UNIVERSITY OF NIŠ
The scientific journal **FACTA UNIVERSITATIS**
Series: **Mechanical Engineering** Vol.1, N° 5, 1998 pp. 555 - 564
Editor of series: Nenad Radojković, e-mail:radojkovic@ni.ac.yu
Address: Univerzitetski trg 2, 18000 Niš, YU
Tel: +381 18 547-095, Fax: +381 18 547-950
http://ni.ac.yu/Facta

OPTIMUM DIMENSIONS OF TRAPEZIUM CROSS-SECTION IN STRUCTURES

UDC:66.011:693.814

Ratko Šelmić, Radomir Mijailović

Faculty of Transport and Traffic Engineering
Vojvode Stepe 305, 11000 Beograd
E-mail: eselmic@ubbg.etf.bg.ac.yu

Abstract. *In this work dimensions of trapezium thin-wall cross-section in structures are being optimized. Objective function is minimum mass, i.e. minimum cross-section area, and constraint function is general function of hardness and stability. Using the method Lagranges's multipliers, optimum dimensions of trapezium cross-section are defined. Then, the box-rectangular cross-section, as a special case of trapezium cross-section is discussed, and its optimum dimensions are also defined.On a numerical example a detailed analysis of the impact of individual parameters is performed, and these two cross-sections are compared in regard to economy. The conclusion is that the box-rectangular cross-section is more economical than the trapezium one.*

Key words: *trapezium, cross-section, optimization*

1. INTRODUCTION

Trapezium, thin-wall cross-section is not a usual type of a cross-section in mechanical structures. However, it is applied in certain cases. The reasons for its application are certain advantages of such cross-section form. An example of a structure with a cross-section of this type is a truck crane telescopic boom. Namely, world manufacturers (such as "KATO" - Japan, "GROVE" - USA, for example) have decided in favour of this cross-section of telescopic booms, particularly in truck cranes of higher capacity. The firm "GROVE" manufactures a basic boom exactly with this cross-section, while cross-sections of a telescope are also of a trapezium form, but with latticed sides. Thus, the advantages of latticed structures are used, and higher capacity is achieved due to savings in boom mass. That is why a special attention is paid to analysis of a such cross-section and its optimization, as this problem is not sufficiently explained in professional literature

Received February 27, 1998

in regard to stress criterion, typical of metal crane structures.

Problems of structure optimization are discussed in a great number of works, monographs and books [1,2,3,4,5,6] by means of various methods with different objective functions and constraint functions. The problem of optimization of various cross-sections (box-rectangular cross-section, "I" profile, "U" profile, cross-section of latticed structures) is discussed applying the method of Lagrange's multipliers for various constraint functions (in respect to stress and deformation) in a series of works [7,8,9,10,11,12,13,14,15,16].

2. DEFINING OF THE PROBLEM

It is, therefore, necessary to define optimum dimensions of thin-wall trapezium cross-section (Fig. 1.). The trapezium is equilateral, with base lines a and b, and a side c . Thickness of bases are t_1 and t_2 , and of sides are t_3. This thickness are determined out of local stability conditions, i.e. the relation between thickness and length of the corresponding side is a constant, the value of which is defined with local stability [1,2]: $t_1 /a = d_1$, $t_2 /b = d_2$, $t_3 /c = d_3$. So, it is necessary to define optimum dimensions of: a, b, c (or h). Cross-section surface will be:

$$A = at_1 + bt_2 + 2ct_3 = \delta_1 a^2 + \delta_2 b^2 + 2\delta_3 c^2 . \tag{1}$$

The expression for cross-section surface can be also written in this form:

$$A = k_1 a^2 + k_2 b^2 + k_3 ab + k_4 h^2 , \tag{2}$$

where these substitutions are introduced: $k_1 = d_1 + 0,5\, d_3$, $k_2 = d_2 + 0,5 d_3$,
$k_3 = - d_3$, $k_4 = 2d_3$.

