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FERMION MASSES AND MIXINGS IN SO(10) GUTS

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Abstract. *I will present a simple, economic and predictive model of Yukawa structures in the context of a renormalizable SO(10) grandunification. The righthanded neutrino mass is generated radiatively. The fermions have Yukawa couplings with one* 10 *and one* 120 *dimensional Higgses. The model predicts in the approxiamate two generation scenario degenerate neutrinos, small quark mixing and* $b-\tau$ *Yukawa unification.*

1. INTRODUCTION

In the following talk I will study some simple example of flavour physics in the context of SO(10) grandunificastion alone, i.e. assuming no flavour symmetry form the beginning. Clearly such a model would be highly undetermined and arbitrary, if we considered all the operators allowed by SO(10). The reason for this is that at least the first generation of fermion masses could be influenced (if not dominated) by the operators suppressed only by $M_{GUT}/M_{Planck} \approx 10^{-3}$. Thus we are forced to limit ourselves to renormalizable terms only, hoping that for some strange and unknown reason Planck physics does not generate this $1/M_{Planck}$ suppressed operators. That this may not be so crazy is enough to remember that in any supersymmetric grandunification a term

$$W = \frac{c}{M_{Planck}} QQQL \quad (\subset 16_F^4) \tag{1}$$

would be generically allowed. Constraints from $d = 5$ proton decay rules out all such theories except those with a negligible coefficient

$$c \leq 10^{-7} \tag{2}$$

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In a similar spirit we assume that all nonrenormalizable operators are negligible.

The model [1, 2] I will present is a counterexample to the following often claimed statements:

- we need the 126_H Higgs representation (or the 16_H plus the $1/M_{Planck}$ suppressed operators), to give mass to the right-handed neutrino
- $b-\tau$ unification of Yukawa couplings follows only in models with 10_H domination
- small mixing angles in the quark sector and at the same time large mixing angles in the leptonic sector seem difficult to achieve and are considered as problems or at least mysteries to be explained
- prediction of hierarchical neutrinos

2. THE MODEL

The most general renormalizable Yukawa terms in SO(10) can be schematically written as

$$\mathcal{L}_Y = 16_F^T \left(10_H Y_{10} + 120_H Y_{120} + 126_H Y_{126}\right) 16_F \ , \tag{3}$$

where from SO(10) algebra alone one can determine the symmetricity and antisymmetricity of the 3×3 Yukawa matrices:

$$Y_{10,126}^T = +Y_{10,126} \ \ , \ \ Y_{120}^T = -Y_{120} \ . \tag{4}$$

The most general case is not restrictive, so one tries simple models, minimal subcases, which could be potentially realistic.

What about the Higgs sector? There are two types of Higgs representations that break the rank of SO(10). The first one is the already mentioned 126_H. We want a counterexample to the first item in the introduction, so we will avoid this representation and choose instead 16_H. The vev of this representation alone breaks SO(10) to SU(5), so it could give in principle a large SU(5) invariant mass to the righthanded neutrino. The problem is that (3) does not contain any 16_H, i.e. this Higgs representation can be coupled to the fermionic 16_F only through a nonrenormalizable operator. Since we assumed these to be absent, at tree level we get the righthanded neutrino massless, $M_{\nu_R} = 0$. We can however avoid failure by recalling the old Witten's idea [3]: at two loop level there are diagrams that generate an effective operator of the form

$$\frac{16_F^2 16_H^2}{M} \ . \tag{5}$$

Witten took as an example the diagram with gauge bosons exchange and 10_H coupled to fermions. The mass scale M in (5) is presumably the heaviest mass in the loop, something like M_{GUT}. Since there is a two loop suppression, the righthanded neutrino mass matrix become

$$M_{\nu_R} \approx \left(\frac{\alpha}{\pi}\right)^2 \frac{M_R^2}{M_{GUT}} Y_{10} \ , \tag{6}$$

where $M_R = \langle 16_H \rangle$ is the scale of the $\mathrm{SU}(2)_R$ breaking.

When is this approximate formula valid?

In a nonsupersymmetric model this will hardly work. The reason is that there is no one-step gauge unification in such theories, so that an intermediate scale is needed. Usually this scale is exactly M_R, and typically it lies few orders of magnitude below M_{GUT}. This would further suppress the already two loop suppressed righthanded neutrino mass, predicting an unacceptably large light neutrino masses, well above the approximate limit of 1 eV. This of course assuming that the Dirac neutrino Yukawa matrix has not all elements small, i.e. that there are similarity relations between the up quark Dirac neutrino mass matrices, as is usually the case in SO(10). The only possible exception is that M_R is still high, but there are some light multiplets at an intermediate scale that make the running of the gauge coupling modify in the appropriate way. We will not consider this case anymore.

