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TOOTH SURFACE FUNDAMENTAL FORMS IN GEAR TECHNOLOGY

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Abstract. *A new approach to investigate the local topology of the involute gear tooth surface has been developed. The approach is based on the fundamental results in differential geometry of surfaces. The method described here may always be applied, but is less efficient than other methods if calculation of only one single parameter is required. However, when multiple parameters of the local geometry (intrinsic or external) of the gear tooth surface must be calculated, this method is more efficient than other methods in use.*

Key words: *gear tooth surface, gear technology*

1. INTRODUCTION

Design and manufacture of gears requires frequent calculation of the parameters of the working surface of a gear tooth. In cases when it is necessary to calculate many parameters of that surface, application of the first and second fundamental forms of the surface (the Gaussian fundamental forms of the surface) is efficient. Application of the first $\Phi_{1.G}$ and the second $\Phi_{2.G}$ fundamental forms of the working surface G of a gear tooth allows simplifying the formulas necessary for calculations and is convenient for detailed study of the geometry of surface G.

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2. Local topology of the gear tooth surface

Let us start from the determination of surface G.

As is well known [2, p.5.3], a kinematic method of creating an involute curve is to use a circular disk of radius r_b with a string wrapped around it. Unwinding the that string produces a series of lines tangent to this base circle r_b (i.e. to the circular disk), and tracing the ends of these tangent lines generates an involute curve (Fig. 1). In a similar manner, the screw involute surface G may be generated by screw motion of a straight line that maintains its direction tangent to the helix on the base cylinder during this motion. Fig. 1 shows the screw involute surface G that is covered with straight lines (straight generatrix) - tangent to the helix on the base cylinder [2, p.1.11; 4; 5; 6].

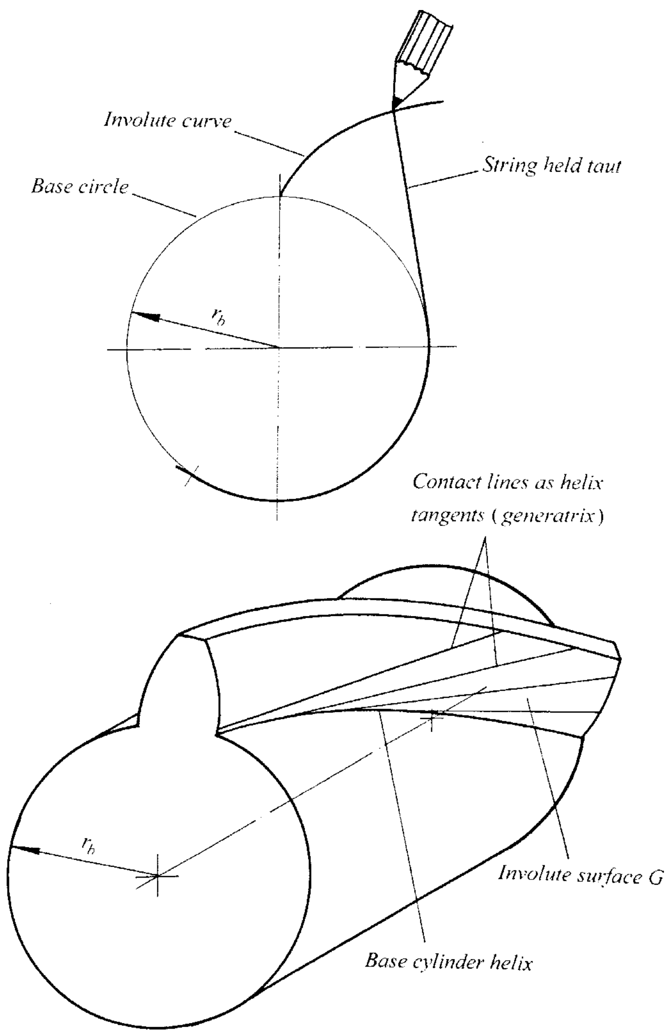

Fig. 1. Generation of an involute curve.