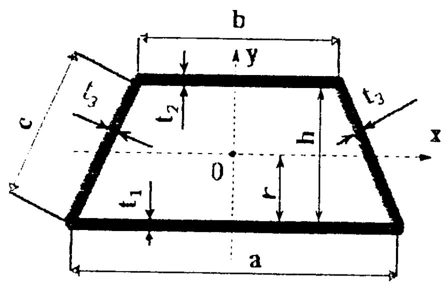

Fig. 1. Trapezium cross-section of a structure

3. OBJECTIVE FUNCTION AND CONSTRAINT FUNCTION

For a certain material and structure length (L), as well as for defined structure coefficients (the relation of real and theoretical cross-section surfaces for objective function (F) the cross-section surface of the structure can be accepted, what in this case is:

$$F \equiv A = k_1 a^2 + k_2 b^2 + k_3 ab + k_4 h^2 \; . \tag{3}$$

For constraint function we accept the stress criterion supposing that there is an axial stress and slant bending, what in crane and many other structures is a usual thing. So, constraint function is represented by general function of hardness and stability:

$$\varphi = \frac{N}{A} + \frac{M_x}{W_x} + \frac{M_y}{W_y} - R_l = 0 \; , \tag{4}$$

where it is designated as:

N – axial force, supposed to act on the cross-section centre,
M_x, M_y – moments of flexion for axes x and y respectively,
W_x, W_y – section modulus for corresponding axes,
R_l – limiting stress.

Correct expressions for moments of inertia and section modulus of trapezium cross-section are very complex for such an analysis. Practically, it is not possible to solve the stated problem with these expressions. Because of that, we shall simplify expressions for section modulus expressing them in the function of the cross-section surface and corresponding sides [7,8,9,10, . . .]:

$$W_x = \alpha_x h A \, , \;\; W_\mathrm{y} = \alpha_y m A = \alpha_y \frac{a+b}{2} A \tag{5}$$

where m - is the middle line of the trapezium, and α_x and α_y - are corresponding coefficients. Function of constraint (4) then achieves a simpler form:

$$\varphi = \frac{N}{A} + \frac{M_x}{\alpha_x h A} + \frac{2M_y}{\alpha_y (a+b) A} - R_l = 0 \; , \tag{6}$$

Coefficients α_x and α_y have analytical values:

$$\begin{aligned} \alpha_x &= \frac{W_x}{hA} = \frac{I_x}{hAy_{\max}} = \frac{at_1 r^2 + bt_2 (h-r)^2 + \frac{1}{6} ct_3 h^2 + \frac{1}{2} ct_3 (h-2r)^2}{hA(h-r)} \\ \alpha_y &= \frac{W_y}{mA} = \frac{2W_y}{(a+b)A} = \frac{2I_y}{(a+b)Ax_{\max}} = 2\,\frac{\frac{1}{12} t_1 a^3 + \frac{1}{12} t_2 b^3 + \frac{1}{24} t_3 c (a-b)^2 + \frac{1}{8} t_3 c (a+b)^2}{(a+b)A\frac{a}{2}} \end{aligned} \tag{7}$$

where r - is the distance between the base of the trapezium and central axis Ox:

$$r = \nu h \, , \;\; \nu = \frac{bt_2 + ct_3}{A} = \frac{\delta_2 b^2 + \delta_3 c^2}{\delta_1 a^2 + \delta_2 b^2 + 2\delta_3 c^2} = \frac{\delta_2 \xi^2 + 0{,}25\delta_3 [4\eta^2 + (1-\xi)^2]}{k_1 + k_2 \xi^2 + k_3 \xi + k_4 \eta^2} \, , \;\; \xi = \frac{b}{a} \, , \;\; \eta = \frac{h}{a} \; .$$

