FACTA UNIVERSITATIS
Series: **Physics, Chemistry and Technology** Vol. 4, N^o 2, 2006, pp 247 -259

DIRAC OPERATORS ON KÄHLERIAN MANIFOLDS

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Abstract. *One presents a report concerning the new type of symmetries generated by the covariantly constant Killing-Yano tensors that play the role of complex or hyper-complex structures of the Kählerian manifolds. Such a Killing-Yano tensor produces simultaneously a Dirac-type operator and the generator of a one-parameter Lie group connecting this operator with the standard Dirac one. The group of these continuous transformations can be only* $U(1)$ *or* $SU(2)$*. It is pointed out that the Dirac-type operators given by a hyper-complex structure form a* $\mathcal{N}=4$ *superalgebra whose automorphisms combine isometries with the* $SU(2)$ *transformation generated by the hyper-complex structure.*

Key words: *Kählerian manifolds, Killing-Yano tensors, Dirac-type operators, isometries, symmetries, supersymmetries.*

1. INTRODUCTION

The quantum physics in curved backgrounds uses operators acting on spaces of vector, tensor or spinor fields whose properties depend on the geometry of the manifolds where these objects are defined. A crucial problem is to find the symmetries having geometrical sources and the related operators. The problem is not trivial since, beside the evident geometrical symmetry given by isometries, there are different types of hidden symmetries frequently associated with supersymmetries that deserve to be carefully studied.

Received: 20 August 2005

The isometries are related to the existence of the Killing vectors that give rise to the orbital operators of the scalar quantum theory commuting with that of the free field equation. In the theories with spin these operators get specific spin terms whose form is strongly dependent on the local non-holonomic frames we choose by fixing the gauge [1, 2]. Recently the theory of isometries was extended allowing one to pick up well-defined conserved quantities in theories with matter fields of *any spin* [3, 4].

Another type of geometrical objects related to the so called hidden symmetries or several specific supersymmetries are the Killing-Yano (K-Y) tensors [5] and the Stäckel-Killing (S-K) tensors of any rank. The K-Y tensors play an important role in theories with spin and especially in the Dirac theory on curved spacetimes where they produce first order differential operators, called Dirac-type operators, which anticommute with the standard Dirac one, D [1, 6]. Another virtue of the K-Y tensors is that they enter as square roots in the structure of several second rank S-K tensors that generate conserved quantities in classical mechanics or conserved operators which commute with D. The construction of Ref. [1] depends upon the remarkable fact that the S-K tensors must have square root in terms of K-Y tensors in order to eliminate the quantum anomaly and produce operators commuting with D [7]. These attributes of the K-Y tensors lead to an efficient mechanism of supersymmetry especially when the S-K tensor is proportional with the metric tensor and the corresponding roots are covariantly constant K-Y tensors. Then each tensor of this type, f^i, gives rise to a Dirac-type operator, D^i, representing a supercharge of a non-trivial superalgebra $\{D^i, D^j\} \propto D^2\delta_{ij}$ [8]. It was shown that D^i can be produced by covariantly constant K-Y tensors having not only real-valued components but also complex ones [9, 10, 11].

In what follows we restrict ourselves only to real-valued K-Y tensors which represent complex structures defining Kählerian geometries. The main part of this paper is devoted to the theory of the Dirac-type operators generated by complex structures.

It is known that in four-dimensional manifolds the standard Dirac operator and the Dirac-type ones can be related among themselves through continuous or discrete transformations [12, 10]. It is interesting that there are only two possibilities, namely either transformations of the $U(1)$ group associated with the discrete group $\mathbb{Z}_4$ or $SU(2)$ transformations and discrete ones of the quaternionic group $\mathbb{Q}$ [12, 10, 11]. The first type of symmetry is proper to Kähler manifolds while the second largest one is characteristic for hyper-Kähler geometries [12]. We have shown that, in general, there are no larger symmetries of this type [11] but here we point out how these could be embedded with the isometries.

