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DIFFERENCE DISCRETE AND FRACTIONAL VARIATIONAL PRINCIPLES

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Abstract. *The discrete Hamiltonian formulation of Lagrangian possessing linear velocities is investigated and the equivalence of Hamilton and Euler-Lagrange equations is obtained. The fractional path integral of damped harmonic oscillator is analyzed in details.*

Key words: *difference equations, fractional calculus, variational principles*

1. INTRODUCTION

The discrete variational theory was subjected to an intense debate during the last years [1, 2]. In [1] the possibility that time can be considered as a discrete dynamical variable was analyzed for classical mechanics and relativistic quantum field theory and the conservation laws of difference equations were investigated in this theory [2]. Recently, the discrete difference theory of constrained systems was found to be a powerful approach in quantum gravity [3]. Besides, the consistent discretization approach [3] to general relativity leaving the spatial slices continuous was analyzed very recently in [3]. An important issue is to construct the discrete canonical Hamiltonian in the presence of primary constraints. This procedure, if we would like to mimic the continuous case construction, will lead us to introduce Lagrange multipliers. In this paper the discrete Euler-Lagrange equations were obtained and the equivalence with Hamilton's equations is discussed.

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The generalization of the concept of derivative and integral to a non-integer order α has been subjected to several approaches and some various alternative definitions o fractional derivatives appeared. During the last three decades the fractional calculus has become a field of growing interest because of its multiple and interesting applications in several fields such as mathematics, physics and engineering [4]. An important issue is to quantize the fractional Hamiltonian corresponding to a given fractional Lagrangian and to find its corresponding path integral [5, 6, 7, 8, 9, 10]. In this study the Riewe's formalism is generalized by considering a Lagrangian involving the Lagrange multipliers. Besides, the corresponding Hamiltonian equations are obtained.

2 DIFFERENCE DISCRETE VARIATIONAL PRINCIPLES

The main aim of this section is to investigate the singular discrete Lagrangians and the corresponding discrete Hamiltonians when the Hessian matrix has rank zero.

The starting point is the continuous action corresponding to a Langrangian possessing only the linear velocities

$$A=\int[a_i(q^1,q^2,...,q^m)\dot{q}^i-V(q^1,q^2,...,q^m)]dt, \tag{1}$$

where the $q^i,\ i=1,2,..,m$ depend only on time and V is the potential.

We will consider the following discrete counterpart of (1)

$$A_D=\sum_{n=1}^{N}[\sum_{i=1}^{m}a_i(q_n^1,q_n^2,...,q_n^m)\frac{\Delta q_{n-1}^i}{\Delta t_{n-1}}-V(q_n^1,q_n^2,...,q_n^m)]\Delta t_{n-1}, \tag{2}$$

where $q^i=q_n^i,\ \dot{q}^i=\dfrac{\Delta q_{n-1}^i}{\Delta t_{n-1}},\ q_n^i=q^i(t_n)$ and $\Delta q_n^i=q_n^i-q_{n-1}^i$.

Using (2) we obtain the Euler-Lagrange equations [12, 13] as follows

$$\sum_{i=1}^{m}\frac{\partial a_i}{\partial q_n^j}\frac{\Delta q_{n-1}^i}{\Delta t_{n-1}}-\frac{\partial V}{\partial q_n^j}=\frac{\Delta a_j}{\Delta t_{n-1}},\quad j=1,2,...,m. \tag{3}$$

In the following we analyze an example described by the following action

$$\int[(\dot{q}^1+\dot{q}^2)q^3+\frac{1}{2}(\dot{q}^3)^2-\frac{1}{2}(q^2)^2]dt. \tag{4}$$

We will consider the following discrete counterpart of the action (4) as follows

$$A_D=\sum_{n=1}^{N}[(\frac{\Delta q^1_{n-1}}{\Delta t_{n-1}}+\frac{\Delta q^2_{n-1}}{\Delta t_{n-1}})q^3_n+\frac{1}{2}(\frac{\Delta q^3_{n-1}}{\Delta t_{n-1}})^2-\frac{1}{2}(q^2_n)^2]\Delta t_{n-1}. \tag{5}$$

The discrete Euler- Lagrange equations for (5) are given by

$$\frac{\Delta q^3_{n-1}}{\Delta t_{n-1}}=0, \tag{6}$$

$$\frac{\Delta q^3_{n-1}}{\Delta t_{n-1}}=-q^2_n, \tag{7}$$

$$\frac{\frac{\Delta q^3_{n-1}}{\Delta t_{n-1}}}{\Delta t_{n-1}}=\frac{\Delta q^1_{n-1}}{\Delta t_{n-1}}+\frac{\Delta q^2_{n-1}}{\Delta t_{n-1}}, \tag{8}$$

where $n=1,2,...,N-1$.

