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CUBIC ALGEBRAS AND GENERALIZED STATISTICS

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Abstract.

This talk is based on a common work [1] *with Todor Popov.*

1. INTRODUCTION

Recently there has been an increased interest in cubic homogeneous algebras in relation to the Matrix model of string theory [2]. The operators of the Matrix model, identified with the covariant coordinates of a noncommutative gauge theory, satisfy equations of motion that have the form of classical Yang-Mills equations. In a gauge field theory the Yang-Mills equations are the equations for connections on bundles over the (pseudo)Euclidean space, expressed in a compact form as cubic homogeneous relations. The gauge covariant operators satisfying the Yang-Mills equations generate an associative algebra, called the Yang-Mills algebra [3]. Thus any solution of the Yang-Mills equation carries a representation of the abstract algebra. The analysis and description of the representations of the abstract cubic algebra will enrich the variety of solutions of the Yang-Mills equations. This is believed to be useful in understanding the problem of gauge/string correspondence which includes relations between string theory null states and solutions of the Yang-Mills equations.

The Yang-Mills algebra has two natural quotients related to parastatistics [4] which has so far provided an application of cubic algebras for the study of generalized types of statistics. Paraquantization [6] was introduced by Green in 1953 as a general quantization method of quantum field theory

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different from the cannonical one. It is based on trilinear exchange relations for two families of creation and annihilation operators whose quantization generalizes the Bose Fermi alternative. The generalized statistics is charactarized by the order of paraquantization p which distinguishes the two families as operators of parabose and parafermi algebras. The physical interpretation of p (integer) is that, for parabosons, it is the maximum number of particles that can occupy an antisymmetric state, while for parafermions, p is the maximum number of particles that can occupy a symmetric state. The states are correspondingly the representations of the symmetric group S_n with at most p rows for parabosons and at most p columns for parafermions. The value $p = 1$ reproduces the usual Bose and Fermi statistics.

At the core of the interest in generalized statistics is (twodimensional) statistical mechanics of phenomena such as fractional Hall effect, high-T_c superconductivity (the experiments on quantum Hall effect confirm the existence of fractionally charged excitations) [5]. Models with fractional statistics and infinite statistics have been explored, termed as anyon statistics, quon statistics, Haldane fractional statistics.

The attempts to develop nonstandard quantum statistics evolved naturally to the study of deformed parastatistics algebras. It appeared as a most natural way to follow the analogy with classical (nondeformed) parastatistics where the parafermi and parabose operators are generators of the univesal enveloping algebras (UEA) of the Lie algebras $so(2n + 1)$ and $osp(1|2n)$ respectively. Analogously the quantized parafermi, parabose algebras are isomorphic *as algebras* to the quantized UEA $U_q(so(2n + 1))$, $U_q(osp(1|2n))$ respectively. We consider a complete basis of defining relations for the deformed parastatistics algebras. The novelty with respect to the known definition of deformed parastatistics are the homogeneous relations. The proposed definition allows to continue the algebras isomorphism to a Hopf morphism which endows the deformed parastatistics algebras with a natural Hopf structure. With this Hopf structure the proposed deformed parafermi and parabose algebras are isomorphic as *Hopf algebras* to $U_q(so(2n + 1))$ and $U_q(osp(1|2n))$ respectively. We then study the deformed analogue of the Green ansatz that follows from the coproduct on the parastatistics Hopf algebra. The noncocommutative coproduct allows for construction of parastatistics Fock-like representations, built out of the simplest deformed bose and fermi representations. The construction gives rise to quadratic algebras of deformed anomalous commutation relations which define the generalized Green ansatz.

2. GREEN PARASTATISTIC ALGEBRAS

Green Parastatistics was introduced as a generalization of the Bose-Fermi alternative.

DEFINITION 1 *The parafermi algebra $pF(n)$ (parabose algebra $pB(n)$) is an associative algebra generated by the creation a^{+i} and annihilation a_i^- operators for $i = 1, \ldots, n$ subject to the relations*

$$\begin{array}{rcr} [[a^{+i}, a_j^-]_\mp, a^{+k}] & = & 2\delta_j^k a^{+i} \\ [[a^{+i}, a^{+j}]_\mp, a^{+k}] & = & 0 \\ [[a^{+i}, a_j^-]_\mp, a_k^-] & = & -2\delta_k^i a_j^- \\ [[a_i^-, a_j^-]_\mp, a_k^-] & = & 0 \end{array} \tag{1}$$

The upper (lower) sign refers to the parafermi algebra $pF(n)$ (parabose algebra $pB(n)$).