The paper is organized as follows. We start in the second section with the construction of a simple version of the Dirac theory in manifolds of any dimensions. In the next section we briefly present the theory of the Dirac-type operators produced by K-Y tensors of any rank following to study the special cases of the Dirac-type operators arising in Kählerian manifolds. Moreover, the continuous symmetries of these Dirac operators are analyzed We point out that the triplets of complex structures give rise to triplets of Dirac-type operators, D^i, $i = 1, 2, 3$ anticommuting with D and among themselves too, forming thus a basis of a $\mathcal{N} = 4$ superalgebra. Furthermore, we show that in the case of the hyper-Kähler manifolds, the automorphisms of these superalgebras combine the mentioned $SU(2)$ specific transformations with those of a representation of the group of isometries induced by the group $SO(3)$, of the rotations among the triplet elements.

We note that an extended study of the symmetries and supersymmetries of the Dirac-type operators, including those produced by complex-valued K-Y tensors, will appear in [13].

2. THE DIRAC FIELD IN ARBITRARY DIMENSIONS

The theory of the Dirac spinors in arbitrary dimensions depends on the choice of the manifold and Clifford algebra. In Refs. [11, 13] we present a simple theory of the Dirac field in arbitrary dimensions. Here we keep all the notations introduced there. We consider a $2l + 1$-dimensional pseudo-Riemannian manifold M_{2l+1} whose flat metric $\tilde{\eta}$ (of its pseudo-Euclidean model) has the signature (m_+, m_-) where $m_+ + m_- = m = 2l+1$. This is the *maximal* manifold that can be associated to the $2l + 1$-dimensional Clifford algebra [14] acting on the 2^l-dimensional space Ψ of the complex spinors $\psi = \tilde{\varphi}_1 \otimes \tilde{\varphi}_2 ... \otimes \tilde{\varphi}_l$ built using complex two-dimensional Pauli spinors $\tilde{\varphi}$. In this algebra we start with the standard Euclidean basis formed by the hermitian matrices $\tilde{\gamma}^A = (\tilde{\gamma}^A)^+$ $(A, B, ... = 1, 2, ..., m)$ that obey $\{\tilde{\gamma}^A, \tilde{\gamma}^B\} = 2\delta^{AB}\mathbf{1}$ where $\mathbf{1}$ is the identity matrix. Then it is not difficult to define a new set of gamma matrices such that

$$\{\gamma^A, \gamma^B\} = 2\tilde{\eta}^{AB}\mathbf{1}\,. \tag{1}$$

In this new form, the first m_+ matrices γ^A remain hermitian while the m_- last ones become anti-hermitian. The unitaryness can be restored replacing the usual Hermitian adjoint with the generalized Dirac adjoint [15].

Definition 1 *We say that* $\overline{\psi} = \psi^{+}\gamma$ *is the generalized Dirac adjoint of the field* ψ *if the hermitian matrix* $\gamma = \gamma^{+}$ *satisfies the condition* $(\gamma)^2 = \mathbf{1}$ *and all the matrices* γ^A *are either self-adjoint or anti self-adjoint with respect to this operation, i.e.* $\overline{\gamma}^A = \gamma(\gamma^A)^{+}\gamma = \pm\gamma^A$.

It is clear that the matrix γ play here the role of *metric operator* giving the generalized Dirac adjoint of any square matrix X as $\overline{X} = \gamma X^{+}\gamma$.

The isometry group $G(\tilde{\eta}) = O(m_+, m_-)$ of the metric $\tilde{\eta}$, with the mentioned signature, is the *gauge* group of the theory defining the principal fiber bundle. This is a pseudo-orthogonal group that admits an universal covering group $\mathbf{G}(\tilde{\eta})$ which is simply connected and has the same Lie algebra we denote by $\mathbf{g}(\tilde{\eta})$. The group $\mathbf{G}(\tilde{\eta})$ is the model of the spinor fiber bundle that completes the spin structure we need. In order to avoid complications due to the presence of these two groups we consider here that the basic piece is the group $\mathbf{G}(\tilde{\eta})$, denoting by $[\omega]$ their elements in the standard *covariant* parametrization given by the skew-symmetric real parameters $\omega_{AB} = -\omega_{BA}$. Then the identity element of $\mathbf{G}(\tilde{\eta})$ is $1 = [0]$ and the inverse of $[\omega]$ with respect to the group multiplication reads $[\omega]^{-1} = [-\omega]$.