After some calculation the solutions of the discrete Euler-Lagrange equations (6), (7) and (8) become

$$q^1_n=c_1,\quad q^2_n=0,\quad q^3_n=c_2, \tag{9}$$

where c_1 and c_2 are constants.

The next step is to define the corresponding discrete Hamiltonian.The canonical momenta are

$$z^j_{n-1}=\frac{\partial L_D}{\partial(\frac{\Delta q^j_{n-1}}{\Delta t_{n-1}})},\ j=1,2,3. \tag{10}$$

By using (10) we obtain

$$z^1_{n-1}=q^3_n,\quad z^2_{n-1}=q^3_n,\quad z^3_{n-1}=\frac{\Delta q^3_{n-1}}{\Delta t_{n-1}}. \tag{11}$$

Having in mind that

$$\frac{\partial z^j_{n-1}}{\partial\frac{\Delta q^i_{n-1}}{\Delta t_{n-1}}}=\frac{\partial^2 L_D}{\partial\frac{\Delta q^i_{n-1}}{\Delta t_{n-1}}\partial\frac{\Delta q^j_{n-1}}{\Delta t_{n-1}}}, \tag{12}$$

is a matrix with rank < 3 we conclude that a discrete Legendre transformation is not valid. To bypass this problem we denote $\lambda_{n-1}^{j} = \frac{\Delta q_{n-1}^{j}}{\Delta t_{n-1}}$, $j = 1,2,3$ and define the discrete Hamiltonian as

$$H_D = \sum_{i=1}^{3} z_{n-1}^{i}\lambda_{n-1}^{i} - L_D. \tag{13}$$

The form of discrete Hamiltonian becomes

$$H_D = \lambda_{n-1}^{1}(z_{n-1}^{1} - q_n^3) + \lambda_{n-1}^{2}(z_{n-1}^{2} - q_n^3) + \lambda_{n-1}^{3}(z_{n-1}^{3} - \frac{1}{2}\lambda_{n-1}^{3}) + \frac{1}{2}(q_n^2)^2. \tag{14}$$

The canonical equations of motions are $\frac{\Delta z_{n-1}^{j}}{\Delta t_{n-1}} = -\frac{\partial H}{\partial q_n^j}$, $j = 1,2,3$ [13] or in the explicit form become

$$\frac{\Delta z_{n-1}^{1}}{\Delta t_{n-1}} = 0, \frac{\Delta z_{n-1}^{2}}{\Delta t_{n-1}} = q_n^2, \frac{\Delta z_{n-1}^{3}}{\Delta t_{n-1}} = \lambda_{n-1}^{1} + \lambda_{n-1}^{2}. \tag{15}$$

By inspection we observed that (15) together with (12) are equivalent to the Euler-Lagrange equations (6) ,(7) and (8) respectively.

3 FRACTIONAL EULER- LAGRANGE AND HAMILTONIAN EQUATIONS

As it is very well known the left Riemann-Liouville fractional derivative is defined as follows

$${}_a\mathbf{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)d\tau, \tag{16}$$

and the corresponding right Riemann-Liouville fractional derivative is given by

$${}_t\mathbf{D}_b^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dt}\right)^n \int_t^b (\tau-t)^{n-\alpha-1} f(\tau)d\tau. \tag{17}$$

Here the order α fulfills $n-1 \leq \alpha < n$ and Γ denotes the Euler's Gamma function. For α being integer, these derivatives are defined in the usual sense, i.e.,

$${}_a\mathbf{D}_t^{\alpha} f(t) = \left(\frac{d}{dt}\right)^{\alpha}, \quad {}_t\mathbf{D}_b^{\alpha} f(t) = \left(-\frac{d}{dt}\right)^{\alpha}, \ \alpha = 1,2,.... \tag{18}$$

Agrawal investigated the Euler-Lagrange equations for fractional variational problems [6]. In the following we summarize his approach.