As with the involute curve (Fig. 1), the position vector of an arbitrary point on the involute surface G in coordinate system XZY can be written as (Fig. 2):

$$\vec{r}_G = \vec{A} + \vec{B} + \vec{C} . \tag{1}$$

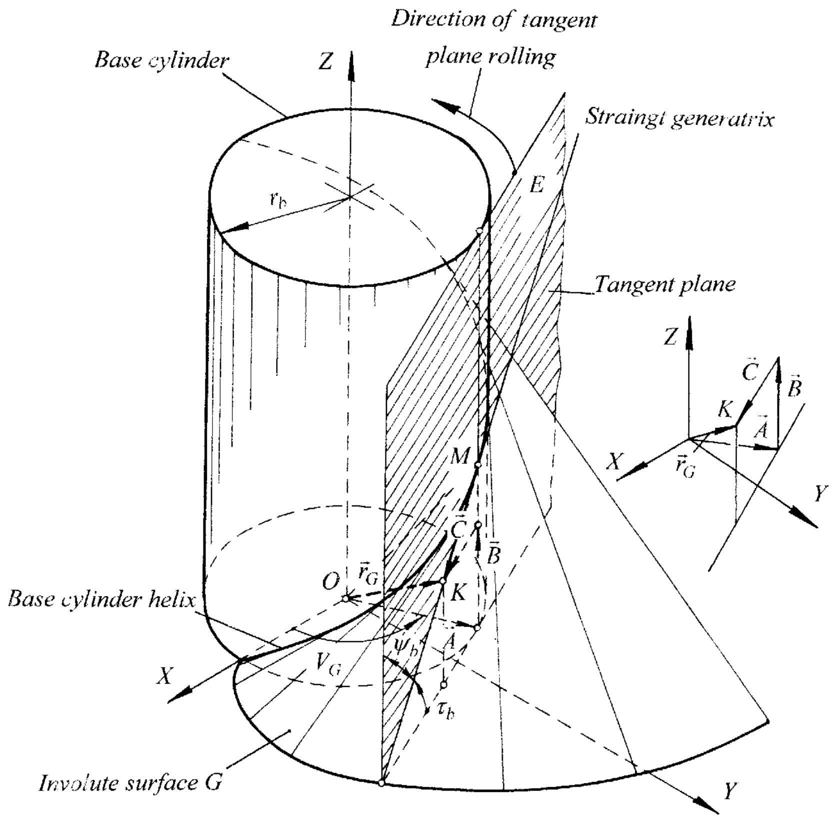

Fig. 2. Generation of surface G.

In (1), vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$, respectively, are equal to (Fig.2):

$$\vec{A} = \vec{i}\, r_b \cos V_G + \vec{j} r_b \sin V_G; \tag{2}$$

$$\vec{B} = \vec{k}(r_b V_G \tan \tau_b - U_G \sin \tau_b) \ ; \tag{3}$$

$$\vec{C} = \vec{i}\, U_G \cos \tau_b \sin V_G - \vec{j} U_G \cos \tau_B \cos V_G \ , \tag{4}$$

where r_b - radius of the base cylinder of the gear;
U_G and V_G - curvilinear (Gaussian) coordinates on surface G;
τ_b - is the base helix angle.

Substituting (2), (3) and (4) to the (1), one can obtain:

$$\begin{aligned}\vec{r}_G = \vec{i}\,(r_b \cos V_G + U_G \sin V_G \cos \tau_b) + \vec{j}\,(r_b \sin V_G - U_G \cos V_G \cos \tau_b) + \\ + \vec{k}\,(r_b \tan \tau_b - \sin \tau_b).\end{aligned} \tag{5}$$

To calculate the local geometrical parameters of surface G: normal and principle radii of curvature, principle directions etc., it is convenient to apply the first and the second fundamental forms, $\Phi_{1.G}$ and $\Phi_{2.G}$, respectively, of surface G.