Expressions (7) can be also written in the form:

$$\alpha_x = f_1(\xi,\eta)\,,\ \alpha_y = f_2(\xi,\eta)\,, \tag{8}$$

that is:

$$\begin{aligned}\alpha_x &= \frac{\delta_1\nu^2 + \xi^2\delta_2(1-\nu)^2 + \frac{1}{24}\delta_3[4\eta^2 + (1-\xi)^2][1+3(1-2\nu)^2]}{(1-\nu)(k_1 + k_2\xi^2 + k_3\xi + k_4\eta^2)} \\ \alpha_y &= \frac{\frac{1}{3}\delta_1 + \frac{1}{3}\delta_2\xi^4 + \frac{1}{24}\delta_3[4\eta^2 + (1-\xi)^2][(1-\xi)^2 + 3(1+\xi)^2]}{(1+\xi)(k_1 + k_2\xi^2 + k_3\xi + k_4\eta^2)}\end{aligned} \tag{9}$$

For recommended values: $\delta_1 = 0{,}014$, $\delta_2 = 0{,}022$, $\delta_3 = 0{,}008$ [17] and $\xi = 0{,}65 - 0{,}80$, $\eta = 1{,}20–1{,}50$, it is obtained that $\nu = 0{,}477 - 0{,}482$, so $0{,}298 \le \alpha_x \le 0{,}313$, $0{,}288 \le \alpha_y \le 0{,}321$, that is

$$\frac{\alpha_{x,\max}}{\alpha_{x,\min}} = 1{,}29\,, \quad \frac{\alpha_{y,\max}}{\alpha_{y,\min}} = 1{,}11$$

so that, with sufficient preciseness, these coefficients can be considered as constants.

4. MATHEMATICAL MODEL

Vector of the given parameters is:

$$\vec{x} = (L, N, M_x, M_y, R_l, \ldots)\ , \tag{10}$$

and vector of variable values is:

$$\vec{y} = (a, b, h)\ . \tag{11}$$

For determination of optimum parameters: a_o, b_o, h_o we shall apply the method of Lagrange's multipliers for extreme functions with three variables. So, to bring function $A = A(a,b,h)$ to have relative maximum or minimum at a certain point, it is necessary, besides function of constraint (6), to satisfy equations:

$$\frac{\partial\Phi}{\partial a} = 0\,,\ \frac{\partial\Phi}{\partial b} = 0\,,\ \frac{\partial\Phi}{\partial h} = 0\,, \tag{12}$$

then, the function of Lagrange is:

$$\Phi(a,b,h,\lambda) = A(a,b,h) + \lambda\varphi(a,b,h)\ , \tag{13}$$

and λ - unknown Lagrange's multiplier. So, the equation system (12) can be also written in the following form:

$$\frac{\partial A}{\partial a} + \lambda\frac{\partial\varphi}{\partial a} = 0\,,\ \frac{\partial A}{\partial b} + \lambda\frac{\partial\varphi}{\partial b} = 0\,,\ \frac{\partial A}{\partial h} + \lambda\frac{\partial\varphi}{\partial h} = 0\ . \tag{14}$$

Combining the first and the second, and then the second and the third equation of the equation system (14), the multiplier λ , is being eliminated:

$$\frac{\partial A}{\partial a}\frac{\partial \varphi}{\partial b}=\frac{\partial A}{\partial b}\frac{\partial \varphi}{\partial a},\quad \frac{\partial A}{\partial b}\frac{\partial \varphi}{\partial h}=\frac{\partial A}{\partial h}\frac{\partial \varphi}{\partial b}\,. \tag{15}$$

Equation systems (6) and (15) determine unknown optimum parameters: a_0, b_0, h_0.

5. Optimum parameters

Substituting expression (6) into the first equation of the system (15) results into:

$$\frac{\partial A}{\partial a}\left[-\theta\frac{\partial A}{A^2\partial b}-\frac{2M_y}{A\alpha_y(a+b)^2}\right]=\frac{\partial A}{\partial b}\left[-\theta\frac{\partial A}{A^2\partial a}-\frac{2M_y}{A\alpha_y(a+b)^2}\right], \tag{16}$$

where the substitution of $\theta = N + M_x/(\alpha_x h) + 2M_y/(\alpha_y(a+b))$ is introduced. Out of the equation (16), after substitution of expression (3), optimum relation of the sides a and b is as follows:

$$\xi_o=\frac{b_o}{a_o}=\frac{\delta_1+\delta_3}{\delta_2+\delta_3}=\frac{2k_1-k_3}{2k_2-k_3}\,. \tag{17}$$

In case that $\delta_1 = \delta_2$, i.e. when thickness of trapezium bases are proportionate to base lengths ($t_1/t_2 = a/b$), it is obtained that $a_o = b_o$ ($\xi_o = 1$), i.e. optimum trapezium cross-section turns into box cross-section.