In the definition only the linearly independent relations are written. Through the (super)Jacobi identity one obtains also the relations

$$[[a^{+i}, a^{+j}]_\mp, a_k^-] = \pm 2\delta_k^j a^{+i} - 2\delta_k^i a^{+j}$$

$$[[a_i^-, a_j^-]_\mp, a^{+k}] = 2\delta_k^j a^- \mp 2\delta_k^i a^- .$$

The bilinear combinations $e_j^i = \frac{1}{2}[a^{+i}, a_j^-]_\mp$ close a linear algebra $gl(n)$. (a^{+i}, (a_j^-) transform as contravariant, (covariant) vectors with respect to the $gl(n)$-action.)

The Hamiltonian $\mathcal{H} = \sum_{i=1}^n \frac{1}{2}[a^{+i}, a_i^-]_\mp$ of the parastatistics system has as eigenvectors the creation a^{+i} and annihilation a_j^- operators: $[\mathcal{H}, a^{+i}] = a^{+i}, \quad [\mathcal{H}, a_j^-] = -a_j^-$.

We accept the superalgebraic point of view:

$$\begin{array}{rcr} [\![[\![a^{+i}, a_j^-]\!], a^{+k}]\!] & = & 2\delta_j^k a^{+i} \\ [\![[\![a^{+i}, a^{+j}]\!], a^{+k}]\!] & = & 0 \\ [\![[\![a^{+i}, a_j^-]\!], a_k^-]\!] & = & -2\delta_k^i a_j^- \\ [\![[\![a_i^-, a_j^-]\!], a_k^-]\!] & = & 0 \end{array} \tag{2}$$

where $[\![a,b]\!] = ab - (-1)^{deg(a)deg(b)}ba$ and $deg(x) \in \{\bar{0}, \bar{1}\}$ is the Z_2 degree of x. Then for the parafermi $pF(n)$ (parabose $pB(n)$) case all the generators are even (odd)

$$\deg(a^{+i}) = deg(a_j^-) = \bar{0}, \qquad (\deg(a^{+i}) = deg(a_j^-) = \bar{1}),$$

The parastatistics algebras admit an antilinear antiinvolution $*$ $(ab)^* = b^* a^*$ (referred to as conjugation)

$$(a^{+i})^* = a_i^-, \quad (a_i^-)^* = a^{+i}.$$

The parafermi algebra $pF(n)$ is isomorphic to UEA of the orthogonal algebra $so(2n+1)$, the parabose algebra $pB(n)$ is isomorphic to UEA of the orthosymplectic algebra $osp(1|2n)$

$$pF(n) \simeq U(so(2n+1)) \qquad pB(n) \simeq U(osp(1|2n))$$

Thus the trilinear relations (1) provide an alternative set of relations for the algebras $so(2n+1)$ and $osp(1|2n)$ in terms of paraoscillators.

3. DEFORMED PARASTATISTIC ALGEBRAS

The idea of quantization of the parastatistics algebras is to "quantize" the isomorphisms, i.e., to deform the trilinear relations (1) so that the arising deformed parafermi $pF_q(n)$ and parabose $pB_q(n)$ algebras are isomorphic to the QUEAs

$$pF_q(n) \simeq U_q(so(2n+1)) \qquad pB_q \simeq U_q(osp(1|2n))$$

and then continue the algebraic isomorphism to a Hopf morphism which endows the deformed parastatistics with a natural Hopf structure.

The Lie superalgebra $osp(1|2n)$ ($B(0|n)$ in Kac notation) has the same Cartan matrix as the simple B_n algebra $so(2n+1)$. Let $H_i, E_{\pm i}$ be the Chevalley basis of $so(2n+1)$ or $osp(1|2n)$

$$H^{\alpha_i} = H_i, \qquad E^{\pm\alpha_i} = E_{\pm i} \qquad 1 \le i \le n. \tag{3}$$

The Lie superalgebra $osp(1|2n)$ has a grading induced by $deg(H_i) = \bar{0}$ and

$$deg(E_{\pm i}) = \bar{0} \quad 1 \le i \le n-1 \qquad deg(E_{\pm n}) = \bar{1} \tag{4}$$

All generators of the Lie algebra $so(2n+1)$ are even.

