial^2 \vec{q}}{\partial x \partial t} = \sum_{j=0}^{r} n_j \alpha_j e^{-(\alpha_j, \vec{q}(x,t))}, \qquad (4)$$

where n_j are the minimal positive integers for which $\sum_{j=0}^{r} n_j \alpha_j = 0$. The present paper extends the ideas of [8] and [9] to the ATFT related to the exceptional simple Lie algebra E_6; for finite Toda chains see [10, **?**, 1, 11].

2. THE REDUCTION GROUP

The operators L and M are invariant with respect to the reduction group $\mathcal{G}_{\mathbb{R}} \simeq \mathbb{D}_{\langle}$ where h is the Coxeter number of $\mathfrak{g}$. It is generated by two elements satisfying $g_1^h = g_2^2 = (g_1 g_2)^2 = \mathbb{1}$ which allow realizations both as elements in $\mathrm{Aut}\,_{\mathfrak{g}}$ and in $\mathrm{Conf}\,\mathbb{C}$. The invariance condition has the form [2]:

$$C_k(U(x,t,\kappa_k(\lambda))) = U(x,t,\lambda), \qquad C_k(V(x,t,\kappa_k(\lambda))) = V(x,t,\lambda), \qquad (5)$$

where $U(x,t,\lambda) = -iq_x(x,t) - \lambda J_0$ and $V(x,t,\lambda) = -\frac{1}{\lambda} I(x,t)$. Here C_k are automorphisms of finite order of $\mathfrak{g}$, i.e. $C_1^h = C_2^2 = (C_1 C_2)^2 = \mathbb{1}$ while $\kappa_k(\lambda)$ are conformal mappings of the complex λ-plane. The algebraic constraints (5) are automatically compatible with the evolution. A number of nontrivial reductions of nonlinear evolution equations can be found in [12, 13].

3. SPECTRAL PROPERTIES OF THE LAX OPERATOR

The reduction conditions (5) lead to rather special properties of the operator L. Along with L we will use also the equivalent system:

$$Lm(x,t,\lambda) \equiv i\frac{dm}{dx} + iq_x m(x,t,\lambda) - \lambda[J_0, m(x,t,\lambda)] = 0, \qquad (6)$$

where $m(x,t,\lambda) = \psi(x,t,\lambda)e^{iJ_0 x\lambda}$. Combining the ideas of [14] with the symmetries of the potential (5) we can construct a set of $2h$ fundamental analytic solutions (FAS) $m_\nu(x,t,\lambda)$ of (6) and prove that (see [12, 7]):

1. the continuous spectrum Σ of L fills up $2h$ rays l_ν passing through the origin: $\lambda \in l_\nu\colon\ \arg\lambda = (\nu-1)\pi/h$;
2. $m_\nu(x,t,\lambda)$ is a FAS of (6) analytic with respect to λ in the sector $\Omega_\nu\colon (\nu-1)\pi/h \le \arg\lambda \le \nu\pi/h$ satisfying $\lim_{\lambda\to\infty} m_\nu(x,t,\lambda) = \mathbb{1}$;

3. to each l_ν one relates a subalgebra $\mathfrak{g}_\nu \subset \mathfrak{g}$ such that $\mathfrak{g}_\nu \cap \mathfrak{g}_\mu = \emptyset$ for $\nu \neq \mu \mod (h)$ and $\cup_{\nu=1}^{h} \mathfrak{g}_\nu = \mathfrak{g}$. The symmetry ensures that each of the subalgebras $\mathfrak{g}_\nu$ is a direct sum of $sl(2)$-subalgebras;

4. on Σ the FAS $m_\nu(x,t,\lambda)$ satisfy

$$m_\nu(x,t,\lambda) = m_{\nu-1}(x,t,\lambda) G_\nu(x,t,\lambda), \qquad \lambda \in l_\nu, \tag{7}$$

$$G_\nu(x,t,\lambda) = e^{-i(\lambda J_0 x + f(\lambda))t} G_{0,\nu}(\lambda) e^{i(\lambda J_0 x + f(\lambda))t} \in \mathcal{G}_\nu, \tag{8}$$

where $\mathcal{G}_\nu$ is the subgroup with Lie algebra $\mathfrak{g}_\nu$ and $f(\lambda)$ is determined by the dispersion law of the NLEE: $f(\lambda) = \sum_{k=0}^{r} E_{-\alpha_k}/\lambda$;

