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GUTS FROM FUZZY EXTRA DIMENSIONS

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Abstract. *We consider gauge theories defined in higher dimensions when the extra dimensions form a fuzzy space. We recall the striking feature of the appearance of non-abelian U_n gauge theories in four dimensions starting with an abelian gauge theory in higher dimensions and discuss the difficulties of extending this property to SO_n gauge theories.*

Key words: *Non-commutative Geometry, Gauge Theories and Fuzzy Spaces.*

1. INTRODUCTION

Non-commutative Geometry [1, 2] has been regarded a promising framework for obtaining finite quantum field theories. Quantization of fields over "spaces" described by infinite-dimensional algebras has proven to be more subtle than was originally expected. The difficulties encountered prompted

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some to look for models with finite-dimensional algebras, for example matrix algebras. It is interesting to consider these non-commutative spaces as extra dimensions in higher dimensional theories [12, 13]. A subsequent dimensional reduction of the extra dimensions could lead to interesting models of low-energy particle physics. The dimensional reduction scheme over coset spaces (CSDR) [3]-[6] gave fruitful results in case of classical commutative spaces [4], [7]-[11]; its use for the case of non-commutative ones may lead to important conclusions. As a sequel to [12, 13] we here consider gauge theories defined in higher dimensions, where the extra dimensions form a fuzzy sphere [14, 2]. We interpret these gauge theories as four-dimensional theories with Kaluza-Klein modes. We recall how U_n gauge theories emerge in four dimensions starting only with a U_1 in higher dimensions and discuss the extension of this result to gauge theories based on orthogonal groups.

2. THE FUZZY SPHERE

The fuzzy sphere, S_F^2, is a matrix approximation[2] of the usual sphere S^2 which has been used [14, 2] as a model of non-commutative gravity. The algebra of functions on S^2 (for example spanned by the spherical harmonics) is truncated at a given frequency; the product is modified so that the resulting vector space becomes an algebra of complex matrices. The "space" described by this non-commutative algebra, endowed with a differential structure and geometry we refer to as a fuzzy sphere. The algebra itself is that of $n \times n$ matrices. More precisely, the algebra of functions on the ordinary sphere can be generated by the coordinates of $\mathbb{R}^3$ modulo the relation $\sum_{\hat{a}=1}^{3} x_{\hat{a}} x_{\hat{a}} = r^2$. Following the notation of previous articles [12, 13] we describe the fuzzy sphere S_F^2 at fuzziness level $n-1$ to be the non-commutative "manifold" whose coordinate "functions" $iX_{\hat{a}}$ are $n \times n$ hermitian matrices proportional to the generators of the n-dimensional representation of SU_2. They satisfy the condition $\sum_{\hat{a}=1}^{3} X_{\hat{a}} X_{\hat{a}} = \alpha r^2$ and the commutation relations $[X_{\hat{a}}, X_{\hat{b}}] = C_{\hat{a}\hat{b}\hat{c}} X_{\hat{c}}$ where $C_{\hat{a}\hat{b}\hat{c}} = \varepsilon_{\hat{a}\hat{b}\hat{c}}/r$ and the proportionality factor α goes as n^2 for n large. Indeed it can be proven that for $n \to \infty$ one obtains the usual commutative sphere. The coordinates $X^{\hat{a}}$ can be also considered as momenta [2] and have been designated as $p^{\hat{a}}$.

On the fuzzy sphere there is a natural SU_2 covariant differential calculus. This calculus is three-dimensional and the derivations $e_{\hat{a}}$ along $X_{\hat{a}}$ of a function f are given by $e_{\hat{a}}(f) = [X_{\hat{a}}, f]$. Accordingly the action of the

[2]For historical details and literature for the construction consult [2].