The QUE algebras $U_q(so(2n+1))$ and $U_q(osp(1|2n))$ are associative algebras generated, in the Chevalley basis, by $q^{\pm H_i}$ and $E_{\pm i}$ subject to the relations

$$\begin{aligned}
&q^{H_i}q^{H_j} = q^{H_j}q^{H_i} && i,j \le n\\
&q^{H_i}E_{\pm j}q^{-H_i} = q^{\pm a_{ij}}E_{\pm j} && i,j \le n\\
&[2][E_i, E_{-j}] = \delta_{i,j}[2H_i] && i \le n-1\\
&[\![E_n, E_{-n}]\!] = [H_n] && \\
&[E_{\pm i}, E_{\pm j}] = 0 && |i-j| \ge 2\\
&[E_{\pm i}, [E_{\pm i}, E_{\pm(i+1)}]_q]_{q^{-1}} = 0 && i \le n-1\\
&[E_{\pm(i+1)}, [E_{\pm(i+1)}, E_{\pm i}]_q]_{q^{-1}} = 0 && i \le n-2\\
&[[\![[E_{\pm(n-1)}, E_{\pm n}]_{q^{-1}}, E_{\pm n}]\!], E_{\pm n}]_q = 0 &&
\end{aligned} \tag{5}$$

where $[x,y]_q = xy - qyx$ is the q-commutator, α_n is the only odd simple root of $osp(1|2n)$, $a_{ij} = (\alpha_i, \alpha_j)$ is the symmetrized Cartan matrix (same for both cases) given by $a_{ij} = 2\delta_{ij} - \delta_{in} - \delta_{i+1j} - \delta_{ij+1}$ and the quantum bracket is chosen to be $[x] := \frac{q^{\frac{x}{2}} - q^{-\frac{x}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$.

We consider a change of basis by using the subset of short roots ε_i related to the simple roots by

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1} \qquad 1 \le i \le n-1, \qquad\qquad \alpha_n = \varepsilon_n \tag{6}$$

Then the corresponding change of basis on the Cartan subalgebra is

$$H_i = h_i - h_{i+1} \qquad 1 \le i \le n-1, \qquad\qquad H_n = h_n. \tag{7}$$

By construction $q^{h_i}q^{h_j} = q^{h_j}q^{h_i}$.

The ladder operators $E^{+\varepsilon_i}$ and $E^{-\varepsilon_i}$ related to the roots ε_i are a^{+i} and a_j^- and therefore the inverse change $\varepsilon_i = \sum_{k=i}^n \alpha_k$ implies

$$\begin{aligned}
a^{+i} &= [E_i, [E_{i+1}, \ldots [E_{n-1}, E_n]_{q^{-1}} \ldots]_{q^{-1}}]_{q^{-1}}\\
a_i^- &= [[\ldots [E_{-n}, E_{-n+1}]_q \ldots, E_{-(i+1)}]_q, E_{-i}]_q
\end{aligned} \tag{8}$$

On the other hand the Chevalley generators are expressed as

$$\begin{aligned}
E_i &= \tfrac{1}{[2]} q^{-h_{i+1}} [\![a^{+i}, a_{i+1}^-]\!]\\
E_{-i} &= \tfrac{1}{[2]} [\![a^{+(i+1)}, a_i^-]\!] q^{h_{i+1}} \qquad i < n\\
E_n &= a^{+n} \qquad\qquad E_{-n} = a_n^-
\end{aligned} \tag{9}$$

One has

$$q^{h_i} a^{+j} q^{-h_i} = q^{\delta_{ij}} a^{+j} \quad q^{h_i} a_j^- q^{-h_i} = q^{-\delta_{ij}} a_j^- \tag{10}$$

The graded commutator of opposite ladder operators

$$[\![a^{+i}, a_i^-]\!] = [2h_i] \tag{11}$$

defines the partial hamiltonian $\mathcal{H}_i$ (of the i-th paraoscillator)

$$\mathcal{H}_i = \frac{1}{[2]} [\![a^{+i}, a_i^-]\!] = \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}} \tag{12}$$

The full hamiltonian $\mathcal{H}$ is the sum over all paraoscillators $\mathcal{H} = \sum_{i=1}^n \mathcal{H}_i$.

The antiinvolution $*$ on the new generators is

$$(a^{+i})^* = a_i^- \qquad (a_i^-)^* = a^{+i} \qquad (q^{\pm h_i})^* = q^{\mp h_i} \tag{13}$$

and also $(q)^* = q^{-1}$, (q on the unit circle). (and for the Chevalley basis $(E_{\pm i})^* = E_{\mp i}$, $H_i^* = H_i$)

The complete basis of relations defining the deformed parastatistics algebras, is given by the following