The supersymmetric version is even worse. The righthanded neutrino mass term comes from a superpotential, and it is well known that in supersymmetry the superpotential does not get renormalized. So if such a term was not present at tree order, as we assumed, it will not appear at any order of perturbation theory. In other words, the righthanded neutrino mass must be proportional to the supersymmetry breaking scale, i.e. to some positive power of the ratio m_{SUSY}/M_{GUT}. In low energy supersymmetry breaking like in MSSM this is obviously far too small.

Although at this point the situation seems hopeless, the above examples give a hint of the direction to follow. What we need is a one step unification together with a large susy breaking scale. Such a model has been proposed last year, and it is nothing else that split supersymmetry [4]. In these scenarios the running of the gauge couplings gets improved by the presence at low energy of new fermions like gauginos and higgsinos. The absence at the low energy of the sfermions is not important since they come along in SU(5) multiplets and thus do not contribute at the one loop level to the determination of the unification scale. In the original proposal these supersymmetric partners of the SM fermions could have any mass between M_W and M_{GUT}. If we want to generate the righthanded neutrino masses with the radiative mechanism, we need them very massive, i.e. at $m_{SUSY} \approx M_{GUT}$ [2], an interesting and important result.

The now missing 126_H representation had also another function, i.e. to correct the Yukawa matrices. In fact the standard model Higgs is in general a linear combination of $SU(2)_L$ doublets coming from different representations, and the two most important contributions were certainly those coming from 10_H and 126_H [5]. This option is now gone, and one needs to add another Higgs representation to the Yukawa sector, on top of 10_H. From (3) we see as the first possibility to add a 120_H. This is what we will do in detail in the next chaper.

3. $10_H + 120_H$

The matrix Y_{120} is very restrictive, since it contains only 3 complex parameters, being antisymmetric (4). Partially one gain some new parameters with respect to the case $10_H + 126_H$ [5] due to some new doublet vevs. In fact we have 6 unknown $SU(2)_L$ breaking vevs that contribute to the light fermion masses:

$$\langle (2,2,1)_{10} \rangle = \begin{pmatrix} v_{10}^u & 0 \\ 0 & v_{10}^d \end{pmatrix} , \tag{7}$$

$$\langle (2,2,1)_{120} \rangle = \begin{pmatrix} v_{120}^u & 0 \\ 0 & v_{120}^d \end{pmatrix} , \tag{8}$$

$$\langle (2,2,15)_{120} \rangle = \begin{pmatrix} w_{120}^u & 0 \\ 0 & w_{120}^d \end{pmatrix} . \tag{9}$$

Can we fit the data with 10_H and 120_H only? We will see that in the two generation case the answer is yes. But let us start more generally. The mass matrices that follow from (3) without 126_H after symmetry breaking are

$$\begin{aligned}
M_D &= v_{10}^d Y_{10} + \left(v_{120}^d + w_{120}^d\right) Y_{120} \ , && (10)\\
M_U &= v_{10}^u Y_{10} + \left(v_{120}^u + w_{120}^u\right) Y_{120} \ , && (11)\\
M_E &= v_{10}^d Y_{10} + \left(v_{120}^d - 3w_{120}^d\right) Y_{120} \ , && (12)\\
M_{\nu_D} &= v_{10}^u Y_{10} + \left(v_{120}^u - 3w_{120}^u\right) Y_{120} \ , && (13)\\
M_{\nu_R} &= v_R Y_{10} \ , && (14)\\
M_N &= -M_{\nu_D}^T M_{\nu_R}^{-1} M_{\nu_D} \ . && (15)
\end{aligned}$$

In the above equation v_R is just the coefficient of the two loop contribution (6). We have used the type I seesaw formula, because the type II seesaw contribution M_{ν_L} is two loop suppressed similarly as M_{ν_R}, while the type I, being inversely proportional to M_{ν_R} gets at the same time enhanced. One could say that the type II over type I contribution is proportional to $(\alpha/\pi)^4$, so I will neglect the type II contribution from now on.