Lie derivatives on functions is given by $\mathcal{L}_{\hat{a}}f = [X_{\hat{a}}, f]$. In the $n \to \infty$ limit the derivations $e_{\hat{a}}$ become $e_{\hat{a}} = C_{\hat{a}\hat{b}\hat{c}}x^{\hat{b}}\partial^{\hat{c}}$ and only in this commutative limit the tangent space becomes two-dimensional. The exterior derivative is given by $df = [X_{\hat{a}}, f]\theta^{\hat{a}}$ with $\theta^{\hat{a}}$ the one-forms dual to the vector fields $e_{\hat{a}}$, $< e_{\hat{a}}, \theta^{\hat{b}} >= \delta^{\hat{b}}_{\hat{a}}$. The space of one-forms is generated by the $\theta^{\hat{a}}$'s in the sense that for any one-form $\omega = \sum_i f_i dh_i\, t_i$ we can always write $\omega = \sum_{\hat{a}=1}^{3} \omega_{\hat{a}}\theta^{\hat{a}}$ with given functions $\omega_{\hat{a}}$ depending on the functions f_i, h_i and t_i. The action of the Lie derivatives $\mathcal{L}_{\hat{a}}$ on the one-forms $\theta^{\hat{b}}$ explicitly reads $\mathcal{L}_{\hat{a}}(\theta^{\hat{b}}) = C_{\hat{a}\hat{b}\hat{c}}\theta^{\hat{c}}$. On a general one-form $\omega = \omega_{\hat{a}}\theta^{\hat{a}}$ we have $\mathcal{L}_{\hat{b}}\omega = \mathcal{L}_{\hat{b}}(\omega_{\hat{a}}\theta^{\hat{a}}) = \left[X_{\hat{b}}, \omega_{\hat{a}}\right]\theta^{\hat{a}} - \omega_{\hat{a}}C^{\hat{a}}{}_{\hat{b}\hat{c}}\theta^{\hat{c}}$ and therefore

$$(\mathcal{L}_{\hat{b}}\omega)_{\hat{a}} = \left[X_{\hat{b}}, \omega_{\hat{a}}\right] - \omega_{\hat{c}}C^{\hat{c}}{}_{\hat{b}\hat{a}}\ ; \tag{1}$$

this formula is fundamental for formulating the CSDR principle on fuzzy cosets [12].

The differential geometry on the product space Minkowski times fuzzy sphere, $M^4 \times S^2_F$, is easily obtained from that on M^4 and on S^2_F. For example a one-form A defined on $M^4 \times S^2_F$ is written as $A = A_\mu dx^\mu + A_{\hat{a}}\theta^{\hat{a}}$ with $A_\mu = A_\mu(x^\mu, X_{\hat{a}})$ and $A_{\hat{a}} = A_{\hat{a}}(x^\mu, X_{\hat{a}})$.

One can also introduce spinors on the fuzzy sphere and study the Lie derivative on these spinors. Similarly, one can study other (higher dimensional) fuzzy spaces (e.g. fuzzy CP^M).

3. FOUR-DIMENSIONAL INTERPRETATION OF HIGHER DIMENSIONAL ACTIONS

First we consider on $M^4 \times (S/R)_F$ a non-commutative gauge theory with gauge group $G = U_P$ and examine its four-dimensional interpretation. $(S/R)_F$ is a fuzzy coset, for example the fuzzy sphere S^2_F (in which case the groups S and R are given by $S = SU(2)$ and $R = U(1)$). The action is

$$\mathcal{A}_{YM} = \frac{1}{4g^2}\int d^4x\, kTr\, tr_G\, F_{MN}F^{MN}, \tag{2}$$

where kTr denotes integration over the fuzzy coset $(S/R)_F$ described by $n \times n$ matrices; here the normalization parameter k is related to the size of the fuzzy coset space. For example for the fuzzy sphere at fuzzyness level $n - 1$ we have $r^2 = \sqrt{n^2 - 1}\pi k$ [2]. In the $n \to \infty$ limit kTr becomes the usual integral on the coset space. For finite n, Tr is a good integral because it has the cyclic property $Tr(f_1 \dots f_{p-1} f_p) = Tr(f_p f_1 \dots f_{p-1})$. It is also invariant under the action of the group S, that is infinitesimally given by the Lie derivative. In the action (2) tr_G is the gauge group G trace. The higher-dimensional field strength F_{MN}, decomposed in four-dimensional space-time and extra-dimensional components, reads as follows $(F_{\mu\nu}, F_{\mu\hat{b}}, F_{\hat{a}\hat{b}})$; explicitly the various components of the field strength are given by