From definition [7], the first fundamental form $\Phi_{1.G}$ of surface G is equal to:

$$\Phi_{1.G} \Rightarrow \acute{\Gamma}_G dU_G^2 + 2F_G dU_G dV_G + G_G dV_G^2 , \tag{6}$$

where the Gaussian coefficients E_G, F_G and G_G are given by:

$$\begin{aligned} E_G &= \left(\frac{\partial X_G}{\partial U_G}\right)^2 + \left(\frac{\partial Y_G}{\partial U_G}\right)^2 + \left(\frac{\partial Z_G}{\partial U_G}\right)^2; \\ F_G &= \frac{\partial X_G}{\partial U_G} \cdot \frac{\partial X_G}{\partial V_G} + \frac{\partial Y_G}{\partial U_G} \cdot \frac{\partial Y_G}{\partial V_G} + \frac{\partial Z_G}{\partial U_G} \cdot \frac{\partial Z_G}{\partial V_G}; \\ G_G &= \left(\frac{\partial X_G}{\partial V_G}\right)^2 + \left(\frac{\partial Y_G}{\partial V_G}\right)^2 + \left(\frac{\partial Z_G}{\partial V_G}\right)^2. \end{aligned} \tag{7}$$

All derivatives necessary to calculate the Gaussian coefficients E_G, F_G and G_G using (7) can be calculated from (5) as:

$$\frac{\partial X_G}{\partial U_G} = \sin V_G; \quad \frac{\partial Y_G}{\partial U_G} = -\cos V_G \cos \tau_b; \quad \frac{\partial Z_G}{\partial U_G} = -\sin \tau_b \tag{8}$$

and

$$\begin{aligned} \frac{\partial X_G}{\partial V_G} &= -r_b \sin V_G + U_G \cos V_G \cos \tau_{b;} \\ \frac{\partial Y_G}{\partial V_G} &= r_b \cos V_G + U_G \sin V_G \cos \tau_b; \\ \frac{\partial Z_G}{\partial V_G} &= r_b \tan \tau_b. \end{aligned} \tag{9}$$

According to (8) and (9), the Gaussian coefficients E_G, F_G and G_G of the first fundamental form $\Phi_{1.G}$ of surface G, are equal to:

$$E_G = 1; \quad F_G = -\frac{r_b}{\cos \tau_b}; \quad G_G = -\frac{r_b^2}{\cos^2 \tau_b} + U_G^2 \cos^2 \tau_b . \tag{10}$$

The second fundamental form $\Phi_{2.G}$ of surface G is determined as follows [7]:

$$\Phi_{2.G} \Rightarrow L_G dU_G^2 + 2M_G dU_G dV_G + N_G dV_G^2 , \tag{11}$$

where the Gaussian coefficients L_G, M_G and N_G of the fundamental form $\Phi_{2.G}$ are given by:

$$L_G = \frac{\begin{vmatrix} \frac{\partial^2 X_G}{\partial U_G^2} & \frac{\partial^2 Y_G}{\partial U_G^2} & \frac{\partial^2 Z_G}{\partial U_G^2} \\ \frac{\partial X_G}{\partial U_G} & \frac{\partial Y_G}{\partial U_G} & \frac{\partial Z_G}{\partial U_G} \\ \frac{\partial X_G}{\partial V_G} & \frac{\partial Y_G}{\partial V_G} & \frac{\partial Z_G}{\partial V_G} \end{vmatrix}}{\sqrt{E_G G_G - F_G^2}} = \frac{\left(\frac{\partial^2 \vec{r}_G}{\partial U_G^2}, \frac{\partial \vec{r}_G}{\partial U_G}, \frac{\partial \vec{r}_G}{\partial V_G} \right)}{\sqrt{E_G G_G - F_G^2}} \;; \tag{12}$$