THEOREM 1 *The quantum parafermi $pF_q(n)$ (parabose $pB_q(n)$) algebra is the associative (super)algebra generated by the even Cartan generators q^{h_i} for $i = 1, \ldots, n$ and the even (odd) raising a^{+i} and lowering a_i^- generators subject to the relations*

$$[\![[\![a^{+i}, a_j^-]\!], a^{+k}]\!]_{q^{-\delta_{ik}\sigma(j,k)}} = [2]\delta_j^k a^{+i} q^{\sigma(i,j)h_j} + (q - q^{-1})\theta(i,j;k) a^{+i} [\![a^{+k}, a_j^-]\!] \tag{14}$$

$$\begin{array}{rcll} [\![[\![a^{+i_1}, a^{+i_3}]\!], a^{+i_2}]\!]_{q^2} + q[\![[\![a^{+i_1}, a^{+i_2}]\!], a^{+i_3}]\!] & = & 0 & i_1 < i_2 < i_3 \\ [\![[\![a^{+i_1}, a^{+i_2}]\!], a^{+i_2}]\!]_q & = & 0 & i_1 < i_2 \\ [\![a^{+i_2}, [\![a^{+i_1}, a^{+i_3}]\!]]\!]_{q^2} + q[\![a^{+i_1}, [\![a^{+i_2}, a^{+i_3}]\!]]\!] & = & 0 & i_1 < i_2 < i_3 \\ [\![a^{+i_2}, [\![a^{+i_2}, a^{+i_3}]\!]]\!]_q & = & 0 & i_2 < i_3 \end{array} \tag{15}$$

as well as their conjugated

$$[\![[\![a^{+i}, a_j^-]\!], a_k^-]\!]_{q^{-\delta_{jk}\sigma(i,k)}} = -[2]\delta_k^i a_j^- q^{-\sigma(i,j)h_i} - (q - q^{-1})\theta(j,i;k)[\![a^{+i}, a_k^-]\!] a_j^- \tag{16}$$

$$\begin{array}{rcll} [\![[\![a_{i_1}^-, a_{i_3}^-]\!], a_{i_2}^-]\!]_{q^2} + q[\![[\![a_{i_1}^-, a_{i_2}^-]\!], a_{i_3}^-]\!] & = & 0 & i_1 < i_2 < i_3 \\ [\![[\![a_{i_1}^-, a_{i_2}^-]\!], a_{i_2}^-]\!]_q & = & 0 & i_1 < i_2 \\ [\![a_{i_2}^-, [\![a_{i_1}^-, a_{i_3}^-]\!]]\!]_{q^2} + q[\![a_{i_1}^-, [\![a_{i_2}^-, a_{i_3}^-]\!]]\!] & = & 0 & i_1 < i_2 < i_3 \\ [\![a_{i_2}^-, [\![a_{i_2}^-, a_{i_3}^-]\!]]\!]_q & = & 0 & i_2 < i_3 \end{array} \tag{17}$$

where $\sigma(i,j) = \epsilon_{ij} + \delta_{ij}$ [2] *or* $\sigma(i,j) = \epsilon_{ij} - \delta_{ij}$ *and* $\theta(i,j;k) = \frac{1}{2}\epsilon_{ij}\epsilon_{ijk}(\epsilon_{jk} - \epsilon_{ik})$[3].

The inhomogeneous relations are related to the adjoint action of a deformed linear algebra. The homogeneous relations describe an ideal which is a $U_q(gl(n))$-module, a deformation of a Schur module $E^{(2,1)}$.

To prove the theorem we make use of the R-matrix FRT-formalism for QUEA $U_q(g)$ of a simple (super-)Lie algebra g (see [7],[8]), introducing the L-functionals for $U_q(g)$ in the form of upper(lower)-triangular matrices $L^{(+)}$ ($L^{(-)}$)

$$R^{(+)}L_1^{(\pm)}L_2^{(\pm)} = L_2^{(\pm)}L_1^{(\pm)}R^{(+)} \qquad R^{(+)}L_1^{(+)}L_2^{(-)} = L_2^{(-)}L_1^{(+)}R^{(+)} \tag{18}$$

where $L_1^{(\pm)} = 1\otimes L^{(\pm)}$, $L_2^{(\pm)} = L^{(\pm)}\otimes 1$ and $R^{(+)} = PRP$ is the corresponding R-matrix for $U_q(g)$.

