FACTA UNIVERSITATIS
Series: **Physics, Chemistry and Technology** Vol. 4, N^o 2, 2006, pp 233 -246

GEOMETRY OF THE FUZZY DOUGHNUT

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Abstract. *The noncommutative extension of a dynamical 2-dimensional space-time is given and some of its properties discussed. Wick rotation to euclidean signature yields a surface which has as commutative limit the donut but in a singular limit in which the radius of the hole tends to zero.*

Key words: *Noncommutative geometry*

1. INTRODUCTION AND NOTATION

There is a very simple argument due to Pauli that the quantum effects of a gravitational field will in general lead to an uncertainty in the measurement of space coordinates. It is based on the observation that two 'points' on a quantized curved manifold can never be considered as having a purely space-like separation. If indeed they had so in the limit for infinite values of the Planck mass, then at finite values they would acquire for 'short time intervals' a time-like separation because of the fluctuations of the light cone. Since the 'points' are in fact a set of four coordinates, that is scalar fields, they would not then commute as operators. This effect could be considered important at least at distances of the order of Planck length, and perhaps greater. This is one motivation to study noncommutative geometry. A second motivation, which is the one we consider ours, is the fact that it is possible to study noncommutative differential geometry, and there is no reason to assume that even classically coordinates commute at all length scales. One can consider for example coordinates as order parameters as in solid-state physics and suppose that singularities in the gravitational field become analogs of

Received: 15 August 2005

core regions; one must go beyond the classical approximation to describe them. A straightforward and conservative way is to represent coordinates by operators. The space-time manifold is thus replaced with an algebra generated by a set of noncommutative 'coordinates'. The essential element which allows us to interpret a noncommutative algebra as a space-time is the possibility [1, 2] to introduce a differential structure on the former.

We define noncommutative 'space' as an associative $*$-algebra $\mathcal{A}$ generated by a set of hermitean 'coordinates' x^i which in some limit tend to the (real) coordinates $\tilde{x}^i$ of a manifold; the latter we identify as the classical limit of the geometry. We suppose that the center of the algebra $\mathcal{A}$ is trivial. The coordinates satisfy a set of commutation relations

$$[x^i, x^j] = i\bar{k} J^{ij}(x^k). \tag{1}$$

The parameter $\bar{k}$ is introduced to describe the fundamental area scale on which noncommutativity becomes important. It is presumably of order of the Planck area $G\hbar$; the commutative limit is defined by $\bar{k} \to 0$. The simplest relation which can be used to define the algebra is

$$\left[x^i, x^j\right] = i\bar{k} J^{ij} \tag{2}$$

where J^{ij} are real numbers; it is called the canonical structure.

In order to define the differential structure on $\mathcal{A}$ we use a noncommutative version of Cartan's frame formalism [3]. In ordinary geometry a vector field can be defined as a derivation of the algebra of smooth functions; this definition can be used also when the algebra is noncommutative. A derivation, we recall, is a linear map which satisfies the Leibniz rule; sometimes this is modified to a 'twisted' Leibniz rule, [4, 5]). The set of all derivations we denote by $\mathrm{Der}(\mathcal{A})$. The classical notion of the moving frame is generalized in the following way. The frame on $\mathcal{A}$ is a set of n inner derivations e_a, generated by 'momenta' p_a:

$$e_a f = [p_a, f]. \tag{3}$$

We assume that the momenta also generate the whole algebra $\mathcal{A}$. For inner derivations, the Leibniz rule is the Jacobi identity. An alternative way to define the frame is to use the 1-forms θ^a dual to e_a such that the relation

$$\theta^a(e_b) = \delta^a_b \tag{4}$$

holds. The module of 1-forms we denote by $\Omega^1(\mathcal{A})$. To define the left hand side of the equation (4), that is the basic forms θ^a, we first define the differential, exactly as in the classical case, by the condition

$$df(e_a) = e_a f, \tag{5}$$

and the multiplication of 1-forms by elements of the algebra $\mathcal{A}$ by

$$f dg = f e_a g \theta^a, \qquad dg f = e_a g f \theta^a. \tag{6}$$

Since every 1-form can be written as sum of such terms, the definition of differential is complete. In particular, since

$$f\theta^a(e_b) = f\delta^a_b = (\theta^a f)(e_b), \tag{7}$$

we conclude that the frame necessarily commutes with all the elements of the algebra $\mathcal{A}$. The 1-form θ defined as

$$\theta = -p_a\theta^a \tag{8}$$

can be considered as an analog of the Dirac operator in ordinary geometry. It implements the action of the exterior derivative on elements of the algebra. That is

$$df = -[\theta, f] = [p_a\theta^a, f] = [p_a, f]\,\theta^a. \tag{9}$$

In the case of the canonical commutation rule (2) for example, the frame is $\theta^a = \delta^a_i dx^i$. From duality we obtain that the momenta are

$$p_a = \frac{1}{i\bar{k}} J^{-1}_{ai} x^i. \tag{10}$$

The equation (10) gives 'Fourirer transformation' between the coordinates and the momenta in this case. The momenta are singular in the limit $\bar{k} \to 0$.