$$M_G = \frac{\begin{vmatrix} \frac{\partial^2 X_G}{\partial U_G \partial V_G} & \frac{\partial^2 Y_G}{\partial U_G \partial V_G} & \frac{\partial^2 Z_G}{\partial U_G \partial V_G} \\ \frac{\partial X_G}{\partial U_G} & \frac{\partial Y_G}{\partial U_G} & \frac{\partial Z_G}{\partial U_G} \\ \frac{\partial X_G}{\partial V_G} & \frac{\partial Y_G}{\partial V_G} & \frac{\partial Z_G}{\partial V_G} \end{vmatrix}}{\sqrt{E_G G_G - F_G^2}} = \frac{\left(\frac{\partial^2 \vec{r}_G}{\partial U_G \partial V_G}, \frac{\partial \vec{r}_G}{\partial U_G}, \frac{\partial \vec{r}_G}{\partial V_G} \right)}{\sqrt{E_G G_G - F_G^2}}; \tag{13}$$

$$N_G = \frac{\begin{vmatrix} \frac{\partial^2 X_G}{\partial V_G^2} & \frac{\partial^2 Y_G}{\partial V_G^2} & \frac{\partial^2 Z_G}{\partial V_G^2} \\ \frac{\partial X_G}{\partial U_G} & \frac{\partial Y_G}{\partial U_G} & \frac{\partial Z_G}{\partial U_G} \\ \frac{\partial X_G}{\partial V_G} & \frac{\partial Y_G}{\partial V_G} & \frac{\partial Z_G}{\partial V_G} \end{vmatrix}}{\sqrt{E_G G_G - F_G^2}} = \frac{\left(\frac{\partial^2 \vec{r}_G}{\partial V_G^2}, \frac{\partial \vec{r}_G}{\partial U_G}, \frac{\partial \vec{r}_G}{\partial V_G} \right)}{\sqrt{E_G G_G - F_G^2}}; \tag{14}$$

The second derivatives of surface *G*, necessary to calculate the Gaussian coefficients L_G, M_G and N_G of the fundamental form $\Phi_{2.G}$, are:

$$\frac{\partial^2 X_G}{\partial U_G^2} = 0; \;\; \frac{\partial^2 Y_G}{\partial U_G^2}; \;\; \frac{\partial^2 Z_G}{\partial U_G^2} \tag{15}$$

and

$$\begin{gathered} \frac{\partial^2 X_G}{\partial V_G^2} = -r_b \cos V_G - U_G \sin V_G \cos \tau_b; \\ \frac{\partial^2 Y_G}{\partial V_G^2} = -r_b \sin V_G + U_G \cos V_G \cos \tau_b; \\ \frac{\partial^2 Z_G}{\partial V_G^2} = 0 \end{gathered} \tag{16}$$

and

$$\frac{\partial^2 X_G}{\partial U_G \partial V_G} = \cos \tau_b\,; \quad \frac{\partial^2 Y_G}{\partial U_G \partial V_G} = -\sin V_G \cos \tau_b\,; \quad \frac{\partial^2 Z_G}{\partial U_G \partial V_G} = 0. \tag{17}$$

According to (15)–(17), the Gaussian coefficients L_G, M_G and N_G of the fundamental form $\Phi_{2.G}$, are:

$$L_G = 0\,; \quad M_G = 0\,; \quad N_G = -U_G \sin \tau_b \cos \tau_b\,. \tag{18}$$

3. Calculations of the radii of the principle curvature of surface G

The radii of the principle curvature $R_{1..G}$ and $R_{2..G}$ of surface G are calculated as a roots of the quadratic equation [1]:

$$(L_G N_G - M_G^2) R_{1,2.G}^2 + (E_G N_G - 2F_G M_G + G_G L_G) R_{1,2.G} + (E_G G_G - F_G^2) = 0, \tag{19}$$

which in this case simplifies to:

$$E_G N_G R_{1,2.G} + E_G G_G - F_G^2 = 0 \tag{20}$$

or, in terms of the parameters of the gear design,

$$-U_G \sin \tau_b \cos \tau_b R_{1,2.G} + U_G^2 \cos^2 \tau_b = 0. \tag{21}$$

Because $(L_G N_G - M_G^2) = 0$, the radii of the principle curvature $R_{1..G}$ and $R_{2..G}$ of surface G are equal to:

$$R_{1.G} = \infty\,; \quad R_{2.G} = U_G \cot \tau_b\,. \tag{22}$$

4. Calculations of the radii of normal curvature R_G of the working surface of gear tooth G.

According to Euler's formula, the radius of normal curvature R_G of any arbitrary section by a normal plane of surface G is equal to:

$$R_G^{-1} = R_{1.G}^{-1} \sin^2 \varphi + R_{2.G}^2 \cos^2 \varphi. \tag{23}$$

Using the result above, ($R_{1.G} = \infty$) (23) simplifies to

$$R_G^{-1} = R_{2.G}^2 \cos^2 \varphi\,. \tag{24}$$

For that reason, the radius of curvature of the curved line of intersection of surface G by an arbitrary normal plane section can be calculated by the formula:

$$R_G = \frac{R_{1.G}}{\cos^2 \varphi}, \tag{25}$$

where the value of the angle φ is unknown.

To determine the value of the central angle φ, consider Fig. 3. First of all, it is obvious that QK on Fig.3 is equal to U_G in (22). So:

$$U_G = QK = \frac{QS}{\sin\psi_b} = \frac{\sqrt{D^2 - D_b^2}}{2\sin\psi_b}, \tag{26}$$

where D – is the pitch diameter of the gear;

D_b – is the base diameter of the gear.

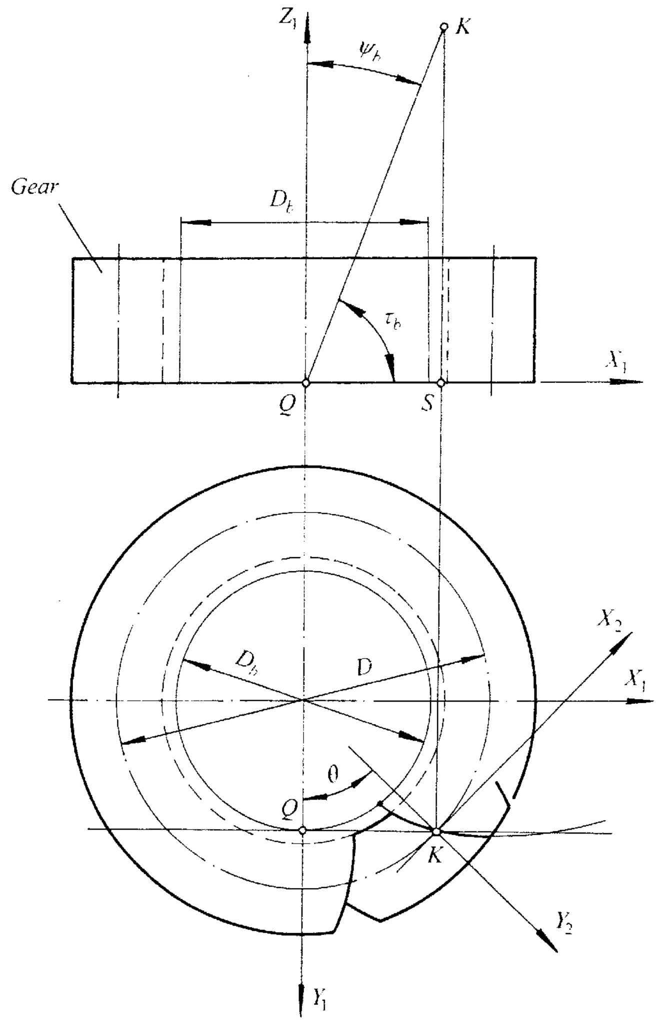

Fig. 3. Determination of the central angle φ.

5. EXAMPLE

It is necessary to determine the radius of normal curvature of the working surface of the gear tooth G in a given direction tangent to the helix on the gear pitch cylinder. According to a drawing of gear the base diameter of the gear is equal to $D_b = 45.318$; pitch diameter $D = 48.708$; pressure angle (normal) $\phi_n = 20^0$; helix angle $\psi = 22.5^0$.