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], && (3) \\ F_{\mu\hat{a}} &= \partial_\mu A_{\hat{a}} - [X_{\hat{a}}, A_\mu] + [A_\mu, A_{\hat{a}}], \\ F_{\hat{a}\hat{b}} &= [X_{\hat{a}}, A_{\hat{b}}] - [X_{\hat{b}}, A_{\hat{a}}] + [A_{\hat{a}}, A_{\hat{b}}] - C^{\hat{c}}{}_{\hat{a}\hat{b}} A_{\hat{c}}. && (4) \end{aligned}$$

Under an infinitesimal G gauge transformation $\lambda = \lambda(x^\mu, X^{\hat{a}})$ we have

$$\delta A_{\hat{a}} = -[X_{\hat{a}}, \lambda] + [\lambda, A_{\hat{a}}] \ , \tag{5}$$

thus F_{MN} is covariant under local G gauge transformations: $F_{MN} \to F_{MN} + [\lambda, F_{MN}]$. This is an infinitesimal abelian U_1 gauge transformation if λ is just an antihermitian function of the coordinates $x^\mu, X^{\hat{a}}$. It is an infinitesimal non-abelian U_P gauge transformation if λ is valued in Lie(U_P), the Lie algebra of hermitian $P \times P$ matrices. In the following we will always assume Lie(U_P) elements to commute with the coordinates $X^{\hat{a}}$. In fuzzy/non-commutative gauge theory and in Fuzzy-CSDR a fundamental role is played by the covariant coordinate, $\varphi_{\hat{a}} \equiv X_{\hat{a}} + A_{\hat{a}}$. This field transforms indeed covariantly under a gauge transformation, $\delta(\varphi_{\hat{a}}) = [\lambda, \varphi_{\hat{a}}]$. In terms of φ the field strength in the non-commutative directions reads,

$$F_{\mu\hat{a}} = \partial_\mu \varphi_{\hat{a}} + [A_\mu, \varphi_{\hat{a}}] = D_\mu \varphi_{\hat{a}}, \quad F_{\hat{a}\hat{b}} = [\varphi_{\hat{a}}, \varphi_{\hat{b}}] - C^{\hat{c}}{}_{\hat{a}\hat{b}} \varphi_{\hat{c}} \ ; \tag{6}$$

and using these expressions the action reads

$$\mathcal{A}_{YM} = \int d^4x \, Tr \, tr_G \left(\frac{k}{4g^2} F_{\mu\nu}^2 + \frac{k}{2g^2} (D_\mu \varphi_{\hat{a}})^2 - V(\varphi) \right), \tag{7}$$

where the potential term $V(\varphi)$ is the $F_{\hat{a}\hat{b}}$ kinetic term (in our conventions $F_{\hat{a}\hat{b}}$ is antihermitian so that $V(\varphi)$ is hermitian and non-negative)

$$\begin{aligned} V(\varphi) &= -\frac{k}{4g^2} Tr \, tr_G \sum_{\hat{a}\hat{b}} F_{\hat{a}\hat{b}} F_{\hat{a}\hat{b}} \\ &= -\frac{k}{4g^2} Tr \, tr_G \left([\varphi_{\hat{a}}, \varphi_{\hat{b}}][\varphi^{\hat{a}}, \varphi^{\hat{b}}] - 4C_{\hat{a}\hat{b}\hat{c}} \varphi^{\hat{a}} \varphi^{\hat{b}} \varphi^{\hat{c}} + 2r^{-2} \varphi^2 \right) . \end{aligned} \tag{8}$$