The differential is real if $(df)^* = df^*$. This is assured if the derivations e_a are real: $e_a f^* = (e_a f)^*$, which is the case if the momenta p_a are antihermitean.

Let us mention further properties of the module structure defined by (5-7). The exterior product is a map from the tensor product of two copies of the module of 1-forms into the module of 2-forms; we shall identify the latter as a subset of the former and write the product as

$$\theta^a\theta^b = P^{ab}{}_{cd}\theta^c \otimes \theta^d. \tag{11}$$

The $P^{ab}{}_{cd}$ are complex numbers which satisfy the projector condition and the hermiticity [6]:

$$P^{ab}{}_{cd}P^{cd}{}_{ef} = P^{ab}{}_{ef}, \qquad \bar{P}^{ab}{}_{cd}P^{dc}{}_{ef} = P^{ba}{}_{ef}. \tag{12}$$

The basis 1-forms anticommute for $P^{ab}{}_{cd} = \frac{1}{2}\,(\delta^a_c\delta^b_d - \delta^b_c\delta^a_d)$. The exterior derivative of θ^a is a 2-form,

$$d\theta^a = -\frac{1}{2}C^a{}_{bc}\,\theta^b\theta^c. \tag{13}$$

The $C^a{}_{bc}$ are called the structure elements. They can be chosen to satisfy

$$C^a{}_{bc}P^{bc}{}_{de} = C^a{}_{de}. \tag{14}$$

The relation $d^2 = 0$ and the consistency of the relation (7) with the differential, $d(f\theta^a - \theta^a f) = 0$, have nontrivial consequences. The structure elements are linear in the momenta

$$C^a{}_{bc} = F^a{}_{bc} - 2p_d P^{(ad)}{}_{bc}. \tag{15}$$

Furthermore, the momenta obey a quadratic relation

$$2p_c p_d P^{cd}{}_{ab} - p_c F^c{}_{ab} - K_{ab} = 0. \tag{16}$$

The $F^a{}_{bc}$ and K_{ab} are complex numbers which can be chosen to satisfy

$$F^a{}_{bc}P^{bc}{}_{de} = F^a{}_{de}, \qquad K_{ab}P^{ab}{}_{ef} = K_{ef}. \tag{17}$$

From (15) it follows immediately that

$$e_a C^a{}_{bc} = 0. \tag{18}$$

This relation must be also satisfied in the commutative limit and constitutes a constraint on the frame. A frame has four degrees of freedom in two dimensions; the constraint subtracts one therefrom.

2. FUZZY DOUGHNUT

Having outlined the main features of the frame formalism, let us discuss it in a simple 2-dim case. Clearly, every choice of a frame θ^a implements a different differential structure. On the other hand, the conditions (15-16) constrain the possible choices quite rigidely. This can easily be seen in low dimensions: one can readily 'solve' a family of 2-dim metrics with one Killing vector. We shall exhibit all possible frames which yield differential calculi based on inner derivations. As a frame we choose

$$\theta^0 = f(x)dt, \quad f > 0, \qquad \theta^1 = dx. \tag{19}$$

The frame relations can be written as

$$\begin{aligned} dx\,x &= x\,dx, \qquad & dx\,t &= t\,dx, \\ dt\,x &= x\,dt, \qquad & dt\,t &= (t + i\bar{k}F)dt, \end{aligned} \tag{20}$$

and imply $dJ^{01} = 0$. We have set

$$F = J^{01}\frac{d}{dx}\log f. \tag{21}$$

The differential structure of the algebra can be written as

$$(dx)^2 = 0, \quad dx\,dt = -dt\,dx, \qquad (dt)^2 = -\tfrac{1}{2}i\bar{k}F'dx\,dt \tag{22}$$

or as the relations

$$(\theta^1)^2 = 0, \quad \theta^0\theta^1 = -\theta^1\theta^0, \tag{23}$$

$$(\theta^0)^2 = \tfrac{1}{2}i\bar{k}fF'\theta^0\theta^1 = 2i\epsilon\theta^0\theta^1. \tag{24}$$

We have introduced here a parameter ϵ define by

$$\epsilon = \bar{k}\mu^2, \qquad \mu^2 = \tfrac{1}{4}fF'. \tag{25}$$

It follows from the frame properties that the mass scale μ is a constant.