The $(n+1)\times(n+1)$ minor $L_{ij}^{(+)}$, $1 \le i,j \le n+1$ of the $(2n+1)\times(2n+1)$ matrix $L^{(+)}$ for the QUEA $U_q(so(2n+1))$ and $U_q(osp(1|2n))$ is very simple when expressed in terms of the generators $a_i^{\pm}$

$$\left(L_{ij}^{(+)}\right)_{1\le i,j\le n+1} = \begin{pmatrix} q^{h_1} & \omega[\![a_1^+, a_2^-]\!] & \omega[\![a_1^+, a_3^-]\!] & \dots & \omega[\![a_1^+, a_n^-]\!] & ca_1^+ \\ 0 & q^{h_2} & \omega[\![a_2^+, a_3^-]\!] & \dots & \omega[\![a_2^+, a_n^-]\!] & ca_2^+ \\ 0 & 0 & q^{h_3} & \dots & \omega[\![a_3^+, a_n^-]\!] & ca_3^+ \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & q^{h_n} & ca_n^+ \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \tag{19}$$

where $\omega = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$. The coefficient $c = q^{-\frac{1}{2}}(q - q^{-1})$. One has $(L_{ij}^{(+)})^* = L_{ji}^{(-)}$.

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[2] Levi-Chevita symbol $\epsilon_{ij} = 1$ for $i < j$

[3] The function $\theta(i,j;k) = -\theta(j,i;k)$ is 0 ; or 1 and -1 for $i < k < j$ and $i > k > j$, respectively.

Hence the inhomogeneous relations (14) ((16)), involving the entries of the minors of $L_{ij}^{(\pm)}$ (19) for $1 \le i,j \le n+1$, follow directly from the RLL-relations(18) with the corresponding R-matrix upon restricting the indices from 1 to $n+1$. The homogeneous relations (15) (as well as (17)) form an irreducible finite-dimensional $U_q(gl_n)$-module $\mathcal{L}$ with respect to the adjoint action of $U_q(gl_n)$. The dimension of the representation is equal to the number of the semistandard Young tableaux which are fillings with numbers $(1,...,n)$ of the diagram $(2,1)$, $\dim\mathcal{L} = \frac{(n+1)n(n-1)}{3}$.

4. HOPF STRUCTURE ON PARASTATISTIC ALGEBRAS

The QUE algebras $U_q(so(2n+1))$ and $U_q(osp(1|2n))$ endowed with the Drinfeld-Jimbo coalgebraic structure

$$\begin{array}{rclrcl} \Delta H_i &=& H_i\otimes 1 + 1\otimes H_i & S(H_i) &=& -H_i \\ \Delta E_i &=& E_i\otimes 1 + q^{H_i}\otimes E_i & S(E_i) &=& -q^{-H_i}E_i \\ \Delta E_{-i} &=& E_{-i}\otimes q^{-H_i} + 1\otimes E_{-i} & S(E_{-i}) &=& -E_{-i}q^{H_i} \end{array} \tag{20}$$

$$\epsilon(H_i) = \epsilon(E_i) = \epsilon(E_{-i}) = 0$$

become Hopf algebra and Hopf superalgebra, respectively. (Superalgebras have a graded Hopf structure with antipode which is a *graded* antihomomorphism

$$S(ab) = (-1)^{deg(a)deg(b)}S(b)S(a). \tag{21}$$

The conjugation $*$ (13) for $|q| = 1$ is a coalgebraic antihomomorphism, $(\Delta x)^* = \sum(x_{(1)}\otimes x_{(2)})^* = \sum x^*_{(2)}\otimes x^*_{(1)}$ and $S(x^*) = S(x)^*$ for $x \in U_q$.

The isomorphisms to the QUEA induce Hopf structure on the deformed parastatistics algebras.

THEOREM 2 *The deformed parafermionic algebra $pF_q(n)$, the deformed parabosonic algebra $pB_q(n)$ is a Hopf algebra, a Hopf superalgebra, respectively when endowed with*

(i) a coproduct Δ defined on the generators by

$$\Delta q^{\pm h_i} = q^{\pm h_i}\otimes q^{\pm h_i} \tag{22}$$

$$\Delta a^{+i} = a^{+i}\otimes 1 + q^{h_i}\otimes a^{+i} + \omega\sum_{i<j\le n} [\![a^{+i}, a_j^-]\!]\otimes a^{+j} \tag{23}$$

$$\Delta a_i^- = a_i^-\otimes q^{-h_i} + 1\otimes a_i^- - \omega\sum_{i<j\le n} a_j^-\otimes [\![a^{+j}, a_i^-]\!] \tag{24}$$

(ii) a counit ϵ defined on the generators by

$$\epsilon(q^{\pm h_i}) = 1 \qquad \epsilon(a^{i+}) = \epsilon(a_i^-) = 0 \tag{25}$$