The action (7) is naturally interpreted as an action in four dimensions. The infinitesimal G gauge transformation with gauge parameter $\lambda(x^\mu, X^{\hat{a}})$ can indeed be interpreted just as an M^4 gauge transformation. We write

$$\lambda(x^\mu, X^{\hat{a}}) = \lambda^\alpha(x^\mu, X^{\hat{a}})\mathcal{T}^\alpha = \lambda^{h,\alpha}(x^\mu)T^h\mathcal{T}^\alpha \ , \tag{9}$$

where $\mathcal{T}^\alpha$ are hermitian generators of U_P, $\lambda^\alpha(x^\mu, X^{\hat{a}})$ are $n \times n$ antihermitian matrices and thus are expressible as $\lambda(x^\mu)^{\alpha,h}T^h$, where T^h are antihermitian generators of U_n. The fields $\lambda(x^\mu)^{\alpha,h}$, with $h = 1, \ldots n^2$, are the Kaluza-Klein modes of $\lambda(x^\mu, X^{\hat{a}})^\alpha$. We now consider on equal footing the indices h and α and interpret the fields on the r.h.s. of (9) as one field valued in the tensor product Lie algebra Lie$(U_n) \otimes$ Lie(U_P). This Lie algebra is indeed Lie(U_{nP}) (the $(nP)^2$ generators $T^h\mathcal{T}^\alpha$ being $nP \times nP$ antihermitian matrices that are linearly independent). Similarly we rewrite the gauge field A_ν as

$$A_\nu(x^\mu, X^{\hat{a}}) = A^\alpha_\nu(x^\mu, X^{\hat{a}})\mathcal{T}^\alpha = A^{h,\alpha}_\nu(x^\mu)T^h\mathcal{T}^\alpha, \tag{10}$$

and interpret it as a Lie(U_{nP}) valued gauge field on M^4, and similarly for $\varphi_{\hat{a}}$. Finally $Tr\, tr_G$ is the trace over U_{nP} matrices in the fundamental representation.

Up to now we have just performed an ordinary fuzzy dimensional reduction. Indeed in the commutative case the expression (7) corresponds to rewriting the initial lagrangian on $M^4 \times S^2$ using spherical harmonics on S^2. Here the space of functions is finite dimensional and therefore the infinite tower of modes reduces to the finite sum given by Tr. The machinery of CSDR [3, 4] can be used afterwards to reduce the number of the field content of the theory and possibly obtain realistic particle physics model. The rules of a non-commutative version of the dimensional reduction in question have been set in ref [12], whereas the renormalisability of these theories have been discussed in [13].

4. NON-TRIVIAL DIMENSIONAL REDUCTION AND FUZZY EXTRA DIMENSIONS

Next we reduce the number of gauge fields and scalars in the action (7) by applying the CSDR scheme. Since SU_2 acts on the fuzzy sphere $(SU_2/U_1)_F$, and more in general the group S acts on the fuzzy coset $(S/R)_F$, we can state the CSDR principle in the same way as in the continuum case, i.e. the fields in the theory must be invariant under the infinitesimal SU_2, respectively S, action up to an infinitesimal gauge transformation

$$\mathcal{L}_{\hat{b}}\phi = \delta_{W_{\hat{b}}}\phi = W_{\hat{b}}\phi\,, \qquad \mathcal{L}_{\hat{b}}A = \delta_{W_{\hat{b}}}A = -DW_{\hat{b}}, \tag{11}$$

where A is the one-form gauge potential $A = A_\mu dx^\mu + A_{\hat{a}}\theta^{\hat{a}}$, and $W_{\hat{b}}$ depends only on the coset coordinates $X^{\hat{a}}$ and (like A_μ, A_a) is antihermitian. We thus write $W_{\hat{b}} = W_{\hat{b}}^{\alpha}\mathcal{T}^{\alpha}$, $\alpha = 1, 2 \ldots P^2$, where $\mathcal{T}^i$ are hermitian generators of $U(P)$ and $(W_{\hat{b}}^i)^\dagger = -W_{\hat{b}}^i$, here † is hermitian conjugation on the $X^{\hat{a}}$'s.