According to an equation in [6, p.30]:

$$\sin\psi_b = \sin\psi\cos\phi_n\,. \tag{27}$$

For that reason, at point K,

$$U_G = \frac{\sqrt{D^2 - D_b^2}}{2\sin\psi\cos\phi_n} = \frac{\sqrt{48.708^2 - 45.318^2}}{2\sin 22.5^o \cos 20^o} = 24.823811\,. \tag{28}$$

Then $\tau_b = 90^0 - \psi_b$. So in (22) sot $\tau_b = \tan\psi_b$, and $R_{2.G} = U_G \tan\psi_n$

As is well known,

$$\tan\mu = \frac{\sin\mu}{\sqrt{1-\sin^2\mu}} \tag{29}$$

for any angle μ. According to (29) one can write:

$$\tan\psi_b = \frac{\sin\psi_b}{\sqrt{1-\sin^2\psi_b}} \tag{30}$$

and

$$\begin{aligned} R_{2.G} &= \frac{U_G\sin\psi_b}{\sqrt{1-\sin^2\psi_b}} = \frac{U_G\sin\psi\cos\phi_n}{\sqrt{1-\sin^2\psi\cos^2\phi_n}} = \\ &= \frac{24.8238\sin 22.5^o\cos 20^o}{\sqrt{1-\sin^2 22.5^o\cos^2 20^o}} = 9.5667. \end{aligned} \tag{31}$$

Consider a plane through point K tangent to surface G. The straight generatrix QK of surface G lies in this tangent plane. A straight line through point K tangent to the helix on the pitch cylinder coincides with this tangent plane as well. The central angle φ sought is equal to the angle between the straight generatrix of surface G (or straight line through the point K tangent to the helix on the base diameter cylinder) and the straight line tangent to the helix on the pitch cylinder. These straight lines are necessary in order to derive the equation of the above-mentioned straight lines in a common coordinate system.

In coordinate system $X_1Y_1Z_1$ attached to the gear at point K, the straight line is determined by the equation

$$\vec{r}_p^{\,*} = \vec{i}\,\frac{D_d}{2}\tan\theta + \vec{j}\,\frac{D_b}{2} + \vec{k}\,\tan\theta\cot\psi_b\,, \tag{32}$$

where θ - is a parameter describing the location of an arbitrary point on the straight line through points Q and K.

In matrix form, (32) is

$$\vec{r}_p^* = \frac{D_d}{2}\begin{bmatrix} \tan\theta \\ 1 \\ \tan\theta c \cot\psi_b \end{bmatrix}. \tag{33}$$

Here in (33), at point K (Fig. 3)

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{1-\cos^2\theta}}{\cos\theta} = \frac{\sqrt{1-\left(\frac{D_b}{D}\right)^2}}{\frac{D_b}{D}} = 0.3940. \tag{34}$$

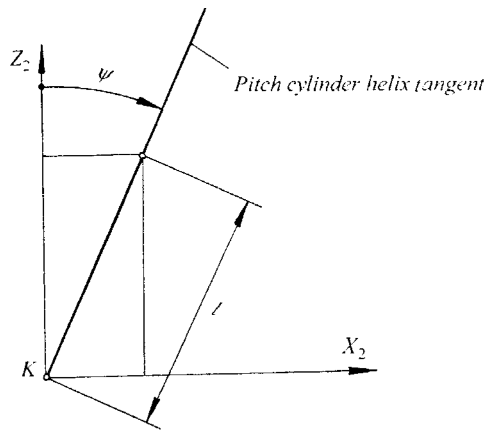

Fig. 4. Derivation of the equation of a straight line tangent to the helix on the base cylinder.