5. the FAS of (6) satisfy:

$$\bar{C}_1(m_\nu(x,t,\omega\lambda)) = m_{\nu-2}(x,t,\lambda), \qquad \lambda \in l_{\nu-2}, \tag{9}$$

where $\bar{C}_1$ is equivalent to the Coxeter automorphism [15]:

$$\bar{C}_1(X) \equiv C_1^{-1} X C_1, \qquad C_1 = e^{\frac{2\pi i}{h} H_\rho}, \qquad \rho = \frac{1}{2}\sum_{\alpha>0} \alpha; \tag{10}$$

obviously $C_1^h = \mathbb{1}$ and $\bar{C}_1(J_0) = \omega^{-1} J_0$;

6. the FAS $m_\nu(x,t,\lambda)$ satisfy one of the following two involutions:

$$\bar{C}_2(m_\nu(x,t,\lambda^*))^\dagger = C_2(m_{2h-\nu+1}^{-1}(x,t,\lambda)), \tag{11}$$

where C_2, $C_2^2 = \mathbb{1}$ is conveniently chosen Weyl group element, or

$$(m_\nu(x,t,-\lambda^*))^* = m_{h-\nu+1}(x,t,\lambda). \tag{12}$$

These relations lead to the following constraints for the sewing functions $G_{0,\nu}(\lambda)$ and the minimal set of scattering data:

$$\bar{C}_1(G_{0,\nu}(\omega\lambda)) = G_{0,\nu-2}(\lambda), \tag{13}$$

$$\bar{C}_2(G_{0,\nu}^\dagger(\lambda^*)) = G_{0,2h-\nu+1}^{-1}(\lambda), \tag{14}$$

$$G_{0,\nu}^*(-\lambda^*) = G_{0,h-\nu+1}(\lambda). \tag{15}$$

If L has no discrete eigenvalues the minimal set of scattering data is provided by the coefficients of $G_{0,1}(\lambda), \lambda \in l_1$ and $G_{0,2}(\lambda), \lambda \in l_2$. All the other sewing functions $G_{0,\nu}(\lambda)$ can be determined from them by eqs. (13)–(15).

4. REAL HAMILTONIAN FORMS

The Lax representations of the ATFT models (see e.g. [2, 3, 4, 16, 6] and the references therein) are related mostly to the normal real form of the Lie algebra $\mathfrak{g}$, see [17].

Our aim here is to:

1. generalize the ATFT to complex-valued fields $\vec{q}^{\mathbb{C}} = \vec{q}^{0} + i\vec{q}^{1}$, and to
2. describe the family of RHF of these ATFT models.

We also provide a tool generalizing of the one in [1] for the construction of new inequivalent RHF's of the ATFT. The ATFT for the algebra $sl(n)$ can be written down as an infinite-dimensional Hamiltonian system as follows:

$$\frac{dq_k}{dt} = \{q_k, H_{\rm ATFT}\}, \qquad \frac{dp_k}{dt} = \{p_k, H_{\rm ATFT}\}, \tag{16}$$

$$H_{\rm ATFT} = \int_{-\infty}^{\infty} dx \, \left(\frac{1}{2}(\vec{p}(x,t), \vec{p}(x,t)) + \sum_{k=0}^{r} e^{-(\vec{q}(x,t),\alpha_k)} \right), \tag{17}$$

where $\vec{q}(x,t)$ and $\vec{p} = \partial \vec{q}/\partial t$ are the canonical coordinates and momenta satisfying canonical Poisson brackets:

$$\{q_k(x), p_j(y)\} = \delta_{jk}\delta(x-y). \tag{18}$$

Next we define the involution C acting on the phase space $\mathcal{M}$as follows:

$$\begin{array}{ll} 1) & C(F(p_k,q_k)) = F(C(p_k), C(q_k)), \\ 2) & C\left(\{F(p_k,q_k), G(p_k,q_k)\}\right) = \{C(F), C(G)\}, \\ 3) & C(H(p_k,q_k)) = H(p_k,q_k). \end{array}$$

Here $F(p_k,q_k)$, $G(p_k,q_k)$ and the Hamiltonian $H(p_k,q_k)$ are functionals on $\mathcal{M}$ depending analytically on the fields $q_k(x,t)$ and $p_k(x,t)$.

