DIFFERENCE DISCRETE AND FRACTIONAL VARIATIONAL PRINCIPLES

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Abstract. The discrete Hamiltonian formulation of Lagrangian possessing linear velocities is investigated and the equivalence of Hamilton and Euler-Lagrange equations is obtained. The fractional path integral of damped harmonic oscillator is analyzed in details.

Key words: difference equations, fractional calculus, variational principles

1. INTRODUCTION

The discrete variational theory was subjected to an intense debate during the last years [1, 2]. In [1] the possibility that time can be considered as a discrete dynamical variable was analyzed for classical mechanics and relativistic quantum field theory and the conservation laws of difference equations were investigated in this theory [2]. Recently, the discrete difference theory of constrained systems was found to be a powerful approach in quantum gravity [3]. Besides, the consistent discretization approach [3] to general relativity leaving the spatial slices continuous was analyzed very recently in [3]. An important issue is to construct the discrete canonical Hamiltonian in the presence of primary constraints. This procedure, if we would like to mimic the continuous case construction, will lead us to introduce Lagrange multipliers. In this paper the discrete Euler-Lagrange equations were obtained and the equivalence with Hamilton’s equations is discussed.

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The generalization of the concept of derivative and integral to a non-
integer order \(\alpha\) has been subjected to several approaches and some various
alternative definitions of fractional derivatives appeared. During the last
three decades the fractional calculus has become a field of growing interest
because of its multiple and interesting applications in several fields such as
mathematics, physics and engineering [4]. An important issue is to quantize
the fractional Hamiltonian corresponding to a given fractional Lagrangian
and to find its corresponding path integral [5, 6, 7, 8, 9, 10]. In this study
the Riewe’s formalism is generalized by considering a Lagrangian involving
the Lagrange multipliers. Besides, the corresponding Hamiltonian equations
are obtained.

2 DIFFERENCE DISCRETE VARIATIONAL PRINCIPLES

The main aim of this section is to investigate the singular discrete La-
grangians and the corresponding discrete Hamiltonians when the Hessian
matrix has rank zero.

The starting point is the continuous action corresponding to a Lan-
grangian possessing only the linear velocities

\[
A = \int \left[ a_i(q^1, q^2, ..., q^m)\dot{q}^i - V(q^1, q^2, ..., q^m) \right] dt,
\]

where the \(q^i, \ i = 1, 2, ..., m\) depend only on time and \(V\) is the potential.

We will consider the following discrete counterpart of (1)

\[
A_D = \sum_{n=1}^{N} \sum_{i=1}^{m} a_i(q^1_n, q^2_n, ..., q^m_n) \frac{\Delta q^i_n}{\Delta t_{n-1}} - V(q^1_n, q^2_n, ..., q^m_n) \Delta t_{n-1},
\]

where \(q^i = q^i_n, \ \dot{q}^i = \frac{\Delta q^i_n}{\Delta t_{n-1}}, \ q^i_n = q^i(t_n)\) and \(\Delta q^i_n = q^i_n - q^i_{n-1}\).

Using (2) we obtain the Euler-Lagrange equations [12, 13] as follows

\[
\sum_{i=1}^{m} \frac{\partial a_i}{\partial q^i_n} \frac{\Delta q^i_{n-1}}{\Delta t_{n-1}} - \frac{\partial V}{\partial q^i_n} = \frac{\Delta a_j}{\Delta t_{n-1}}, \ j = 1, 2, ..., m.
\]

In the following we analyze an example described by the following action

\[
\int \left[ (q^1 + q^2)^2 + \frac{1}{2}(q^3)^2 - \frac{1}{2}(q^2)^2 \right] dt.
\]
We will consider the following discrete counterpart of the action (4) as follows

\[ A_D = \sum_{n=1}^{N} \left( \frac{\Delta q^1_n}{\Delta t_{n-1}} + \frac{\Delta q^2_n}{\Delta t_{n-1}} \right) q^3_n + \frac{1}{2} \left( \frac{\Delta q^3_n}{\Delta t_{n-1}} \right)^2 - \frac{1}{2} \left( q^2_n \right)^2 \Delta t_{n-1}. \] (5)