Definition 2 *We say that the gauge group is the* vector *representation of* $\mathbf{G}(\tilde{\eta})$ *and denote* $G(\tilde{\eta}) = vect[\mathbf{G}(\tilde{\eta})]$. *The representation* $spin[\mathbf{G}(\tilde{\eta})]$ *carried by the space* Ψ *and generated by the spin operators*

$$S^{AB} = \frac{i}{4}\left[\gamma^A, \gamma^B\right] \tag{2}$$

is called the spinor *representation of* $\mathbf{G}(\tilde{\eta})$. *The spin operators are the basis generators of the spinor representation* $spin[\mathbf{g}(\tilde{\eta})]$ *of the Lie algebra* $\mathbf{g}(\tilde{\eta})$.

In what follows we consider the general case of the Dirac theory on any submanifold $M_n \subset M_m$ of dimension $n \le m$ whose flat metric η is a part (or restriction) of the metric $\tilde{\eta}$, having the signature (n_+, n_-), with $n_+ \le m_+$, $n_- \le m_-$ and $n_+ + n_- = n$, such that the gauge group is $G(\eta) = vect[\mathbf{G}(\eta)] = O(n_+, n_-)$. In M_n we choose a local chart (i.e. natural frame) with coordinates x^μ, $\alpha, ..., \mu, \nu, ... = 1, 2, ..., n$, and introduce local orthogonal non-holonomic frames using the gauge fields (or "vilbeins") $e(x)$ and $\hat{e}(x)$, whose components are labeled by local (hated) indices, $\hat{\alpha}, ...\hat{\mu}, \hat{\nu}, ... = 1, 2, ..., n$, that represent a subset of the Latin capital ones, eventually renumbered. The local indices have to be raised or lowered by the metric η. The fields e and $\hat{e}$ accomplish the conditions

$$e^{\mu}_{\hat{\alpha}}\hat{e}^{\hat{\alpha}}_{\nu} = \delta^{\nu}_{\mu}, \quad e^{\mu}_{\hat{\alpha}}\hat{e}^{\hat{\beta}}_{\mu} = \delta^{\hat{\beta}}_{\hat{\alpha}} \tag{3}$$

and orthogonality relations as $g_{\mu\nu}e^{\mu}_{\hat\alpha}e^{\nu}_{\hat\beta}=\eta_{\hat\alpha\hat\beta}$.

The next step is to choose a suitable representation of the n matrices $\gamma^{\hat\alpha}$ obeying Eq. (1) and to calculate the spin matrices $S^{\hat\alpha\hat\beta}$ defined by Eq. (2). Now these are the basis generators of the spinor representation $spin[\mathbf{g}(\eta)]$ of the Lie algebra $\mathbf{g}(\eta)$, corresponding to the metric η. If $n<m$ there are many matrices, $\gamma^{n+1},...,\gamma^{m}$, which anticommutes with all the n matrices $\gamma^{\hat\alpha}$ one uses for the Dirac theory in M_n. We can select one of these extra gamma-matrices denoting it by γ^{ch} and matching its phase factor such that $(\gamma^{ch})^2=\mathbf{1}$ and $(\gamma^{ch})^{+}=\gamma^{ch}$. This matrix obeying

$$\left\{\gamma^{ch},\,\gamma^{\hat\mu}\right\}=0\,,\quad \hat\mu=1,2,...,n\,,\tag{4}$$

is called the *chiral* matrix since it plays the same role as the matrix γ^5 in the usual Dirac theory, helping us to distinguish between even and odd matrices or matrix operators.