The complexification of the ATFT is rather straightforward. The resulting complex ATFT (CATFT) can be written down as standard Hamiltonian system with twice as many fields $\vec{q}^{a}(x,t)$, $\vec{p}^{a}(x,t)$, $a = 0, 1$:

$$\vec{p}^{\mathbb{C}}(x,t) = \vec{p}^{0}(x,t) + i\vec{p}^{1}(x,t), \qquad \vec{q}^{\mathbb{C}}(x,t) = \vec{q}^{0}(x,t) + i\vec{q}^{1}(x,t), \tag{19}$$

$$\{q_k^0(x,t), p_j^0(y,t)\} = -\{q_k^1(x,t), p_j^1(y,t)\} = \delta_{kj}\delta(x-y). \tag{20}$$

The densities of the corresponding Hamiltonian and symplectic form equal

$$\begin{aligned} H^{\mathbb{C}}_{\rm ATFT} &\equiv \operatorname{Re} H_{\rm ATFT}(\vec{p}^{0} + i\vec{p}^{1}, \vec{q}^{0} + i\vec{q}^{1}) \\ &= \frac{1}{2}(\vec{p}^{0}, \vec{p}^{0}) - \frac{1}{2}(\vec{p}^{1}, \vec{p}^{1}) + \sum_{k=0}^{r} e^{-(\vec{q}^{0},\alpha_k)} \cos((\vec{q}^{1}, \alpha_k)), \qquad (21) \\ \omega^{\mathbb{C}} &= (d\vec{p}^{0} \wedge id\vec{q}^{0}) - (d\vec{p}^{1} \wedge d\vec{q}^{1}). \qquad (22) \end{aligned}$$

The family of RHF then are obtained from the CATFT by imposing an invariance condition with respect to the involution $\tilde{C} \equiv C \circ *$ where by $*$ we denote the complex conjugation. The involution $\tilde{C}$ splits the phase space $M^{\mathbb{C}}$ into a direct sum $M^{\mathbb{C}} \equiv M^{\mathbb{C}}_{+} \oplus M^{\mathbb{C}}_{-}$ where $M^{\mathbb{C}}_{+} = M_0 \oplus iM_1$, $M^{\mathbb{C}}_{-} = iM_0 \oplus M_1$,. The phase space of the RHF is $M_{\mathbb{R}} \equiv M^{\mathbb{C}}_{+}$. By M_0 and M_1 we denote the eigensubspaces of C, i.e. $C(u_a) = (-1)^a u_a$ for any $u_a \in M_a$.

Thus to each involution $\mathcal{C}$ satisfying 1) - 3) one can relate a RHF of the ATFT. Due to the condition 3) $\mathcal{C}$ must preserve the system of admissible roots of $\mathfrak{g}$; such involutions can be constructed from the $\mathbb{Z}_2$-symmetries of the extended Dynkin diagrams of $\mathfrak{g}$ studied in [6].

5. EXAMPLE: $\mathbf{E}_6^{(1)}$ TODA FIELD THEORIES

The set of admissible roots for this algebra is

$$\begin{aligned}
\alpha_1 &= \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8), \qquad \alpha_2 = e_1 + e_2, \\
\alpha_3 &= e_2 - e_1, \qquad \alpha_4 = e_3 - e_2, \qquad \alpha_5 = e_4 - e_3, \qquad \alpha_6 = e_5 - e_4, \\
\alpha_0 &= -\frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7 + e_8),
\end{aligned} \tag{23}$$