Substituting expression (6) into the second equation of the system (15), it is obtained:

$$\frac{\partial A}{\partial b}\left[-\theta\frac{\partial A}{A^2\partial h}-\frac{M_x}{A\alpha_x h^2}\right]=\frac{\partial A}{\partial b}\left[-\theta\frac{\partial A}{A^2\partial b}-\frac{2M_y}{A\alpha_y(a+b)^2}\right]. \tag{18}$$

When the expression for surface (3) is substituted into the equation (18), optimum relation of height h and side a is obtained:

$$\eta_o=\frac{h_o}{a_o}=\sqrt[3]{\frac{\alpha_y(2k_2\xi_o+k_3)(1+\xi_o)^2 M_x}{4\alpha_x k_4 M_y}}\,. \tag{19}$$

Finally, if the obtained relations (17) and (19) are substituted into constraint function (6), algebraic cube equation is obtained, the solution of which represents optimum length of the side a_o:

$$\rho_3 a^3+\rho_1 a+\rho_o=0\,, \tag{20}$$

where:

$$\rho_3=R_l(k_1+k_2\xi_o^2+k_3\xi_o+k_4\eta_o^2),\quad \rho_1=-N,\quad \rho_o=-\frac{M_x}{\alpha_x\eta_o}-\frac{2M_y}{\alpha_y(1+\xi_o)}\,. \tag{21}$$

The other side and height of trapezium cross-section are:

$$b_o=\xi_o a_o,\quad h_o=\eta_o a_o. \tag{22}$$

Relation of the sides $\xi_o = b_o/a_o$, depends only on limited relations δ_1, δ_2, δ_3, the values of which evolve out of local stability conditions of the corresponding sides. If the values δ_1, δ_2, δ_3 are considered constants, then it appears that also ξ_o = *const*. Further, it means that coefficients α_x and α_y disperse even less. For already accepted values for δ_1, δ_2, δ_3 it will be that $\xi_o = 0{,}73$, so: $0{,}294 \le \alpha_x \le 0{,}325$, $0{,}287 \le \alpha_y \le 0{,}318$, that is

$$\frac{\alpha_{x,\max}}{\alpha_{x,\min}} = 1{,}10 \qquad \frac{\alpha_{y,\max}}{\alpha_{y,\min}} = 1{,}11$$

and the mean values of these coefficients are: $\overline{\alpha}_x = 0{,}310\ \overline{\alpha}_y = 0{,}302$.

6. Box-rectangular cross-section

It is interesting, from the point of economical use of material, to compare the discussed trapezium and box-rectangular cross-sections (Fig. 2). In previous author's works optimization of box-rectangular cross-section [7,8,10] was discussed.

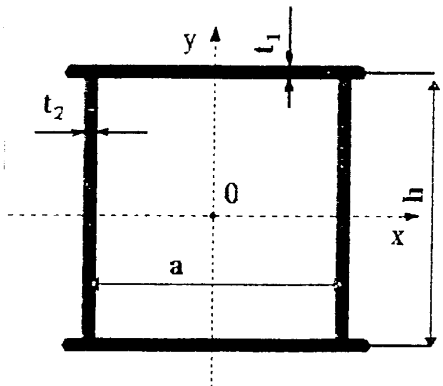

Fig. 2. Box-rectangular cross-section

Considering box-rectangular cross-section as a separate case of trapezium cross-section ($\xi = 1$), optimum relation of the sides is obtained from (19)

$$\eta_o = \frac{h_o}{a_o} = \sqrt[3]{\frac{\alpha_y \delta_1 M_x}{\alpha_x \delta_2 M_y}}\ , \tag{23}$$

what coincides with the results obtained in [7,8,10] , where $\delta_1 = t_1/a$, $\delta_2 = t_2/h$.