Suppose now that the dual momenta exist. The duality relations are

$$\begin{aligned} [p_0, t] &= f^{-1}, \qquad & [p_0, x] &= 0, \\ [p_1, t] &= 0, \qquad & [p_1, x] &= 1. \end{aligned} \tag{26}$$

These relations allow us to identify p_1 with the partial derivative with respect to x. If $\phi = \phi(x)$ then

$$[p_1, \phi] = [p_1, x]\partial_x\phi = \partial_x\phi. \tag{27}$$

On the other hand, for $\phi = \phi(t, x)$ we can write to first order

$$[p_0, \phi] = [p_0, t]\partial_t\phi = f^{-1}\partial_t\phi. \tag{28}$$

If we denote $[p_0, p_1] = L_{01}$, the Jacobi identities imply the relations

$$\begin{aligned} &[p_0, J^{01}] = 0, \qquad &&[p_1, J^{01}] = 0, \\ &[t, L_{01}] = -f' f^{-2}, \qquad &&[x, L_{01}] = 0. \end{aligned} \tag{29}$$

One can conclude again that J^{01} is constant and also that L_{01} is a function of x alone. We set $J^{01} = 1$. It follows that, neglecting the integration constants, the 'Fourier transformation' between the position and momentum generators is given by

$$p_0 = -\frac{1}{i\bar{k}} \int f^{-1}, \qquad p_1 = -\frac{1}{i\bar{k}}\, t. \tag{30}$$

Each of the pairs (t, x) and (p_0, p_1) generates the algebra.

The array $P^{ab}{}_{cd}$ we write as

$$P^{ab}{}_{cd} = \tfrac{1}{2}\delta^{[a}_c \delta^{b]}_d + i\epsilon Q^{ab}{}_{cd}. \tag{31}$$

In dimension two, if we assume that metric depends on x, that is on p_0 only, we find that

$$P^{ab}{}_{cd} p_a p_b = \frac{1}{2}[p_c, p_d] + i\epsilon Q^{00}{}_{cd} p_0^2 \tag{32}$$

and therefore L_{01} is given by

$$L_{01} = K_{01} + p_0 F^0{}_{01} - 2i\epsilon p_0^2 Q^{00}{}_{01}. \tag{33}$$

The structure elements are

$$C^0{}_{01} = F^0{}_{01} - 4i\epsilon p_0 Q^{00}{}_{01}. \tag{34}$$

Symmetry and reality of the product (12) imply that $Q^{ab}{}_{cd}$ has the following non-vanishing elements:

$$Q^{10}{}_{00} = -Q^{01}{}_{00} = 1, \qquad Q^{00}{}_{01} = -Q^{00}{}_{10} = 1. \tag{35}$$

We set also

$$K_{01} = \frac{1}{i\bar{k} J^{01}} = \frac{1}{i\bar{k}}, \qquad F^0{}_{01} = -ib\mu, \tag{36}$$

while $C^0{}_{10}$ is determined by the constraint

$$C^0{}_{ab} P^{ab}{}_{01} = C^0{}_{01}, \qquad C^0{}_{01} + C^0{}_{10} = -2i\epsilon C^0{}_{00}. \tag{37}$$

We have then finally the expressions

$$L_{01} = (i\bar{k})^{-1}(1 - b\mu^{-1}(i\epsilon p_0) - 2\mu^{-2}(i\epsilon p_0)^2), \tag{38}$$

$$C^0{}_{01} = -ib\mu - 4i\epsilon p_0, \tag{39}$$

and a differential equation for p_0:

$$-i\epsilon \frac{dp_0}{dx} = \mu^2 - i\epsilon b\mu p_0 - 2(i\epsilon p_0)^2. \tag{40}$$

There are three cases to be considered. The simplest is the case with $\mu^2 \to \infty$. The equation (40) reduces to

$$-i\bar{k}\,\frac{dp_0}{dx} = 1. \tag{41}$$

One finds the relations

$$i\bar{k}p_0 = -x, \qquad f(x) = 1. \tag{42}$$

This is noncommutative Minkowski space.