The discrete Euler-Lagrange equations for (5) are given by

\[ \frac{\Delta q^3_n}{\Delta t_{n-1}} = 0, \] (6)

\[ \frac{\Delta q^3_n}{\Delta t_{n-1}} = -q^2_n, \] (7)

\[ \frac{\Delta q^3_n}{\Delta t_{n-1}} = \frac{\Delta q^1_n}{\Delta t_{n-1}} + \frac{\Delta q^2_n}{\Delta t_{n-1}}, \] (8)

where \( n = 1, 2, ..., N - 1. \)

After some calculation the solutions of the discrete Euler-Lagrange equations (6), (7) and (8) become

\[ q^1_n = c_1, \quad q^2_n = 0, \quad q^3_n = c_2, \] (9)

where \( c_1 \) and \( c_2 \) are constants.

The next step is to define the corresponding discrete Hamiltonian. The canonical momenta are

\[ z^j_{n-1} = \frac{\partial L_D}{\partial \left( \frac{\Delta q^j_n}{\Delta t_{n-1}} \right)}, \quad j = 1, 2, 3. \] (10)

By using (10) we obtain

\[ z^1_{n-1} = q^3_n, \quad z^2_{n-1} = q^3_n, \quad z^3_{n-1} = \frac{\Delta q^3_n}{\Delta t_{n-1}}. \] (11)

Having in mind that

\[ \frac{\partial z^j_{n-1}}{\partial \Delta q^j_n} = \frac{\partial^2 L_D}{\partial \Delta q^j_n \partial \Delta q^j_n}, \] (12)
is a matrix with rank < 3 we conclude that a discrete Legendre transformation is not valid. To bypass this problem we denote \( \lambda^j_{n-1} = \frac{\Delta q^j_{n-1}}{\Delta t_{n-1}} \), \( j = 1, 2, 3 \) and define the discrete Hamiltonian as

\[
H_D = \sum_{i=1}^{3} z^i_{n-1} \lambda^i_{n-1} - L_D. \tag{13}
\]

The form of discrete Hamiltonian becomes

\[
H_D = \lambda^1_{n-1}(z^1_{n-1} - q^3_n) + \lambda^2_{n-1}(z^2_{n-1} - q^3_n) + \lambda^3_{n-1}(z^3_{n-1} - \frac{1}{2} \lambda^3_{n-1}) + \frac{1}{2} (q^2_n)^2. \tag{14}
\]

The canonical equations of motions are

\[
\frac{\Delta z^j_{n-1}}{\Delta t_{n-1}} = -\frac{\partial H}{\partial q^j_n}, \quad j = 1, 2, 3 \tag{15}
\]

By inspection we observed that (15) together with (12) are equivalent to the Euler-Lagrange equations (6), (7) and (8) respectively.

### 3 FRACTIONAL EULER-LAGRANGE AND HAMILTONIAN EQUATIONS

As it is very well known the left Riemann-Liouville fractional derivative is defined as follows

\[
a D^\alpha_t f(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n t \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau, \tag{16}
\]

and the corresponding right Riemann-Liouville fractional derivative is given by

\[
b D^\alpha_t f(t) = \frac{1}{\Gamma(n-\alpha)} \left( -\frac{d}{dt} \right)^n b \int_t^b (\tau-t)^{n-\alpha-1} f(\tau) d\tau. \tag{17}
\]

Here the order \( \alpha \) fulfills \( n - 1 \leq \alpha < n \) and \( \Gamma \) denotes the Euler’s Gamma function. For \( \alpha \) being integer, these derivatives are defined in the usual sense, i.e.,

\[
a D^\alpha_t f(t) = \left( \frac{d}{dt} \right)^\alpha, \quad b D^\alpha_t f(t) = \left( -\frac{d}{dt} \right)^\alpha, \quad \alpha = 1, 2, \ldots. \tag{18}
\]
Agrawal investigated the Euler-Lagrange equations for fractional variational problems [6]. In the following we summarize his approach.