(iii) an antipode S (graded for $pB_q(n)$) defined on the generators by

$$S(q^{\pm h_i}) = q^{\mp h_i} \tag{26}$$

$$S(a^{+i}) = -q^{-h_i}a^{+i} - \sum_{s=1}^{n-i}(-\omega)^s \sum_{i<j_1<\ldots<j_s\leq n} W^{+i}_{\ j_1} W^{+j_1}_{\ j_2} \ldots W^{+j_{s-1}}_{\ j_s} q^{-h_{j_s}} a^{+j_s} \tag{27}$$

$$S(a_i^-) = -a_i^- q^{h_i} - \sum_{s=1}^{n-i}(\omega)^s \sum_{n\geq j_s>\ldots>j_1>i} a_{j_s}^- q^{h_{j_s}} W^{-j_s}_{\ j_{s-1}} \ldots W^{-j_2}_{\ j_1} W^{-j_1}_{\ i} \tag{28}$$

where $W^{+i}_{\ j} = q^{-h_i}[\![a^{+i}, a_j^-]\!]$, $W^{-j}_{\ i} = [\![a^{+j}, a_i^-]\!]q^{h_i}$ and $\omega = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$.

The proof of the theorem follows from the Hopf structure on the elements of $L^{(+)}$ and $L^{(-)}$, given by the coproduct $\Delta L^{\pm}$, the counit $\epsilon(L^{(\pm)})$

$$\Delta L_k^{i(\pm)} = \sum_j L_j^{i(\pm)} \otimes L_k^{j(\pm)} \qquad \epsilon(L_k^{i(\pm)}) = \delta_k^i \tag{29}$$

and the antipode $S(L^{(\pm)})$

$$\sum_j L_j^{i(\pm)} S(L_k^{j(\pm)}) = \delta_k^i = \sum_j S(L_j^{i(\pm)}) L_k^{j(\pm)}. \tag{30}$$

5. THE OSCILLATOR REPRESENTATIONS

The unitary representations π_p of the parastatistics algebras $pB(n)$ and $pF(n)$ with unique vacuum state are indexed by a non-negative integer p . The representation π_p is the lowest weight representation with a unique vacuum state $|0\rangle$ annihilated by all a_i^- and labelled by the *order of parastatistics* p

$$\pi_p(a_i^-)|0\rangle = 0 \qquad \pi_p(a_i^-)\pi_p(a^{+j})|0\rangle = p\delta_i^j|0\rangle\,. \tag{31}$$

The *vacuum representation* (the trivial one with $p = 0$) is given by the counit

$$\pi_0(x)|0\rangle = \epsilon(x)|0\rangle \qquad x \in pB(n), pF(n). \tag{32}$$

In the representation π_p of the nondeformed parastatistics algebras the hamiltonian $h_i = \frac{1}{2}[a^{+i}, a_i^-]_\mp$ and the number operator $N_i = a^{+i}a_i^-$ of the i-th paraoscillator are related by

$$h_i = N_i \mp \frac{p}{2} \tag{33}$$

where the upper (lower) sign is for parafermions (parabosons). In the representation π_p of the deformed parastatistics algebras the quantum analogue of the relation holds

$$[a^{+i}, a_i^-]_\mp = [2]\mathcal{H}_i = [2h_i] = [2N_i \mp p]$$

which implies the deformed analogue of the π_p defining condition

$$a_i^-(p)a^{+j}(p)|0\rangle^{(p)} = [p]\delta_i^j|0\rangle^{(p)}\,. \tag{34}$$

The constant $\mp[p]/[2]$ plays the role of energy of the vacuum (as well as $\mp\frac{p}{2}$ in the nondeformed case)

$$\mathcal{H}_i|0\rangle^{(p)} = \mp\frac{[p]}{[2]}|0\rangle^{(p)}\,.$$

The algebra of the q-deformed fermionic (bosonic) oscillators $F_q(n)$ $(B_q(n))$ arises as a representation π of order $p = 1$ of the $pF_q(n)(pB_q(n))$

$$\left.\begin{array}{rclrcll}
\underline{a}_i^+\underline{a}_j^+ \pm q^{\mp\epsilon_{ij}}\underline{a}_j^+\underline{a}_i^+ & = & 0 & \underline{a}_i^-\underline{a}_j^- \pm q^{\mp\epsilon_{ij}}\underline{a}_j^-\underline{a}_i^- & = & 0 & \\
\underline{a}_i^-\underline{a}_i^+ \pm q\underline{a}_j^+\underline{a}_i^- & = & q^{\pm\underline{N}_i} & \underline{a}_i^-\underline{a}_i^+ \pm q^{-1}\underline{a}_i^+\underline{a}_i^- & = & q^{\mp\underline{N}_i} & \\
\underline{a}_i^+\underline{a}_j^- \pm q^{\mp\epsilon_{ji}}\underline{a}_j^-\underline{a}_i^+ & = & 0 & \underline{a}_i^-\underline{a}_j^+ \pm q^{\mp\epsilon_{ji}}\underline{a}_j^+\underline{a}_i^- & = & 0 & i \neq j
\end{array}\right\} \tag{35}$$

where $\pi(x) = \underline{x}$ and $\underline{N}_i = \underline{h}_i \mp \frac{1}{2}$.