In terms of the covariant coordinate $\varphi_{\hat{d}} = X_{\hat{d}} + A_{\hat{d}}$ and of $\omega_{\hat{a}} \equiv X_{\hat{a}} - W_{\hat{a}}$, the CSDR constraints assume a particularly simple form, namely

$$[\omega_{\hat{b}}, A_\mu] = 0\,, \tag{12}$$

$$C_{\hat{b}\hat{d}\hat{e}}\varphi^{\hat{e}} = [\omega_{\hat{b}}\,, \varphi_{\hat{d}}]\,. \tag{13}$$

In addition we have a consistency condition following from the relation $[\mathcal{L}_{\hat{a}}, \mathcal{L}_{\hat{b}}] = C_{\hat{a}\hat{b}}{}^{\hat{c}}\mathcal{L}_{\hat{c}}$:

$$[\omega_{\hat{a}}, \omega_{\hat{b}}] = C_{\hat{a}\hat{b}}{}^{\hat{c}}\omega_c, \tag{14}$$

where $\omega_{\hat{a}}$ transforms as $\omega_{\hat{a}} \to \omega'_{\hat{a}} = g\omega_{\hat{a}}g^{-1}$. One proceeds in a similar way for the spinor fields [12].

Solving the CSDR constraints for the fuzzy sphere

We consider $(S/R)_F = S_F^2$, i.e. the fuzzy sphere, and to be definite at fuzziness level $n-1$ ($n\times n$ matrices). We study here the basic example where the gauge group is $G = U_1$. In this case the $\omega_{\hat{a}} = \omega_{\hat{a}}(X^{\hat{b}})$ appearing in the consistency condition (14) are $n \times n$ antihermitian matrices and therefore can be interpreted as elements of Lie(U_n). On the other hand the $\omega_{\hat{a}}$ satisfy the commutation relations (14) of Lie(SU_2). Therefore in order to satisfy the consistency condition (14) we have to embed Lie(SU_2) in Lie(U_n). Let T^h with $h = 1, \ldots, n^2$ be the generators of Lie(U_n) in the fundamental representation, we can always use the convention $h = (\hat{a}, u)$ with $\hat{a} = 1, 2, 3$ and $u = 4, 5, \ldots, n^2$ where the $T^{\hat{a}}$ satisfy the SU_2 Lie algebra, $[T^{\hat{a}}, T^{\hat{b}}] = C^{\hat{a}\hat{b}}_{\hat{c}}T^{\hat{c}}$. Then we define an embedding by identifying

$$\omega_{\hat{a}} = T_{\hat{a}}\,. \tag{15}$$

The constraint (12), $[\omega_{\hat{b}}, A_\mu] = 0$, then implies that the four-dimensional gauge group K is the centralizer of the image of SU_2 in U_n, i.e.

$$K = C_{U_n}(SU_2) = SU_{n-2} \times U_1 \times U_1\ ,$$

where the last U_1 is the U_1 of $U_n \simeq SU_n \times U_1$. The functions $A_\mu(x, X)$ are arbitrary functions of x but the X dependence is such that $A_\mu(x, X)$ is Lie(K) valued instead of Lie(U_n), i.e. eventually we have a four-dimensional gauge potential $A_\mu(x)$ with values in Lie(K). Concerning the constraint (13), it is satisfied by choosing

$$\varphi_{\hat{a}} = r\varphi(x)\omega_{\hat{a}} \ , \tag{16}$$

i.e. the unconstrained degrees of freedom correspond to the scalar field $\varphi(x)$ which is a singlet under the four-dimensional gauge group K.

The choice (15) defines one of the possible embedding of Lie(SU_2) in Lie(U_n). For example we could also embed Lie(SU_2) in Lie(U_n) using the irreducible n-dimensional rep. of SU_2, i.e. we could identify $\omega_{\hat{a}} = X_{\hat{a}}$. The constraint (12) in this case implies that the four-dimensional gauge group is U_1 so that $A_\mu(x)$ is U_1 valued. The constraint (13) leads again to the scalar singlet $\varphi(x)$.

In general, we start with a U_1 gauge theory on $M^4 \times S^2_F$. We solve the CSDR constraint (14) by embedding SU_2 in U_n. There exist p_n embeddings, where p_n is the number of ways one can partition the integer n into a set of non-increasing positive integers [14]. Then the constraint (12) gives the surviving four-dimensional gauge group. The constraint (13) gives the surviving four-dimensional scalars and eq. (16) is always a solution but in general not the only one. By setting $\phi_{\hat{a}} = \omega_{\hat{a}}$ we obtain always a minimum of the potential. This minimum is given by the chosen embedding of SU_2 in U_n.