Optimum width of box-rectangular cross-section a_o is defined by equation:

$$\rho_3 a^3 + \rho_1 A + \rho_o = 0 \tag{24}$$

where:

$$\rho_3 = 2R_l(\delta_1 + \eta_o^2 \delta_2)\,,\ \ \rho_1 = -N\,,\ \ \rho_o = -\frac{M_x}{\alpha_x \eta_o} - \frac{M_y}{\alpha_y}\,. \tag{25}$$

7. Numerical example. Analysis of the results

Verification of the obtained theoretical results is performed on a numerical example. For a truck crane telescopic boom the following data are known:

- moments of flexion for the corresponding axes: $M_x = 550$ [kNm], $M_y = yM_x$, where $\psi = 0{,}40 - 0{,}75$;
- axial force is $N = 115$ [kN];
- limiting stress is $R_l = 196\Delta$ [MPa], where coefficient of stress variation is $\Delta = 0{,}75 - 1{,}25$.

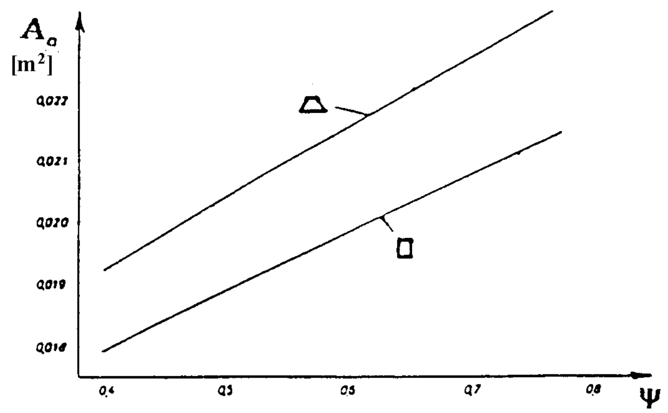

Fig. 3. Dependence of optimum cross-section surface (A_o) on the moment of flexion (ψ)

According to formulae obtained, dependence of optimum cross-section surface is first determined from the moment of flexion, i.e. $A_o = f(\psi)$, for trapezium and box-rectangular cross-sections (Fig. 3).

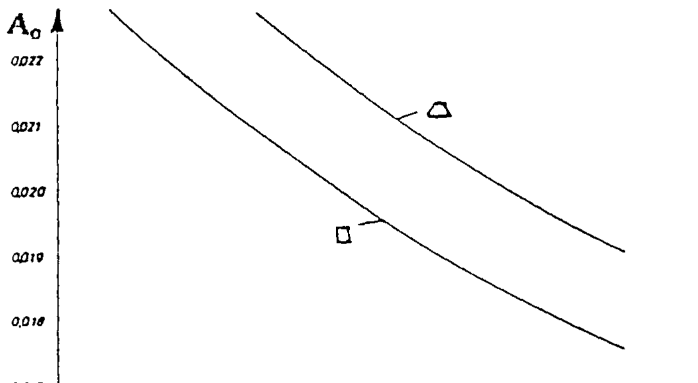

Fig. 4. Dependence of optimum cross-section surface (A_o) on limiting stress (Δ)

That dependence is almost linear, and in the whole scope, the trapezium surface is considerably larger. This difference in surface becomes greater if the moment of flexion is higher. Diagram confirms that box-rectangular cross-section is more economical, than the trapezium cross-section.

Dependence of optimum cross-section surface A_o on the limiting stress, i.e. on the coefficients Δ is shown on Fig. 4. It is a hyperbola dependence where the advantage of box-rectangular cross-section is noticed again.