where $\alpha_1, \ldots, \alpha_6$ form the set of simple roots of E_6 and α_0 is the minimal root of the algebra. This is the standard definition of the root system of E_6 embedded into the 8-dimensional Euclidean space $\mathbb{E}^8$. The root space $\mathbb{E}_6$ of the algebra E_6 is the 6-dimensional subspace of $\mathbb{E}^8$ orthogonal to the vectors $e_7 + e_8$ and $e_6 + e_7 + 2e_8$. Thus any vector $\vec{q}$ belonging to $\mathbb{E}_6$ has only 6 independent coordinates and can be written as:

$$\vec{q} = \sum_{k=1}^{5} q_k e_k + q_6 e_6', \qquad e_6' = \frac{1}{\sqrt{3}}(e_6 + e_7 - e_8). \tag{24}$$

Let us fix up the action of the involution $\mathcal{C}$ on a generic vector $\vec{q}$ in $\mathbb{E}^8$ by:

$$C(q_k) = -q_{5-k} + \frac{1}{2}\sum_{m=1}^{4} q_m, \qquad \text{for} \quad k = 1, \ldots, 4 \tag{25}$$

$$= q_{13-k} - \frac{1}{2}\sum_{m=5}^{8} q_m, \qquad \text{for} \quad k = 5, \ldots, 8. \tag{26}$$

This action is compatible with the $\mathbb{Z}_2$-symmetry $\mathcal{C}^{\#}$ of the extended Dynkin diagram (see fig. 1) and reflects an involution of the Kac-Moody algebra

$\mathbf{E}_6^{(1)}$, see [18]. It acts on the root space as follows:

$$
\begin{aligned}
C^{\#} e_k &= -e_{5-k} + \frac{1}{2}\sum_{m=1}^{4} e_m, \qquad \text{for } \ k = 1, \dots, 4 &\quad (27)\\
&= e_{13-k} - \frac{1}{2}\sum_{m=5}^{8} e_m, \qquad \text{for } \ k = 5, \dots, 8. &\quad (28)\\
C^{\#}\alpha_1 &= \alpha_6, \qquad \mathcal{C}^{\#}\alpha_{\ni} = \alpha_{\bigtriangledown}, \qquad \mathcal{C}^{\#}\alpha_{\|} = \alpha_{\|}, \qquad \| = \prime, \in, \triangle. &
\end{aligned}
$$

The involution $\mathcal{C}^{\#}$ splits the root space $\mathbb{E}_6$ into a direct sum of its eigensubspaces: $\mathbb{E}_6 = \mathbb{E}_+ \oplus \mathbb{E}_-$ with $\dim \mathbb{E}_+ = 4$, $\dim \mathbb{E}_- = 2$. The vectors:

$$
\begin{aligned}
\widetilde{e}_1 &= \frac{1}{2}(e_5 - \sqrt{3}e_6'), \qquad \widetilde{e}_2 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4),\\
\widetilde{e}_3 &= \frac{1}{2}(-e_1 - e_2 + e_3 + e_4), \qquad \widetilde{e}_4 = \frac{1}{2}(-e_1 + e_2 - e_3 + e_4), \qquad (29)\\
\widetilde{e}_5 &= \frac{1}{2}(-e_1 + e_2 + e_3 - e_4), \qquad \widetilde{e}_6 = \frac{1}{2}(\sqrt{3}e_5 + e_6').
\end{aligned}
$$

form an orthonormal basis in $\mathbb{E}_6$. The first four satisfy $C^{\#}\widetilde{e}_k = \widetilde{e}_k$, $k = 1, \dots, 4$, so they span $\mathbb{E}_+$; the last two span $\mathbb{E}_-$ because $C^{\#}\widetilde{e}_j = -\widetilde{e}_j$, $j = 5, 6$. In terms of $\widetilde{e}_k$ the admissible root system of $\mathbf{F}_4^{(1)}$ takes the standard form:

$$
\begin{aligned}
\beta_0 &= -\widetilde{e}_2 - \widetilde{e}_1, \qquad \beta_1 = \frac{1}{2}(\widetilde{e}_1 - \widetilde{e}_2 - \widetilde{e}_3 - \widetilde{e}_4),\\
\beta_2 &= \widetilde{e}_2 - \widetilde{e}_3, \qquad \beta_3 = \widetilde{e}_4, \qquad \beta_3 = \widetilde{e}_3 - \widetilde{e}_4. \qquad (30)
\end{aligned}
$$