5. ATTEMPTS TO GENERATE SO_n GAUGE GROUP FROM DIMENSIONAL REDUCTION OVER FUZZY SPACES

5.1. $U_n(\mathbb{C})$ Lie algebra and real matrix representation

We first notice that the algebra $\mathcal{C}$ of complex numbers can be thought of as a subalgebra M_τ of the algebra $M_2(\mathbb{R})$ of 2×2 real matrices. The subalgebra M_τ is generated by the identity and by the Pauli matrix $\tau = i\sigma_2$. This element has been chosen because of the relation $\tau^T = -\tau$. The map is given by

$$x + iy \mapsto \mathbf{1}(x) + \tau(y) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \tag{17}$$

It is constructed simply by the replacement $i \mapsto \tau$. One sees that the map (17) is an algebra isomorphism of $\mathbb{C}$ with M_τ which respects involution and norm. In particular we notice that $M_2(\mathbb{R})$ is not simple.

Next we would like to examine the global SO_2 transformation. Just as i generates the Lie algebra of U_1 so generates τ that of SO_2. Let $\xi \in \mathbb{C}^2$ be a spinor and consider the map

$$\xi \mapsto \Lambda\xi, \qquad \Lambda = e^{\tau\alpha}, \quad \alpha \in \mathbb{R}. \tag{18}$$

Since $\Lambda^T = \Lambda^{-1}$ we see that $\xi^T\xi$ is invariant.

To motivate better the notation we recall the U_n case. In the U_1 case one replaces a complex number by a complex matrix, both of which have a star operation with which the corresponding embedding is consistent; if

$$f \in \mathbb{C} \mapsto F \in M_n(\mathbb{C}) = \mathbb{C} \otimes M_n(\mathbb{R}) \tag{19}$$

then

$$f^* \in \mathbb{C} \mapsto F^*. \tag{20}$$

In the real case one must replace the adjoint by the transpose and $\mathbb{C}$ by M_τ; if

$$f \in \mathbb{R} \mapsto F \in M_{2n\tau}(\mathbb{R}) = M_\tau \otimes M_n(\mathbb{R}) \tag{21}$$

then

$$f^T \in \mathbb{R} \mapsto F^T. \tag{22}$$

According to the prescription the gauge group consists of elements of the algebra $M_{2n\tau}(\mathbb{R})$ which is a real subalgebra of $M_{2n}(\mathbb{R})$ and spanned by generators of the form

$$g = (M_n^{(A)}(\mathbb{R}) \otimes \mathbf{1}_n) + (M_n^{(S)}(\mathbb{R}) \otimes \tau), \tag{23}$$

with S (A) superscripts denoting the symmetric (antisymmetric) square matrices $M_n(\mathbb{R})$ and $\tau = i\sigma_2$ the SO_2 generator. The gauge group generated by g is actually a subgroup of $SO_{2n}(\mathbb{R})$, describable by n^2 independent components in total, i.e. U_n. Therefore we did not manage to generate the SO_n gauge group structure we were looking for.

5.2. Algebra of functions and non-closure of their space

We would like to consider a U_1 (SO_2) gauge theory in $4{+}E$ dimensions, where E is the number of extra dimensions. Then, as done in the case of the fuzzy sphere (there $E = 2$), we would like to reinterpret the higher dimensional theory as a 4-dimensional gauge thery with an enhanced gauge group (U_n for the fuzzy sphere at level $n-1$). However now we want to obtain an orthogonal gauge group rather than a unitary one. Orthogonal gauge groups are more interesting than unitary ones in the context of obtaining phenomenological lagrangians via CSDR. In this section we give some general comments describing why it is difficult to achieve this result.