6. GREEN ANSATZ

The Green ansatz states - The parafermi (parabose) oscillators a^{+i} and a_i^- can be represented as sums of p fermi (bose) oscillators

$$\pi_p(a^{+i}) = \sum_{r=1}^{p} a^{+i}_{(r)} \qquad \pi_p(a_i^-) = \sum_{r=1}^{p} a^-_{i(r)} \tag{36}$$

satisfying quadratic commutation relations of the same type (i.e., fermi for parafermi and bose for parabose) for equal indices (r)

$$[a^-_{i(r)}, a^{+k}_{(r)}]_\pm = \delta_i^k, \quad [a^-_{i(r)}, a^-_{k(r)}]_\pm = [a^{+i}_{(r)}, a^{+k}_{(r)}]_\pm = 0, \tag{37}$$

and of the opposite type for the different indices

$$[a^-_{i(r)}, a^-_{k(s)}]_\mp = [a^{+i}_{(r)}, a^{+k}_{(s)}]_\mp = [a^-_{i(r)}, a^{+k}_{(s)}]_\mp = 0, r \neq s. \tag{38}$$

The upper (lower) signs stay for the parafermi (parabose) case.

The coproduct endows the tensor product of $\mathcal{A}$-modules of the Hopf algebra $\mathcal{A}$ with the structure of an $\mathcal{A}$-module. Thus one can use the coproduct for constructing a representation out of simple ones. The simplest representations of the parastatistics algebras are the oscillator representations π (with $p = 1$). A parastatistics algebra representation of arbitrary order arises through the iterated coproduct.

Let us denote the (p-fold) iteration of the coproduct by

$$\Delta^{(0)} = id, \Delta^{(1)} = \Delta, \Delta^{(p)} = (\underbrace{\Delta \otimes 1 \ldots \otimes 1}_{p-1}) \circ \Delta^{(p-1)} \tag{39}$$

and π denotes the projection from the (deformed) parafermi and parabose algebra onto the (deformed) fermionic F (F_q) and bosonic B (B_q) Fock representation, respectively. Then we have

$$\begin{aligned}\pi_p(a^{+i}) = \pi^{\otimes p}\circ\Delta^{(p)}(a^{+i}) &:= \sum_{r=1}^{p} a^{+i}_{(r)}\\ \pi_p(a_i^-) = \pi^{\otimes p}\circ\Delta^{(p)}(a_i^- &:= \sum_{r=1}^{p} a^{-}_{i(r)}\end{aligned} \tag{40}$$

Consistency of the vacuum condition with the deformed Green ansatz requires the vacuum state $|0\rangle^{(p)}$ of the representation π_p to be identified with the tensor power of the oscillator ($p=1$) vacuum, $|0\rangle^{(p)} = |0\rangle^{\otimes p}$. Evaluating the iterated graded commutator

$$\Delta^{(p)}[\![a^{+i},\, a_i^-]\!] = \frac{(q^{h_i})^{\otimes p} - (q^{-h_i})^{\otimes p}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \tag{41}$$

on the vacuum state $|0\rangle^{\otimes p}$ in the oscillator representations $\pi^{\otimes p}$ we get the defining condition of the deformed π_p

$$\mp\pi^{\otimes p}\circ\Delta^{(p)}[\![a^{+i},\, a_i^-]\!]|0\rangle^{(p)} = \pi_p(a_i^-)\pi_p(a^{+i})|0\rangle^{(p)} = [p]|0\rangle^{(p)} \quad (= \frac{q^{\frac{p}{2}} - q^{-\frac{p}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}|0\rangle^{(p)})$$

since $\pi(q^{h_i}) = q^{N_i \mp \frac{1}{2}}$, which proves the consistency.