Let us consider the action function

$$S[q_0^1, ...q_0^R] = \int_a^b L(\{q_n^r, Q_{n'}^r\}, t)dt, \tag{19}$$

subject to the independent constraints

$$\Phi_m(t, q_0^1, ..., q_0^R, q_n^r, Q_{n'}^r) = 0, \; m < R. \tag{20}$$

Here the generalized coordinates are defined as follows

$$q_n^r = ({}_a\mathbf{D}_t^\alpha)^n x_r(t), \; Q_{n'}^r = ({}_t\mathbf{D}_b^\beta)^{n'} x_r(t). \tag{21}$$

Then, the necessary condition for the curves $q_0^1, ..., q_0^R$ with the boundary conditions $q_0^r(a) = q_0^{ra}, \; q_0^r(b) = q_0^{rb}, r = 1, 2, , ..., R$, to be an extremal of the functional given by equation (19) is that the functions q_0^r satisfy the following Euler-Lagrange equations [6]:

$$\frac{\partial \bar{L}}{\partial q_0^r} + \sum_{n=1}^{N} ({}_t\mathbf{D}_b^\alpha)^n \frac{\partial \bar{L}}{\partial q_n^r} + \sum_{n'=1}^{N'} ({}_a\mathbf{D}_t^\alpha)^{n'} \frac{\partial \bar{L}}{\partial Q_{n'}^r} = 0, \tag{22}$$

where $\bar{L}$ is given by [6]

$$\bar{L}(\{q_n^r, Q_{n'}^r\}, t, \lambda_m(t)) = L(\{q_n^r, Q_{n'}^r\}, t) + \lambda_m(t)\Phi_m(t, q_0^1, ..., q_0^R, q_n^r, Q_{n'}^r). \tag{23}$$

Here the multiple $\lambda_m(t) \in R^m$ represent continuous functions on $[a, b]$.

The next step is to obtain the Hamilton's equations for the the fractional variational problems.Fore these reasons we re-define the left and the right canonical momenta as follows

$$\begin{aligned} p_n^r &= \sum_{k=n+1}^{N} ({}_t\mathbf{D}_b^\alpha)^{k-n-1} \frac{\partial \bar{L}}{\partial q_k^r}, \\ \pi_{n'}^r &= \sum_{k=n'+1}^{N'} ({}_a\mathbf{D}_t^\alpha)^{k-n'-1} \frac{\partial \bar{L}}{\partial Q_k^r}. \end{aligned} \tag{24}$$

By taking into account (24), the canonical Hamiltonian becomes

$$\bar{H} = \sum_{r=1}^{R} \sum_{n=0}^{N-1} p_n^r q_{n+1}^r + \sum_{r=1}^{R} \sum_{n'=0}^{N'-1} \pi_{n'}^r Q_{n'+1}^r - \bar{L}. \tag{25}$$

Therefore, the modified canonical equations of motion are obtained as follows

$$\{q_n^r, \bar{H}\} = {}_t\mathbf{D}_b^\alpha p_n^r, \ \{Q_{n'}^r, \bar{H}\} = {}_a\mathbf{D}_t^\alpha \pi_{n'}^r, \tag{26}$$
$$\{q_0^r, \bar{H}\} = {}_t\mathbf{D}_b^\alpha p_0^r + {}_a\mathbf{D}_t^\alpha \pi_0^r, \tag{27}$$

where, $n = 1, ..., N, \ n' = 1, ..., N'$.

The remaining set of equations of motion are given by

$$\{p_n^r, \bar{H}\} = q_{n+1}^r = {}_a\mathbf{D}_t^\alpha q_n^r, \ \{\pi_{n'}^r, \bar{H}\} = Q_{n+1}^r = {}_t\mathbf{D}_b^\alpha Q_{n'}^r, \tag{28}$$
$$\frac{\partial \bar{H}}{\partial t} = -\frac{\partial \bar{L}}{\partial t}. \tag{29}$$

Here, $n = 0, ..., N, \ n' = 1, ..., N'$ and the commutator $\{,\}$ represents the Poisson's bracket defined as

$$\{A, B\}_{q_n^r, p_n^r, Q_{n'}^r, \pi_{n'}^r} = \frac{\partial A}{\partial q_n^r}\frac{\partial B}{\partial p_n^r} - \frac{\partial B}{\partial q_n^r}\frac{\partial A}{\partial p_n^r} + \frac{\partial A}{\partial Q_{n'}^r}\frac{\partial B}{\partial \pi_{n'}^r} - \frac{\partial B}{\partial Q_{n'}^r}\frac{\partial A}{\partial \pi_{n'}^r}, \tag{30}$$

where, $n = 0, ..., N, \ n' = 1, ..., N'$.