An equally degenerate case is the case $\mu^2 \to \infty$ and $\epsilon b = c\mu$. Equation (40) can be written in the form

$$-i\bar{k}\,\frac{dp_0}{dx} = 1 - icp_0. \tag{43}$$

One finds the solution

$$ip_0 = c^{-1}(e^{-\bar{k}^{-1}cx} - 1), \qquad f(x) = e^{\bar{k}^{-1}cx}. \tag{44}$$

This is noncommutative de Sitter space; it can be brought to the usual form by the change of variables

$$t' = 2t, \qquad \mu x' = 2c^{-1}(e^{-cx} - 1). \tag{45}$$

The case which interests us the most is that with μ finite. With $b = 0$ (that is, with $F^a{}_{bc} = 0$) the equation (40) becomes

$$-i\epsilon\,\frac{dp_0}{dx} = \mu^2 - 2(i\epsilon p_0)^2. \tag{46}$$

If we fix $\beta^2 = 2\mu^2 > 0$, the equation for p_0 becomes

$$\frac{1}{\beta}\frac{d}{dx}\left(-2i\epsilon\beta^{-1}p_0\right) = 1 - \left(-2i\epsilon\beta^{-1}p_0\right)^2. \tag{47}$$

The solution to this equation is given by

$$i\bar{k}p_0 = -\beta^{-1}\tanh(\beta x), \tag{48}$$

with

$$f(x) = \cosh^2(\beta x) \tag{49}$$

and

$$F = -2i\beta^2\bar{k}p_0 = 2\beta\tanh(\beta x). \tag{50}$$

We find therefore the identity

$$F' + F^2 = f^{-1}f'' = 2\beta^2(1 + \tanh^2(\beta x)). \tag{51}$$

The frame corresponding to this solution is given by

$$\theta^0 = \cosh^2(\beta x)dt = \frac{1}{2}(1 + \cosh(2\beta x))dt, \qquad \theta^1 = dx. \tag{52}$$

Frames of similar type have appeared [7, 8, 9] in 2-dimensional dilaton gravity. The connection and the curvature of the analogous commutative moving frame are

$$\tilde{\omega}^0{}_1 = \tilde{\omega}^1{}_0 = F\tilde{\theta}^0, \tag{53}$$

$$\tilde{\Omega}^0{}_1 = \tilde{\Omega}^1{}_0 = -(F' + F^2)\theta^0\theta^1 = -f^{-1}f''\tilde{\theta}^0\tilde{\theta}^1. \tag{54}$$

The solution is a completely regular manifold of Minkowski signature. In the limit $\beta \to 0$

$$i\bar{k}p_0 = -x, \qquad f = 1, \tag{55}$$

one finds Minkowski space. In 'tortoise' coordinate x^*, $x^* = \int \frac{dx}{f(x)}$, the frame is given by

$$\theta^0 = \frac{1}{1 - x^{*2}}\,dt, \qquad \theta^1 = \frac{1}{1 - x^{*2}}\,dx^*. \tag{56}$$

From (30) we see that $x^* = -i\bar{k}p_0$.

Under a Wick rotation

$$u = 2i\beta x, \qquad v = t, \tag{57}$$

the frame (19) becomes

$$\theta^0 = \frac{1}{2}(1 + \cos u)dv, \qquad \theta^1 = \frac{1}{2i\beta}\,du, \tag{58}$$

and the corresponding commutative line element has the form

$$d\tilde{s}^2 = \frac{1}{4}(1+\cos\tilde{u})^2 d\tilde{v}^2 + \frac{1}{4}\beta^{-2}d\tilde{u}^2. \tag{59}$$

This is the surface of the torus embedded in $\mathbf{R}^3$:

$$\tilde{x} = \frac{1}{2}(1+\cos\tilde{u})\cos\tilde{v}, \quad \tilde{y} = \frac{1}{2}(1+\cos\tilde{u})\sin\tilde{v}, \quad \tilde{z} = \frac{1}{2}\beta^{-1}\sin\tilde{u}, \tag{60}$$

and for this reason we call this metric the 'fuzzy doughnut'. It is a singular axially-symmetric surface of Gaussian curvature

$$\tilde{K} = 2\beta^2(1-\tan^2\frac{1}{2}\tilde{u}). \tag{61}$$

The doughnut is defined by the coordinate range $0 \leq \tilde{u} \leq 2\pi$, $0 \leq \tilde{v} \leq 2\pi$, with a singularity at the point $\tilde{u} = \pi$. In spite of the singularity, the Euler characteristic is given by

$$e[\mathcal{A}] = \frac{1}{4\pi}\epsilon_{ab}\int \tilde{\Omega}^{ab} = -\frac{1}{2\pi}\int \tilde{\Omega}^0{}_1 = -\frac{1}{2\pi}\int d\tilde{\omega}^0{}_1 = 0 \tag{62}$$

as it should be. If we suppose the same domain in the Wick rotated real-t region, then

$$0 \leq x \leq \beta^{-1}\pi, \qquad 0 \leq t \leq 2\pi. \tag{63}$$

As $\beta \to \infty$ the doughnut becomes more and more squashed, and this domain becomes an elementary domain in the limiting Minkowski space.