satisfying $\beta_0 + 2\beta_2 + 3\beta_4 + 4\beta_3 + 2\beta_1 = 0$. Let us take the complex vector $\vec{q}(x,t) = \vec{q}^{\,0}(x,t) + i\vec{q}^{\,1}(x,t) \in \mathbb{E}_6$ (i.e., of the form (24)) and let $\vec{p}(x,t) = \partial\vec{q}/\partial x$. Let us denote their projections onto $\mathbb{E}_\pm$ by $\vec{q}_\pm$ and $\vec{p}_\pm$ respectively. Then the densities $\mathcal{H}_\infty^{\mathbb{R}}$, $\omega_1^{\mathbb{R}}$ for the RHF of AFTF equal:

$$
\begin{aligned}
\mathcal{H}_\infty^{\mathbb{R}} &= \frac{1}{2}\left((\vec{p}_+^{\,0}(x,t), \vec{p}_+^{\,0}(x,t)) - (\vec{p}_-^{\,0}(x,t), \vec{p}_-^{\,0}(x,t))\right) + \mathrm{e}^{-(\vec{q}_+^{\,0}(x,t),\beta_0)}\\
&+ 2\mathrm{e}^{-(\vec{q}_+^{\,0}(x,t),\beta_1)} \cos((\vec{q}_-^{\,1}(x,t), \widetilde{e}_5 + \sqrt{3}\widetilde{e}_6)) + 2\mathrm{e}^{-(\vec{q}_+^{\,0}(x,t),\beta_2)} \qquad (31)\\
&+ 4\mathrm{e}^{-(\vec{q}_+^{\,0}(x,t),\beta_3)} \cos((\vec{q}_-^{\,1}(x,t), \widetilde{e}_5)) + 3\mathrm{e}^{-(\vec{q}_+^{\,0}(x,t),\beta_4)},\\
\omega_1^{\mathbb{R}} &= \left(\delta\vec{p}_+(x) \mathop{\wedge}_{,} \delta\vec{q}_+(x)\right) - \left(\delta\vec{p}_-(x) \mathop{\wedge}_{,} \delta\vec{q}_-(x)\right), \qquad (32)
\end{aligned}
$$

If we put $\vec{q}_-(x,t) = 0$ then also $\vec{p}_-(x,t) = 0$ and we get the reduced ATFT related to the Kac-Moody algebra $\mathbf{F}_4^{(1)}$ [6].

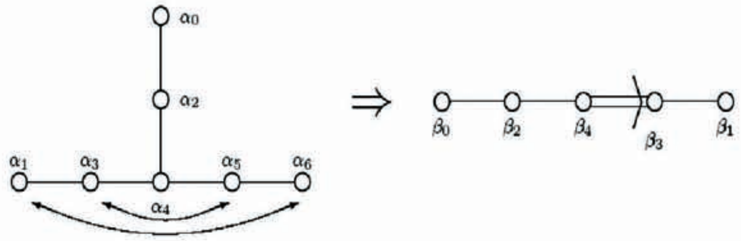

Figure 1: $\mathbf{E}_6^{(1)} \to \mathbf{F}_4^{(1)}$.

6. CONCLUSIONS

The RHF of the ATFT model related to the exceptional Kac-Moody algebras $\mathbf{E}_6^{(1)}$ is constructed. This model generalizes the ATFT in [6] since it contains two types of fields $\vec{q}_+(x,t)$ and $\vec{q}_-(x,t)$ with different properties with respect to the involution $\mathcal{C}$. The models in [6] contain only fields invariant with respect to $\mathcal{C}$. Some additional problems concern: i) the complete classification of all RHF of ATFT; ii) the derivation of their Hamiltonian properties using the classical R-matrix approach [19]; iii) the solution of the inverse scattering problem for L and iv) proof of the complete integrability of ATFT models.

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