The above formulae can be rewritten in a useful way:

$$\begin{aligned}
M_D &= M_0 + M_2 \ , && (16)\\
M_U &= c_0 M_0 + c_2 M_2 \ , && (17)\\
M_E &= M_0 + c_3 M_2 \ , && (18)\\
M_{\nu_D} &= c_0 M_0 + c_4 M_2 \ , && (19)\\
M_{\nu_R} &= c_R M_0 \ , && (20)
\end{aligned}$$

where M_0 is symmetric and M_2 is antisymmetric.

The full analysis is hard and time consuming. We will show first what happens in the approximate world of two generations. This in order to get some analytic results. Consider then the 2^{nd} and 3^{rd} generations only. In all generality we can choose a basis, in which M_0 is diagonal with positive real eigenvalues:

$$M_0 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \ , \ M_2 = \begin{pmatrix} 0 & -i\alpha \\ i\alpha & 0 \end{pmatrix} \ . \qquad (21)$$

In general a, b and $c_{0,R}$ can be taken real $(a, b > 0)$, while α, $c_{2,3,4}$ are complex.

4. $b - \tau$ UNIFICATION

This prediction is relatively easy to obtain. In practice it follows because both M_D and M_E has the same diagonal elements, and because the second generation of fermions is much lighter than the third one.

Since the above matrices are not Hermitian in general, one better calculates

$$M_D M_D^\dagger = \begin{pmatrix} a^2+|\alpha|^2 & -i(a\alpha^*+b\alpha) \\ i(a\alpha+b\alpha^*) & b^2+|\alpha|^2 \end{pmatrix} . \tag{22}$$

From the trace and the determinant of the above matrix one finds for the down quark masses

$$a^2+b^2+2|\alpha|^2 = m_b^2+m_s^2 , \tag{23}$$
$$|ab-\alpha^2|^2 = m_b^2 m_s^2 . \tag{24}$$

Similar relations can be obtained for the charged leptons, just change $\alpha \to c_3\alpha$, $m_b \to m_\tau$, $m_s \to m_\mu$.

Writing $\alpha = |\alpha| e^{i\phi}$ and taking into account that $\cos(2\phi) \leq 1$, one finds that the sum $a+b$ is constrained:

$$m_b - m_s \leq a+b \leq m_b + m_s . \tag{25}$$

For the charged lepton sector one gets a similar bound,

$$m_\tau - m_\mu \leq a+b \leq m_\tau + m_\mu . \tag{26}$$

Clearly, in order to satisfy both requirements (25) and (26) the intervals $[m_b-m_s, m_b+m_s]$ and $[m_\tau-m_\mu, m_\tau+m_\mu]$ must have at least some region in common. Thus the bottom and tau masses can differ by less than

$$m_b - m_\tau < m_s + m_\mu , \tag{27}$$

i.e. we get $b-\tau$ unification with a quite good precision.

4. DEGENERATE NEUTRINOS

Let's first prove, that a large leptonic mixing angle forces the neutrinos to be nearly degenerate. The main point is the particular form of M_2, which is just the second Pauli matrix. This makes the calculation of M_N particularly easy, with the surprising result that the neutrino mass matrix is diagonal:

$$M_N = c_N M_0 \ \ , \ \ c_N = \frac{c_4^2 \alpha^2 - c_0^2 ab}{abc_R} \ . \tag{28}$$

This was obtained without any approximation, except for working in the two generation case.

Similarly as before we calculate

$$M_E M_E^\dagger \ = \ \begin{pmatrix} a^2 + |c_3\alpha|^2 & -i(ac_3^*\alpha^* + bc_3\alpha) \\ i(ac_3\alpha + bc_3^*\alpha^*) & b^2 + |c_3\alpha|^2 \end{pmatrix} , \tag{29}$$

$$M_N M_N^\dagger \ = \ |c_N|^2 \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} . \tag{30}$$

From the trace and the determinants of the above matrices one finds for the lepton masses (ϕ_3 is the phase of $c_3\alpha$)

$$a^2 + b^2 + 2|c_3\alpha|^2 \ = \ m_\tau^2 + m_\mu^2 \ , \tag{31}$$

$$a^2 b^2 + |c_3\alpha|^4 - 2ab|c_3\alpha|^2 \cos{(2\phi_3)} \ = \ m_\tau^2 m_\mu^2 \ , \tag{32}$$

$$|c_N|^2 a^2 \ = \ m_2^2 \ , \tag{33}$$

$$|c_N|^2 b^2 \ = \ m_3^2 \ . \tag{34}$$

The relative angle between the matrices in (22) is nothing else than the neutrino atmospheric mixing angle, and it is easily calculated from the invariant

$$\begin{aligned} Tr\left(M_E M_E^\dagger M_N M_N^\dagger\right) \ &= \ m_\tau^2 m_3^2 + m_\mu^2 m_2^2 \\ &- \ \sin^2\theta_A (m_\tau^2 - m_\mu^2)(m_2^2 - m_1^2) \end{aligned} \tag{35}$$