The gauge-covariant theory of the free spinor field $\psi\in\Psi$ of the mass m_0, defined on M_n, is based on the gauge invariant action

$$\mathcal{S}[e,\psi]=\int d^nx\sqrt{g}\left\{\frac{i}{2}[\overline{\psi}\gamma^{\hat\alpha}\nabla_{\hat\alpha}\psi-(\overline{\nabla_{\hat\alpha}\psi})\gamma^{\hat\alpha}\psi]-m_0\overline{\psi}\psi\right\}\,,\tag{5}$$

where $g=|\det(g_{\mu\nu})|$ and $\nabla_\mu=\hat e^{\hat\alpha}_{\mu}\nabla_{\hat\alpha}=\tilde\nabla_\mu+\Gamma^{spin}_{\mu}$ are the covariant derivatives formed by the usual ones, $\tilde\nabla_\mu$ (acting in natural indices), and the spin connection

$$\Gamma^{spin}_{\mu}=\frac{i}{2}e^{\beta}_{\hat\nu}(\hat e^{\hat\sigma}_{\alpha}\Gamma^{\alpha}_{\beta\mu}-\hat e^{\hat\sigma}_{\beta,\mu})S^{\hat\nu\,\cdot}_{\cdot\,\hat\sigma}\,,\tag{6}$$

giving $\nabla_\mu\psi=(\partial_\mu+\Gamma^{spin}_{\mu})\psi$. The action (5) produces the Dirac equation $D\psi=m_0\psi$ involving the *standard* Dirac operator that can be expressed in terms of point-dependent Dirac matrices as

$$D=i\gamma^\mu\nabla_\mu\,,\quad \gamma^\mu(x)=e^{\mu}_{\hat\alpha}(x)\gamma^{\hat\alpha}\,.\tag{7}$$

Now we can convince ourselves that our definition of the generalized Dirac adjoint is correct since $\overline{\gamma^\mu}=\gamma^\mu$ and $\overline{\Gamma}^{spin}_{\mu}=-\Gamma^{spin}_{\mu}$ such that the Dirac operator results to be self-adjoint, $\overline{D}=D$. Moreover, the quantity $\overline{\psi}\psi$ has to be derived as a scalar, i.e. $\nabla_\mu(\overline{\psi}\psi)=\overline{\nabla_\mu\psi}\,\psi+\overline{\psi}\,\nabla_\mu\psi=\partial_\mu(\overline{\psi}\psi)$, while the quantities $\overline{\psi}\gamma^\alpha\gamma^\beta...\psi$ behave as tensors of different ranks.

Using the standard notations for the Riemann-Christoffel curvature tensor, $R_{\alpha\beta\mu\nu}$, Ricci tensor, $R_{\alpha\beta}=R_{\alpha\mu\beta\nu}g^{\mu\nu}$, and scalar curvature, $R=$

$R_{\mu\nu}g^{\mu\nu}$, we recover the useful formulas

$$\nabla_\mu(\gamma^\nu\psi) = \gamma^\nu\nabla_\mu\psi\,, \tag{8}$$

$$[\nabla_\mu,\, \nabla_\nu]\psi = \tfrac{1}{4}R_{\alpha\beta\mu\nu}\gamma^\alpha\gamma^\beta\psi\,, \tag{9}$$

and the identity $R_{\alpha\beta\mu\nu}\gamma^\beta\gamma^\mu\gamma^\nu = -2R_{\alpha\nu}\gamma^\nu$ that allow one to calculate

$$D^2 = -\nabla^2 + \tfrac{1}{4}R\,\mathbf{1}\,, \quad \nabla^2 = g^{\mu\nu}\nabla_\mu\nabla_\nu\,. \tag{10}$$

It remains to complete the operator algebra with new observables from which we have to select complete sets of commuting observables for defining quantum modes.

3. DIRAC-TYPE OPERATORS RELATED TO K-Y TENSORS

In the classical theory, the hidden symmetries are arising from more general isometries defined in the whole phase space which cannot be reduced to pure coordinate transformations. In a quantum theory it is interesting to construct new conserved quantities or operators commuting with D, produced by the S-K or K-Y tensors fields. Here new specific mechanisms have to be exploited for analyzing the hidden symmetries or several new types of supersymmetries.