Let us consider the action function

$$S[q^1_0, ..., q^R_0] = \int_a^b L(\{q^r_n, Q^r_n\}, t) dt,$$

subject to the independent constraints

$$\Phi_m(t, q^1_0, ..., q^R_0, q^r_n, Q^r_n) = 0, \ m < R.$$  

Here the generalized coordinates are defined as follows

$$q^r_n = (aD^\alpha_t)^n x_r(t), \ Q^r_n = (D^\beta_t)^n x_r(t),$$

Then, the necessary condition for the curves $q^1_0, ..., q^R_0$ with the boundary conditions

$q^r_0(a) = q^a_0, \ q^r_0(b) = q^b_0, r = 1, 2, ..., R,$

to be an extremal of the functional given by equation (19) is that the functions $q^0_n$ satisfy the following Euler-Lagrange equations [6]:

$$\frac{\partial \bar{L}}{\partial q^r_n} + \sum_{n=1}^N (aD^\alpha_t)^n \frac{\partial \bar{L}}{\partial q^r_k} + \sum_{n'=1}^{N'} (D^\beta_t)^n' \frac{\partial \bar{L}}{\partial Q^r_{n'}} = 0,$$

where $\bar{L}$ is given by [6]

$$\bar{L}(\{q^r_n, Q^r_n\}, t, \lambda_m(t)) = L(\{q^r_n, Q^r_n\}, t) + \lambda_m(t)\Phi_m(t, q^1_0, ..., q^R_0, q^r_n, Q^r_n).$$

Here the multiple $\lambda_m(t) \in R:\alpha$ represent continuous functions on $[a, b].$

The next step is to obtain the Hamilton’s equations for the fractional variational problems. For these reasons we re-define the left and the right canonical momenta as follows

$$p^r_n = \sum_{k=n+1}^N (aD^\alpha_t)^{k-n-1} \frac{\partial \bar{L}}{\partial q^r_k},$$

$$\pi^r_{n'} = \sum_{k=n'+1}^{N'} (aD^\alpha_t)^{k-n'-1} \frac{\partial \bar{L}}{\partial Q^r_k}.$$  

By taking into account (24), the canonical Hamiltonian becomes

$$\bar{H} = \sum_{r=1}^R \sum_{n=0}^{N-1} p^r_n q^r_{n+1} + \sum_{r=1}^R \sum_{n'=0}^{N'-1} \pi^r_{n'} Q^r_{n'+1} - \bar{L}.$$
Therefore, the modified canonical equations of motion are obtained as follows

\begin{align}
\{q_n, \bar{H}\} &= i \mathcal{D}^\alpha_t p_n, \quad \{Q_{n'}, \bar{H}\} = a \mathcal{D}^\alpha_t \pi_{n'}, \quad \text{(26)} \\
\{q_0, \bar{H}\} &= i \mathcal{D}^\alpha_t p_0 + a \mathcal{D}^\alpha_t \pi_0, \quad \text{(27)}
\end{align}

where, \(n = 1, \ldots, N\), \(n' = 1, \ldots, N'\).

The remaining set of equations of motion are given by

\begin{align}
\{p_n, \bar{H}\} &= q_{n+1} = a \mathcal{D}^\alpha_t q_n, \quad \{\pi_{n'}, \bar{H}\} = Q_{n+1} = i \mathcal{D}^\alpha_t Q_{n'}, \quad \text{(28)} \\
\frac{\partial \bar{H}}{\partial t} &= -\frac{\partial L}{\partial t}. \quad \text{(29)}
\end{align}

Here, \(n = 0, \ldots, N\), \(n' = 1, \ldots, N'\) and the commutator \(\{,\}\) represents the Poisson’s bracket defined as

\begin{equation}
\{A, B\}_{q_n, p_n, Q_{n'}, \pi_{n'}} = \frac{\partial A}{\partial q_n} \frac{\partial B}{\partial p_n} - \frac{\partial A}{\partial p_n} \frac{\partial B}{\partial q_n} + \frac{\partial A}{\partial Q_{n'}} \frac{\partial B}{\partial \pi_{n'}} - \frac{\partial A}{\partial \pi_{n'}} \frac{\partial B}{\partial Q_{n'}}, \quad \text{(30)}
\end{equation}

where, \(n = 0, \ldots, N\), \(n' = 1, \ldots, N'\).