Using (29)-(30) one gets

$$\frac{b^2 - a^2}{m_\tau^2 - m_\mu^2} = \cos{(2\theta_A)} \ . \tag{36}$$

One can calculate $\cos{(2\phi_3)}$ from (32) and similarly as in the previous section require that its square is smaller than 1. This brings to the consistency relation

$$(m_\tau - m_\mu)^2 \frac{\sin^2{(2\theta_A)}}{4} \leq |c_3\alpha|^2 \leq (m_\tau + m_\mu)^2 \frac{\sin^2{(2\theta_A)}}{4} \ . \tag{37}$$

Putting all together

$$\frac{m_3^2 - m_2^2}{m_3^2 + m_2^2} = \frac{(m_\tau^2 - m_\mu^2)\cos(2\theta_A)}{(m_\tau^2 + m_\mu^2) - (m_\tau + \xi m_\mu)^2 \sin(2\theta_A)^2/2} \;, \tag{38}$$

where $\xi^2 \leq 1$ parametrizes the value of $|c_3\alpha|$ in the range given by (37).

This shows that for a nearly maximal atmospheric mixing angle the neutrinos tend to be degenerate. Of course one needs a better numerical check [6], but the final word can be given only after the three generation analysis. Irrespectively of the numerical fit, it is however interesting that in this model a connection exists between the large atmospheric mixing angle and the neutrino degeneracy.

5. SMALL QUARK MIXING ANGLES

Another interesting connection exists between the large atmospheric mixing angle and the small quark mixing angle. This relation is a bit different from the ones we derived before. In fact, so far, although we kept all the masses nonzero, basically very similar results are obtained in the limit of massless second generation charged fermions. In such a limit however the quark mixing angle becomes exactly zero [1]. Although this is a good approximation in the leading order, it would be interesting to get the main nonzero contribution, as it was done in [6]

To find that the quark mixing angle θ_{cb}, one can use the quark sector analogue of (35):

$$\begin{aligned} Tr\left(M_U M_U^\dagger M_D M_D^\dagger\right) &= m_t^2 m_b^2 + m_c^2 m_s^2 \\ &- \sin^2\theta_{cb}(m_t^2 - m_c^2)(m_b^2 - m_s^2)\;. \end{aligned} \tag{39}$$

A tedious, but straightforward calculation gives at leading order

$$\sin\theta_{cb} = \xi_d \frac{m_s}{m_b}\cos(2\theta_A)\;, \tag{40}$$

where $\xi_d^2 \leq 1$ parametrizes $a + b$ in the range (25). The corrections to this formula are of higher powers of the small parameters m_c/m_t, m_s/m_b and $\cos(2\theta_A)$.

We found thus a relation between the large atmospheric mixing angle and the small quark mixing angle, together with the further suppression due to the small ratio of second generation masses to third generation masses. This is interesting per se, irrespectively of whether it fits the data numerically or not. In fact, the quark mixing angle comes out to be numerically too small. But, as before, further numerical analysis of the three generation case are needed.

6. CONCLUSIONS

There is by now a well defined minimal renormalizable supersymmetric SO(10) model [7], in which the matter 16_F couple to the Higgs 10_H and 126_H. It has been proved [8] to be consistent with all experimental data on light fermion masses and mixings. In the case of a type II seesaw dominance it predicts a relatively large $|U_{e3}| \geq 0.1$ [9] and gives an interesting correlation between large atmospheric neutrino mixing angle and $b-\tau$ unification [10].

However it could happen, that supersymmetry is broken at a very large scale, but that some fermionic partners are nevertheless close to the electroweak scale. In this case I offered a simple alternative model, in which the righthanded neutrino gets its mass through a radiative mechanism. The simplest Yukawa structure has been shown to be consistent with the second and third generations, predicting almost degenerate neutrinos, a correlation between large atmospheric mixing angle and small quark mixing angle, and the equality of b and τ Yukawa couplings.

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