The Green components $a^{+i}_{(r)}$ and $a^-_{i(r)}$ in a $pF_q(n)$ or $pB_q(n)$ representation π_p of parastatistics of order p will be chosen to be

$$\begin{aligned}a^{+i}_{(r)} &= \Delta^{(r-1)}\otimes 1\otimes\Delta^{(p-r)}\left(\textstyle\sum_{k=1}^{n} L_k^{i(+)}\otimes a^{+k}\otimes 1\right)\\ a^-_{i(r)} &= \Delta^{(r-1)}\otimes 1\otimes\Delta^{(p-r)}\left(\textstyle\sum_{k=1}^{n} 1\otimes a_k^-\otimes L_i^{k(-)}\right)\end{aligned} \tag{42}$$

Note that the conjugation $*$ acts as reflection on the Green indices (r)

$$(a^{+i}_{(r)})^* = a^-_{i(r^*)} \quad (a^-_{i(r)})^* = a^{+i}_{(r^*)} \quad r^* = p - r + 1.$$

The exchange relations of the Green components of the deformed Green ansatz close quadratic algebras .

For different Green indices:
($[x,y]_{\pm q} = xy \pm qyx$) ($r > s$)

$$\begin{array}{lllllll} [a^{+i}_{(r)}, a^{+j}_{(s)}]_{\mp} & = & \mp(q-q^{-1})a^{+j}_{(r)}a^{+i}_{(s)} & \quad [a^{-}_{i(r)}, a^{-}_{j(s)}]_{\mp} & = & 0 & \quad i<j \\ [a^{-}_{i(r)}, a^{-}_{j(s)}]_{\mp} & = & \pm(q-q^{-1})a^{-}_{j(r)}a^{-}_{i(s)} & \quad [a^{+i}_{(r)}, a^{+j}_{(s)}]_{\mp} & = & 0 & \quad i>j \end{array} \tag{43}$$

$$[a^{+i}_{(r)}, a^{+i}_{(s)}]_{\mp q} = 0 \qquad\qquad [a^{-}_{i(r)}, a^{-}_{i(s)}]_{\mp q^{-1}} = 0 \tag{44}$$

$$[a^{-}_{i(r)}, a^{+j}_{(s)}]_{\mp} = 0 \qquad \text{for} \qquad r \neq s \tag{45}$$

For equal Green indices:

$$\begin{array}{lllllll} [a^{+i}_{(r)}, a^{+j}_{(r)}]_{\pm q^{\mp\epsilon_{ij}}} & = & 0 & \quad [a^{-}_{i(r)}, a^{-}_{j(r)}]_{\pm q^{\mp\epsilon_{ij}}} & = & 0 \\ [a^{-}_{i(r)}, a^{+j}_{(r)}]_{\pm q^{\mp 1}} & = & q^{\mp\frac{1}{2}}Q^{j(-)}_{i(r)} & \quad [a^{-}_{i(r)}, a^{+j}_{(r)}]_{\pm q^{\pm 1}} & = & q^{\pm\frac{1}{2}}Q^{j(+)}_{i(r)} \end{array} \tag{46}$$

where the operators $Q^{j(+)}_{i(r)}$ and $Q^{j(-)}_{i(r)}$ are quadratic in the Green components

$$\begin{array}{llll} q^{\mp\frac{1}{2}}Q^{j(-)}_{i(r)} = & (q-q^{-1})\sum_{s=1}^{r-1} q^{\mp(r-s)}a^{+j}_{(s)}a^{-}_{i(s)} & i>j \\ q^{\mp\frac{1}{2}}Q^{j(-)}_{i(r)} = & -(q-q^{-1})\sum_{s=r}^{p} q^{\mp(r-s)}a^{+j}_{(s)}a^{-}_{i(s)} & i<j \end{array} \tag{47}$$

$$\begin{array}{llll} q^{\pm\frac{1}{2}}Q^{i(+)}_{i(r)} = q^{\mp(r-\frac{p}{2}-\frac{1}{2})}(q^{N_i})^{\otimes p} & - & (q-q^{-1})\sum_{s=r+1}^{p} & q^{\mp(r-s)}a^{+i}_{(s)}a^{-}_{i(s)} \\ q^{\mp\frac{1}{2}}Q^{i(-)}_{i(r)} = q^{\mp(r-\frac{p}{2}-\frac{1}{2})}(q^{-N_i})^{\otimes p} & + & (q-q^{-1})\sum_{s=1}^{r-1} & q^{\mp(r-s)}a^{+i}_{(s)}a^{-}_{i(s)} \end{array} \tag{48}$$

The upper (lower) signs are for the parafermi (parabose) case. For the

parafermi algebra one has in addition

$$(a^{+i}_{(r)})^2 = 0 \quad (a^{-}_{i(s)})^2 = 0 \quad \text{for} pF_q(n) \tag{49}$$

The system of relations (44-50) defines the generalized Green ansatz of the deformed parastatistics algabras.

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