3.1 OPERATORS CONSTRUCTED FROM K-Y TENSORS

The K-Y tensors, $\tilde{f}^{(r)}$, are completely skew-symmetric tensors of rank r for which the Killing equation reads

$$\tilde{f}^{(r)}_{\mu_1\mu_2\ldots(\mu_r;\mu)} \equiv \tilde{f}^{(r)}_{\mu_1\mu_2\ldots\mu_r;\mu} + \tilde{f}^{(r)}_{\mu_1\mu_2\ldots\mu;\mu_r} = 0\,. \tag{11}$$

It was surprising to see that the K-Y tensors are naturally related to the Dirac theory in curved manifolds since all of them are able to produce first-order differential operators which commutes or anticommutes with D.

Theorem 1 *Given a K-Y tensor $\tilde{f}^{(r)}$ of an arbitrary rank $r = 1, 2, \ldots$, the operator*

$$\begin{aligned} Y[\tilde{f}^{(r)}] \;&=\; (-1)^r i\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{r-1}}\Big(\tilde{f}^{(r)\;\;\;\;\;\;\;\;\;\mu_r}_{\mu_1\mu_2\ldots\mu_{r-1}\cdot}\nabla_{\mu_r} \\ &\quad -\frac{1}{2(r+1)}\tilde{f}^{(r)}_{\mu_1\mu_2\ldots\mu_r;\mu}\gamma^{\mu_r}\gamma^\mu\Big) \end{aligned} \tag{12}$$

commute with D if r is odd and anticommute with D if r is even.

Proof: We delegate the proof to Ref. [6]. ∎
In general, one can construct new operators commuting with D using the operators (12) built with the help of arbitrary K-Y tensors. Indeed, given two K-Y tensors of any rank, $\tilde{f}^{(r_1)}$ and $\tilde{f}^{(r_2)}$, the new second order operator $K^{(2)} = \{Y[\tilde{f}^{(r_1)}],\, Y[\tilde{f}^{(r_2)}]\}$ commutes with D whenever $r_1 + r_2$ is an even number. Moreover, in this way we obtain the corresponding factorized S-K tensor of the second rank that gives rise to the operator $K^{(2)}$ freely of quantum anomaly. In this manner one can generate new types of operators that help one to investigate the hidden symmetries and to obtain large sets of conserved operators that may constitute new (super)algebras. In other respects, the implication of the K-Y tensors in the quantum theory suggests us that such tensors with complex-valued components would be also useful even if from the classical viewpoint these are pointless.

Of a particular interest are the operators built with the help of the second rank K-Y tensors, $\tilde{f}$, with real or complex-valued components $\tilde{f}_{\mu\nu} = -\tilde{f}_{\nu\mu}$ which satisfies the equation (11) for $r = 2$.

Definition 3 *The operators*

$$D_{\tilde{f}} = i\gamma^{\mu}\left(\tilde{f}_{\mu\cdot}^{\cdot\,\nu}\nabla_{\nu} - \tfrac{1}{6}\tilde{f}_{\mu\nu;\rho}\gamma^{\nu}\gamma^{\rho}\right), \tag{13}$$

given by the second rank K-Y tensors, $\tilde{f}$, are called Dirac-type operators.

These are non-standard Dirac operators which obey $\{D_{\tilde{f}},\, D\} = 0$ and can be involved in new types of genuine or hidden (super)symmetries. Remarkable superalgebras of Dirac-type operators can be produced by special second-order K-Y tensors that represent square roots of the metric tensor.