We remind that a proper description of a fuzzy manifold can be achieved by a finite order polynomial expansion over its fuzzy coordinates, together with a redefinition of the product operation in a way that the closure of the algebra will be satisfied. This is the case, for instance in the usual S_F^2. An arbitrary function $f(x^\mu, X^{\hat{a}})$ is expressed as a completely symmetrized polynomial expansion over the fuzzy coordinates $iX^{\hat{a}}$.

$$f(x^\mu, X^{\hat{a}}) = f_0(x^\mu) + f_{\hat{a}}(x^\mu)X^{\hat{a}} + \frac{1}{2!}f_{\hat{a}\hat{b}}(x^\mu)X^{\hat{a}}X^{\hat{b}} + \cdots + \frac{1}{r!}f_{\hat{a}_1,\cdots,\hat{a}_r}(x^\mu)X^{\hat{a}_1}\cdots X^{\hat{a}_r}\,, \quad (24)$$

where the extra i has been absorbed by the complex components of the traceless symmetric tensors $f_{\hat{a}_1,\cdots,\hat{a}_r}$ that are the coefficients of the above expansion.

In particular the gauge potential $A_\mu(x, X)$ is an antihermitian function. However now we want $A_\mu(x, X)$ to be also antisymmetric so that it can be considered Lie($\mathrm{SO_n}$) valued. It is natural to require the gauge potential A to be odd in the fuzzy coordinates iX and the fuzzy coordinates iX to be antisymmetric. In this way $A_\mu(x, X)$ is an antisymmetric. However antisymmetric matrices X cannot be anymore thought as coordinates of a fuzzy space. Indeed the product of two antisymmetric matrices is no more an antisymmetric matrix. Antisymmetric matrices do not form an algebra under matrix multiplication.

6. CONCLUSIONS

The Fuzzy-CSDR has different features from the ordinary CSDR leading therefore to new four-dimensional particle models.

A major difference between fuzzy and ordinary CSDR is that in the fuzzy case one always embeds S in the gauge group G instead of embedding just R in G. This is due to the fact that the differential calculus on the fuzzy coset space is based on $dimS$ derivations instead of the restricted $dimS - dimR$ used in the ordinary one. As a result the four-dimensional gauge group $H = C_G(R)$ appearing in the ordinary CSDR after the geometrical breaking and before the spontaneous symmetry breaking due to the four-dimensional Higgs fields does not appear in the Fuzzy-CSDR. In Fuzzy-CSDR the spontaneous symmetry breaking mechanism takes already place by solving the Fuzzy-CSDR constraints. Therefore in four dimensions appears only the physical Higgs field that survives after a spontaneous symmetry breaking. Moreover, we see that if one would like to describe the spontaneous symmetry breaking of the SM in the present framework, then one would be naturally led to large extra dimensions.

A fundamental difference between the ordinary CSDR and its fuzzy version is the fact that a non-abelian gauge group G is not really required in high dimensions. Indeed the presence of a U_1 in the higher-dimensional theory is enough to obtain non-abelian gauge theories in four dimensions.

A very exciting point that should be stressed [13] is the question of the renormalizability of the gauge theory defined on $M_4 \times (S/R)_F$. First we notice that the theory exhibits certain features so similar to a higher-dimensional gauge theory defined on $M_4 \times S/R$ that naturally it could be considered as a higher-dimensional theory too. For instance the isometries of the spaces $M_4 \times S/R$ and $M_4 \times (S/R)_F$ are the same. It does not matter if the compact space is fuzzy or not. For example in the case of the fuzzy sphere, i.e. $M_4 \times S_F^2$, the isometries are $SO_{3,1} \times SO_3$ as in the case of the continuous space, $M_4 \times S^2$. Similarly the coupling of a gauge theory defined on $M_4 \times S/R$ and on $M_4 \times (S/R)_F$ are both dimensionful and have exactly the same dimensionality. On the other hand the first theory is clearly non-renormalizable, while the latter is renormalizable (in the sense that divergencies can be removed by a finite number of counterterms). So from this point of view one finds a partial justification of the old hopes for considering quantum field theories on non-commutative structures. If this observation can lead to finite theories too, it remains as an open question.

Our hope is that we may be able to construct realistic four-dimensional theories in case we manage to generate SO_n gauge groups from dimensional reduction over fuzzy spaces. Unfortunately this hope is not realized yet.

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