Relation of the bending moment considerably influences the relation of sides in trapezium and box-rectangular cross-section, respectively (Fig. 5). The relation between the height and the basis (η) decreases with the increase of the moment of bending (ψ) according to hyperbola dependence, and it is larger in box/rectangular cross-section.

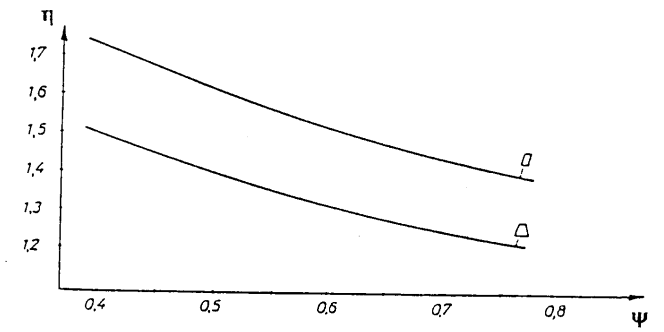

Fig. 5. Dependence of sides relation (η) on the moment of bending (ψ)

Relation of height and basis (η) does not depend on the type of structure material, and it is greater for a defined material in case of box/rectangular cross-section (Fig. 6.).

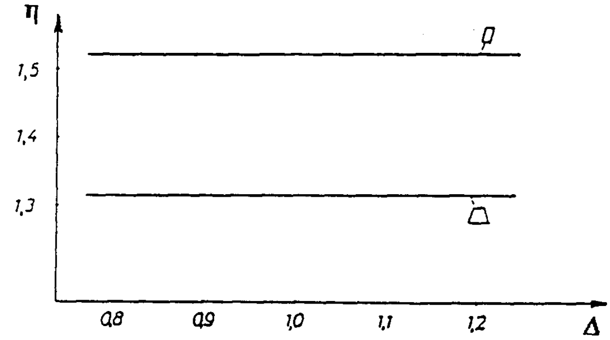

Fig. 6. - Dependence of sides relation (η) on the type of material (Δ)

Analysis of numerical results on an example of a crane telescopic boom allows for comparison between the trapezium and box thin-wall cross-sections. Namely, taking the criterion of material consumption at the same load and kind of material, the box-rectangular cross-section has an undoubtful advantage. But, application of trapezium cross-section, in our opinion, has its relevance in certain cases, especially where excentral pressure is considerably expressed, as for instance, in telescopic booms of high capacity which have their own specifics. Asymmetry of trapezium cross-section becomes evident here, but such a case of load demands a separate analysis.

8. CONCLUSION

In this paper optimum dimensions of a hollow trapezium cross-section are defined, and then, as a special case, optimum dimensions of a box-rectangular cross-section are also given. Simple formulae are obtained in the analytical form which is very suitable for practical use. Comparison between the trapezium and box-rectangular cross-sections is performed from the economical point of view (economical material consumption). Formulae can be very useful to an engineer-designer in the first stage of the designing procedure, when he is faced with the problem of defining basic structure dimensions that would be close to optimum ones.

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OPTIMALNE DIMENZIJE TRAPEZNOG POPREČNOG PRESEKA KONSTRUKCIJA

Ratko Šelmić, Radomir Mijailović

U radu se optimiziraju dimenzije trapeznog, tankozidnog poprečnog preseka konstrukcija. Funkcija cilja je minimalna masa, odnosno minimalna površina poprečnog preseka, a funkcija ograničenja opšta jednačina čvrstoće i stabilnosti. Korišćenjem metode sa Lagranžovim množiteljima, određuju se optimalne dimenzije trapeznog poprečnog preseka. Zatim se razmatra sandučasto-pravougaoni poprečni presek, kao specijalan slučaj trapeznog poprečnog preseka pa određuju i njegove optimalne dimenzije. Kroz brojni primer vrši se detaljna analiza uticaja pojedinih parametara i porede ova dva poprečna preseka sa aspekta ekonomičnosti. Zaključuje se da je sandučasto-pravougaoni poprečni presek ekonomičniji od trapeznog.