3.2 ROOTS AND THEIR DIRAC-TYPE OPERATORS

Let us start with some technical details and the basic definitions. Given ρ an arbitrary tensor field of rank 2 defined on a domain of M_n, we denote with the same symbol $\langle\rho\rangle$ the equivalent matrices with the elements $\rho^{\mu\,\cdot}_{\cdot\,\nu}$ in natural frames and $\rho^{\hat{\alpha}\,\cdot}_{\cdot\,\hat{\beta}} = \hat{e}^{\hat{\alpha}}_{\mu}\rho^{\mu\,\cdot}_{\cdot\,\nu}e^{\nu}_{\hat{\beta}}$ in local frames. We say that ρ is non-singular on M_n if $\det\langle\rho\rangle \neq 0$ on a domain of M_n where the metric is non-singular. This tensor is said irreducible on M_n if its matrix is irreducible.

Definition 4 *The non-singular real or complex-valued K-Y tensor f of rank 2 defined on M_n which satisfies*

$$f^{\mu\,\cdot}_{\cdot\,\alpha}f_{\mu\beta} = g_{\alpha\beta}\,, \tag{14}$$

is called an unit root of the metric tensor of M_n, or simply an unit root of M_n.

It was shown that any K-Y tensor that satisfy Eq. (14) is covariantly constant [9]

$$f_{\mu\nu;\sigma} = 0\,. \tag{15}$$

Since Eq. (14) can be written as $f^{\mu\cdot}_{\cdot\alpha} f^{\alpha\cdot}_{\cdot\nu} = -\delta^{\mu}_{\nu}$ this takes the matrix form

$$\langle f \rangle^2 = -1_n\,, \tag{16}$$

where the notation 1_n stands for the $n \times n$ identity matrix. Hereby we see that the complex structure behave as *complex units* (e.g. $\langle f \rangle^{-1} = -\langle f \rangle$. The complex structures represent automorphisms of the tangent fiber bundle $\mathcal{T}(M_n)$ of M_n. In local frames these appear as particular point-dependent transformations of the gauge group $G(\eta) = vect[\mathbf{G}(\eta)]$.

The K-Y tensor gives rise to Dirac-type operators of the form (13) which have an important property formulated in [9].

Theorem 2 *The Dirac-type operator D_f produced by the K-Y tensor f satisfies the condition*

$$(D_f)^2 = D^2\,. \tag{17}$$

if and only if f is a complex structure.

Proof: The arguments of Ref. [9] show that the condition Eq. (17) is equivalent with Eqs. (14) and (15). Moreover the square of the Dirac-type operator

$$D_f = i f^{\cdot\,\nu}_{\mu\cdot} \gamma^{\mu} \nabla_{\nu}\,, \tag{18}$$

has to be calculated exploiting the identity $0 = f_{\mu\nu;\alpha;\beta} - f_{\mu\nu;\beta;\alpha} = f_{\mu\sigma} R^{\sigma}_{\cdot\,\nu\alpha\beta} + f_{\sigma\nu} R^{\sigma}_{\cdot\,\mu\alpha\beta}$, which gives

$$R_{\mu\nu\alpha\beta} f^{\mu\cdot}_{\cdot\sigma} f^{\nu\cdot}_{\cdot\tau} = R_{\sigma\tau\alpha\beta} \tag{19}$$

and leads to Eq. (17). ∎

Thus we conclude that the equivalence of the condition (17) with Eqs. (14) and (15) holds in any geometry of dimension $n = 2k$ allowing complex structures.

Another interesting operator related to f can be defined as follows.

Definition 5 *Given the complex structure f, the matrix*

$$\Sigma_f = \frac{1}{2} f_{\mu\nu} S^{\mu\nu} \tag{20}$$

is the spin-like operator associated to f.

This is a matrix that acts on the space of spinors Ψ and, therefore, can be interpreted as a generator of the spinor representation $spin[\mathbf{G}(\eta)]$. It has the obvious property $\overline{\Sigma_f} = \Sigma_f$ while from (8) and (14) one obtains that it is covariantly constant in the sense that $\nabla_\nu(\Sigma_f \psi) = \Sigma_f \nabla_\nu \psi$. Hereby we find that the Dirac-type operator (18) can be written as

$$D_f = i\,[D,\, \Sigma_f]\ , \tag{21}$$

where D is the standard Dirac operator defined by Eq. (7). Moreover, one can deduce that $[\Sigma_f,\, D^2] = [\Sigma_f,\, (D_f)^2] = 0$.