As an example of the application of the fractional derivatives in Hamiltonian mechanics, let us consider the dissipative force proportional to $(q_1)^{1/2}$. If we assume an object of mass m with initial velocity v_0 subjected s to a resistive force proportional to $(q_1)^{1/2}$, the corresponding differential equation of motion reads [11]

$$F(q_1) = -c(q_1)^{1/2} = m\frac{dq_1}{dt} = mq_2, \tag{31}$$

where c represents a positive constant. By solving (31) the above equation with respect to the time t, we obtain

$$t = -\frac{2m}{c}\left[(q_1)^{1/2} - (v_0)^{1/2}\right], \tag{32}$$

which is equivalent to

$$(q_1)^{1/2} = (v_0)^{1/2} - \frac{c}{2m}t. \tag{33}$$

In the above equations we have assumed, $q_0 = x, D^\beta q_0 = \frac{d^\beta x}{d(t-b)^\beta}$

Substituting in equation (31), we obtain

$$F(q_1) = -c\left[(v_0)^{1/2} - \frac{c}{2m}t\right]. \tag{34}$$

Following reference [11], the potential energy becomes

$$U = -\frac{2ic}{\sqrt{\pi}}\left[t^{1/2}(v_0)^{1/2} - \frac{c}{3m}t\right] q_{1/2}. \tag{35}$$

The corresponding Lagrangian of this system takes the following form

$$L = \frac{1}{2}mq_1^2 + \frac{2ic}{\sqrt{\pi}}\left[t^{1/2}(v_0)^{1/2} - \frac{c}{3m}t\right] q_{1/2}. \tag{36}$$

The momenta p_0 and $p_{1/2}$ read as

$$p_0 = \frac{2ic}{\sqrt{\pi}}\left[t^{1/2}(v_0)^{1/2} - \frac{c}{3m}t\right] + imD^{1/2}[q_1], \tag{37}$$

$$p_{1/2} = mq_1. \tag{38}$$

Therefore, the Hamiltonian of the system takes the form

$$H = \frac{p_{\frac{1}{2}}^2}{2m} + q_{\frac{1}{2}}p_0 - \frac{2ic}{\sqrt{\pi}}\left[[t^{1/2}(v_0)^{1/2} - \frac{c}{3m}t\right] q_{1/2}. \tag{39}$$

Making use of (39) the canonical action function is calculated as

$$S = \int \left(q_1 p_{\frac{1}{2}} - \frac{p_{\frac{1}{2}}^2}{2m} + \frac{2ic}{\sqrt{\pi}}\left[t^{1/2}(v_0)^{1/2} - \frac{c}{3m}t\right] q_{1/2}\right) dt. \tag{40}$$

The path integral representation for the above system is given by

$$K = \int \; dq_0 \; dp_0 \; dq_{\frac{1}{2}} \; dp_{\frac{1}{2}}$$

$$\times \exp i \left[\int \left(q_1 p_{\frac{1}{2}} - \frac{p_{\frac{1}{2}}^2}{2m} + \frac{2ic}{\sqrt{\pi}}\left[t^{1/2}(v_0)^{1/2} - \frac{c}{3m}t\right] q_{1/2}\right) dt \right]. \tag{41}$$

The path integral representation (41) is an integration over the canonical phase space coordinates (q_0, p_0) and $(q_{\frac{1}{2}}, p_{\frac{1}{2}})$.

Integrating over $p_{\frac{1}{2}}$ and p_0, we arrive at the following result

$$K = \int \; dq_0 \; dq_{\frac{1}{2}} \; \exp i \int \left(\frac{1}{2}mq_1^2 + \frac{2ic}{\sqrt{\pi}}\left[t^{1/2}(v_0)^{1/2} - \frac{c}{3m}t\right] q_{1/2}\right) dt. \tag{42}$$

Finally, the equation (42) can be express in a compact form as follows

$$K=\int\ dq_0\ e^{i\int\left(\frac{1}{2}mq_1^2\right))dt}\ \int\ dq_{\frac{1}{2}}\ \exp i\int\left(\frac{2ic}{\sqrt{\pi}}\left[t^{1/2}(v_0)^{1/2}-\frac{c}{3m}t\right]q_{1/2}\right)dt. \tag{43}$$

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