3. CONCLUSIONS

Several models have been found which illustrate a close relation between noncommutative geometry in its 'frame-formalism' version and classical gravity. Heuristically, but incorrectly, one can formulate the relation by stating that gravity is the field which appears when one quantizes the coordinates much as the Schrödinger wave function encodes the uncertainty resulting from the quantization of phase space.

The first and simplest of these is the fuzzy sphere [10] which is a noncommutative geometry which can be identified with the 2-dimensional (euclidean) 'gravity' of the 2-sphere. The algebra in this case is an $n \times n$ matrix algebra; if the sphere has radius r then the parameter r/n can be interpreted as a lattice length. With the identification this model illustrates how gravity can act as an ultraviolet cutoff, a regularization which is very similar to the 'point splitting' technique which has been used when quantizing a field in classical curved backgrounds. It can also be compared with the screening of electrons in plasma physics, which gives rise to a Debye length proportional to the inverse of the electron-number density n. The analogous 'screening' of an electron by virtual electron-positron pairs is responsible for the reduction of the electron self-energy from a linear to logarithmic dependence on the classical electron radius. Other models have been found which illustrate the identification including an infinite series in all dimensions.

In the present paper yet another model is given, one which although representing a classical manifold of dimension 2 is of interest because the classical 'gravity' which arises has a varying Gaussian curvature. The authors will leave to a subsequent article the delicate task of explaining exactly which property of the metric makes it 'quantizable'. This geometry could furnish a convenient model to study noncommutative effects, for example in the colliding-D-brane description of the Big-Bang proposed by Turok & Steinhardt [11]. The 2-space describing the time evolution of the separation of the branes has been shown to be conveniently described using Rindler coordinates. One can blur this geometry by using the metric and connection described here. The flat geometry would have to be replaced by the one given in this section; in the limit $q \to 0$ it would become flat.

The doughnut example is of importance in that is is the first explicit construction of an algebra and differential calculus which is singularity-free in the Minkowski-signature domain and which has a non-constant curvature. There are two aspects of this problem. To construct a classical manifold from a differential calculus is relatively simple once one has constructed the frame. One takes formally the limit and uses the so constructed moving frame to define the metric. This is contained in the upper right of the following little diagram

$$
\begin{tikzcd}
\begin{matrix}\text{Fuzzy}\\ \text{Frame}\end{matrix} \arrow[r] \arrow[d] & \begin{matrix}\text{Classical}\\ \text{Frame}\end{matrix} \arrow[d] \\
\begin{matrix}\textit{Fuzzy}\\ \textit{Geometry}\end{matrix} \arrow[r] & \begin{matrix}\text{Classical}\\ \text{Geometry}\end{matrix}
\end{tikzcd}
\tag{64}
$$

More difficult is the construction of a 'fuzzy geometry' which would fill in the lower left of the diagram and would be such that the classical geometry is a limit thereof. But this step is very important since it gives an extension of the right-hand side into what could eventually be a domain of quantum geometry. It is the box in the to-be-constructed lower left corner where possibly one can find an interesting extension of the metric containing correction terms which describe the noncommutative structure.

We have not succeeded however to completely extend this geometry to all orders in the noncommutativity parameter $i\epsilon$. This will be considered in a subsequent article. There is evidence that the extension will involve a non-vanishing value of the torsion 2-form. The metric is extended into the noncommutative domain so as to maintain such formal properties as reality and symmetry. The interpretation however as a length requires more attention when the 'coordinates' do not commute.

Last, but not least, our example illustrates even better than the fuzzy sphere the way in which quantum mechanics is modified by geometry and the important role which *noncommutative* geometry plays in understanding the relation between the two. The 'momenta' which we introduce are the natural curved-space generalization of the canonical momentum operators of ordinary quantum mechanics. In the present formalism they generate the algebra as well as do the coordinates. Once the algebra is given the noncommutative structure of space-time is manifest in the commutation relations $[x^i, x^j]$ and the appropriate curved-space version of quantum mechanics is defined by the relations (3). The two structures are intimately enmeshed by the Fourier transform as well as Jacobi identities. If the right-hand side of (3) reduces to the Kronecker symbol when $f = x^i$ then the space is flat; because of the Jacobi identities only in this case can quantum mechanics be consistent with a commutative space-time structure.

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