With the help of this operator we can built the theory of a symmetry relating D and D_f to each other.

Definition 6 *We say that* $G_f = \{[\rho]\,|\,\rho = \alpha f,\, \alpha \in \mathbb{R}\} \subset [\mathbf{G}(\eta)]$ *is the one-parameter Lie group associated to the complex structure* f.

The spinor representation of this group, $spin(G_f)$, is formed by all the transformation matrices

$$T(\alpha f) = e^{-i\alpha\Sigma_f} \in spin[\mathbf{G}(\eta)] \tag{22}$$

depending on the group parameter $\alpha \in \mathbb{R}$.

3.3 DIRAC-TYPE OPERATORS IN HYPER-KÄHLER MANIFOLDS

A higher symmetry given by a non-abelian Lie group arises in the case of the hyper-Kähler geometries.

Definition 7 *The triplet* $\mathbf{f} = \{f^1, f^2, f^3\}$ *of complex structures which satisfy*

$$\langle f^i\rangle\,\langle f^j\rangle = -\delta_{ij}1_n + \varepsilon_{ijk}\langle f^k\rangle\,, \quad i,j,k... = 1,2,3\,, \tag{23}$$

represents a hyper-complex structure. A hyper-Kähler manifold is a manifold whose metric is Kählerian with respect to each different complex structure f^1, f^2 *and* f^3.

The results we know indicate that the hyper-Kähler manifolds must be of dimension $n = 4k$, $k = 1, 2, 3, \ldots$. Moreover these are a very important feature as given by the following theorem.

Theorem 3 *If a manifold* M_n *allows a triplet of complex structures then this must be Ricci flat (having* $R_{\mu\nu} = 0$*).*

Proof: As in the case of any hyper-Kähler manifold, using Eqs. (19) and (23) we calculate the expression $R_{\mu\nu\alpha\beta}f^{1\,\alpha\beta} = R_{\mu\nu\sigma\beta}f^{3\,\sigma\cdot}_{\,\cdot\alpha}(\langle f^3\rangle\,\langle f^1\rangle)^{\alpha\beta} = R_{\mu\nu\sigma\beta}f^{3\,\sigma\cdot}_{\,\cdot\alpha}f^{2\,\alpha\beta} = -R_{\mu\nu\alpha\beta}f^{1\,\alpha\beta}$ which vanishes. Furthermore, permutating the first three indices of R we find the identity

$$2R_{\mu\alpha\nu\beta}f^{1\,\alpha\beta} = R_{\mu\nu\alpha\beta}f^{1\,\alpha\beta} = 0\,. \tag{24}$$

Finally, writing $R_{\mu\nu} = R_{\mu\alpha\nu\beta}f^{1\,\alpha\cdot}_{\,\cdot\tau}f^{1\,\beta\tau} = -R_{\mu\alpha\sigma\beta}f^{1\,\sigma\cdot}_{\,\cdot\nu}f^{1\,\alpha\beta} = 0$, we draw the conclusion that the manifold is Ricci flat. The same procedure holds for f^2 or f^3 leading to identities similar to (24). Note that the manifolds possessing only single complex structure (as the Kähler ones) are not forced to be Ricci flat. ■

Starting with a triplet $\mathbf{f} = \{f^1, f^2, f^3\}$ satisfying (23) one can construct a rich set of Dirac-type operators of the form $D(\vec{\nu}) = \nu_i D^i$ where $\vec{\nu}$ is an unit vector (with $\vec{\nu}^2 = 1$) and $D^i = D_{f^i} = i[D,\,\Sigma^i]$, $i = 1, 2, 3$, play the role of a *basis*.

3.4 SUPERSYMMETRIES AND ISOMETRIES

Beside the types of continuous symmetries we have studied, the presence of the complex structures gives rise to supersymmetries related to the isometries in an interesting manner.