As an example of the application of the fractional derivatives in Hamiltonian mechanics, let us consider the dissipative force proportional to \(q_1^{1/2}\). If we assume an object of mass \(m\) with initial velocity \(v_0\) subject to a resistive force proportional to \(q_1^{1/2}\), the corresponding differential equation of motion reads [11]

\begin{equation}
F(q_1) = -c(q_1^{1/2}) = m \frac{dq_1}{dt} = m q_2, \quad \text{(31)}
\end{equation}

where \(c\) represents a positive constant. By solving (31) the above equation with respect to the time \(t\), we obtain

\begin{equation}
t = -\frac{2m}{c} \left[ (q_1)^{1/2} - (v_0)^{1/2} \right], \quad \text{(32)}
\end{equation}

which is equivalent to

\begin{equation}
(q_1)^{1/2} = (v_0)^{1/2} - \frac{c}{2m} t. \quad \text{(33)}
\end{equation}

In the above equations we have assumed, \(q_0 = x, \mathcal{D}^\beta q_0 = \frac{dx}{dt(-\beta)^\beta}\).

Substituting in equation (31), we obtain

\begin{equation}
F(q_1) = -c \left[ (v_0)^{1/2} - \frac{c}{2m} t \right]. \quad \text{(34)}
\end{equation}
Following reference [11], the potential energy becomes

\[ U = -\frac{2ic}{\sqrt{\pi}} \left( t^{1/2}(v_0)^{1/2} - \frac{c}{3m} t \right) q_{1/2}. \]  

(35)

The corresponding Lagrangian of this system takes the following form

\[ L = \frac{1}{2} mq_1^2 + \frac{2ic}{\sqrt{\pi}} \left[ t^{1/2}(v_0)^{1/2} - \frac{c}{3m} t \right] q_{1/2}. \]  

(36)

The momenta \( p_0 \) and \( p_{1/2} \) read as

\[ p_0 = \frac{2ic}{\sqrt{\pi}} \left[ t^{1/2}(v_0)^{1/2} - \frac{c}{3m} t \right] + \text{im} D^{1/2}[q_1], \]  

(37)

\[ p_{1/2} = m q_1. \]  

(38)

Therefore, the Hamiltonian of the system takes the form

\[ H = \frac{p_{1/2}^2}{2m} + q_1 p_0 - \frac{2ic}{\sqrt{\pi}} \left[ t^{1/2}(v_0)^{1/2} - \frac{c}{3m} t \right] q_{1/2}. \]  

(39)

Making use of (39) the canonical action function is calculated as

\[ S = \int \left( q_1 p_{1/2} - \frac{p_{1/2}^2}{2m} + \frac{2ic}{\sqrt{\pi}} \left[ t^{1/2}(v_0)^{1/2} - \frac{c}{3m} t \right] q_{1/2} \right) dt. \]  

(40)

The path integral representation for the above system is given by

\[ K = \int dq_0 \ dq_{1/2} \ dp_0 \ dp_{1/2} \times \exp i \left[ \int \left( q_1 p_{1/2} - \frac{p_{1/2}^2}{2m} + \frac{2ic}{\sqrt{\pi}} \left[ t^{1/2}(v_0)^{1/2} - \frac{c}{3m} t \right] q_{1/2} \right) dt \right]. \]  

(41)

The path integral representation (41) is an integration over the canonical phase space coordinates \( (q_0, p_0) \) and \( (q_{1/2}, p_{1/2}) \).

Integrating over \( p_{1/2} \) and \( p_0 \), we arrive at the following result

\[ K = \int dq_0 \ dq_{1/2} \ \exp i \int \left( \frac{1}{2} m q_{1/2}^2 + \frac{2ic}{\sqrt{\pi}} \left[ t^{1/2}(v_0)^{1/2} - \frac{c}{3m} t \right] q_{1/2} \right) dt. \]  

(42)
Finally, the equation (42) can be expressed in a compact form as follows

\[ K = \int dq_0 \ e^{i \int (\frac{1}{2} mq^2) dt} \ \int dq_1 \ \exp i \int \left( \frac{2ic}{\sqrt{\pi}} \left[ t^{1/2}(v_0)^{1/2} - \frac{c}{3m} t \right] q_1^{1/2} \right) dt. \]

(43)

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References


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