In a Kähler manifold, a complex structure f generates its own $\mathcal{N} = 2$ *real* superalgebra, $\mathbf{d}_f = \{D(\lambda)\,|\,D(\lambda) = \lambda_0 D + \lambda_1 D_f\}$ where D and D_f (obeying $\{D, D_f\} = 0$, $(D_f)^2 = D^2$) form a basis.

The case of the hyper-Kähler manifolds is more complicated since a triplet $\mathbf{f}$ gives rise to self-adjoint Dirac-type operators $D^i = \overline{D}^i$ which anticommute with D and present the continuous symmetry discussed in the previous section. In these conditions a new algebraic structure is provided by

Theorem 4 *If a triplet* $\mathbf{f}$ *accomplishes Eqs. (23) then the corresponding Dirac-type operators satisfy*

$$\left\{D^i,\,D^j\right\} = 2\delta_{ij}D^2\,,\quad \left\{D^i,\,D\right\} = 0\,. \tag{25}$$

Proof: If $i = j$ we take over the result of Theorem 2. For $i \neq j$ we take into account that M_n is Ricci flat finding that D^i and D^j anticommute. The second relation was demonstrated earlier for any complex structure. ■
Thus it is clear that the operators D and D^i ($i = 1, 2, 3$) form a basis of a four-dimensional real superalgebra of Dirac operators.

Definition 8 *The set* $\mathbf{d_f} = \{D(\lambda)|D(\lambda) = \lambda_0 D + \lambda_i D^i; \lambda_0, \lambda_i \in \mathbb{R}\}$ *is the* $\mathcal{N} = 4$ *superalgebra generated by the triplet* $\mathbf{f}$.

This superalgebra contains the subset $\mathbf{d}_\mathbf{f}^1 = \{D(\nu)\,|\,{\nu_0}^2 + \vec{\nu}^2 = 1\}$ of the Dirac operators which have the property $D(\nu)^2 = D^2$.

Furthermore, it is natural to study the group of automorphisms of this superalgebra, $Aut(\mathbf{d_f})$, and its Lie algebra, $aut(\mathbf{d_f})$. Obviously, these automorphisms have to be linear transformations among D and D^i preserving their anticommutation rules. These supplemental automorphisms must transform the operators D^i among themselves preserving their anticommutators as well as the form of D. Therefore, these may be produced by the isometries of M_n since these leave the operator D invariant.

Theorem 5 *Let* M_n *be a hyper -Kähler manifold having the hypercomplex structure* $\mathbf{f} = \{f^1, f^2, f^3\}$ *and a non-trivial isometry group* $I(M_n)$ *with the corresponding Killing vectors* $k_a(x)$. *Then the basis-generators*

$$X_a = -ik_a^\mu \nabla_\mu + \frac{1}{2} k_{a\,\mu;\nu} S^{\mu,\nu}, \tag{26}$$

of the spinor representation of $I(M_n)$ *and* $\hat{s}_i = \frac{1}{2}\Sigma^i \in spin(g_\mathbf{f}) \sim su(2)$ *satisfy*

$$[X_a,\, \hat{s}_i] = i\hat{c}_{aij}\hat{s}_j\,, \quad a = 1, 2, ..., N\,, \tag{27}$$

where $\hat{c}_{aij}$ *are* point-independent *structure constants.*

Proof: We delegate the proof to Ref. [13]. ■

An important consequence of this theorem is given by

Corollary 1 *The basis generators* X_a *and the Dirac-type operators of the* $\mathcal{N} = 4$ *superalgebra* $\mathbf{d_f}$ *obey*

$$[X_a, D^i] = i\hat{c}_{aij} D^j\,. \tag{28}$$

Proof: This formula results commuting Eq. (27) with D. ■

Acknowledgments

M. V. is grateful to the organizers of the BW2005 meeting, in particular to Goran Djordjevic, for hospitality extended to him. This research was supported in part by Romanian MEC-AEROSPATIAL Program.

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