In coordinate system $X_2Y_2Z_2$ with origin at point K on surface G, the straight line tangent to the helix on the gear pitch cylinder is equal to (Fig. 4):

$$\vec{r}_p = \vec{i}_2 l \sin\psi + \vec{k}_2 l \cos\psi \tag{35}$$

or in matrix form

$$\vec{r}_p = \begin{bmatrix} l\sin\psi \\ 1 \\ l\cos\psi \end{bmatrix}. \tag{36}$$

Now it is possible to write (33) in coordinate system $X_2Y_2Z_2$ or to write (36) in coordinate system $X_1Y_1Z_1$. Choose the first.

In this case, matrix $[M_{1\to2}]$ of the coordinate system transformation is equal to

$$[M_{1\to2}] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{37}$$

In (37) we have neglected the translations of the coordinate systems along the coordinate axes because the central angle φ sought depends on coordinate system rotations only and is independent of translations of the origin of the coordinate systems.

Taking into account (37), the straight line tangent to the helix on the base cylinder (see (33)) is determined by:

$$\vec{r}_p[M_{1\to 2}]\vec{r}_p^{\,*} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \tan\theta \\ 1 \\ \tan\theta\cot\psi_b \end{bmatrix} = \begin{bmatrix} 2\sin\theta \\ -\sin\theta\tan\theta + \cos\theta \\ \tan\theta\cot\psi_b \end{bmatrix}. \quad (38)$$

From (36) and (38) and in accordance with [1, p.269], it follows that:

$$\cos\varphi = \frac{\vec{r}_p^{\,*}\cdot\vec{r}_p}{|\vec{r}_p^{\,*}\cdot\vec{r}_p|} =$$

$$= \frac{2l\sin\theta\sin\psi + \tan\theta\tan\psi_b\cos\psi}{\sqrt{[4\sin^2\theta + (\cos\theta - \sin\theta\tan\theta)^2 + \tan^2\theta\cot^2\psi_b]\cdot(l^2\sin^2\psi + \cos^2\psi)}}. \quad (39)$$

At point K, the parameter $l = 0$. For that reason, (39) simplifies to

$$\cos\varphi = \frac{\tan\theta\tan\psi_b}{\sqrt{4\sin^2\theta + (\cos\theta - \sin\theta\tan\theta)^2 + \tan^2\theta\cot^2\psi_b}}. \quad (40)$$

Taking into consideration (26) and (30), one can obtain:

$$\cos\varphi = \frac{\tan\theta\sin\psi\cos\phi_n}{\sqrt{(1-\sin^2\psi\cos^2\phi_n)\cdot[4\sin^2\theta + (\cos\theta - \sin\theta\tan\theta)^2] + \dfrac{1-\sin^2\psi\cos^2\phi_n}{\sin^2\psi\cos^2\phi_n}\tan\theta}} =$$

$$= \frac{0.3939609\sin 22.5^{\circ}\cos 20^{\circ}}{\sqrt{(1-\sin^2 22.5^{\circ}\cos^2 20^{\circ})\cdot(0.5374115 + 0.6177934 + 2.6525486)}} = 0.077806. \quad (41)$$

It follows that $R_G = 1580.292$.

This result shows the normal radius of curvature of surface G in a given direction tangent to the helix on the gear pitch cylinder, which is needed in order to calculate the parameters of the design of the cutting tool for manufacturing the gear. There is also another way to solve the problem. In that method, it is necessary to consider the curved line of intersection of surface G of the gear tooth with a plane tangent to the pitch cylinder of the gear, and to calculate the curvature of the line of intersection.

The method described here may always be applied, but is less efficient than other methods if calculation of only one single parameter is required. However, when multiple parameters of the local geometry (internal or external) of the gear tooth surface must be calculated, this method is more efficient than other methods in use.

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OSNOVNI OBLICI POVRŠINE ZUBACA U TEHNOLOGIJI ZUPČANIKA

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U radu je prikazan jedan novi pristup istraživanja lokalne topologije involute površine zubaca. Pristup je zasnovan na fundamentalnim rezultatima diferencijalne geometrije površina. Ovaj metod se može uvek primeniti, ali je manje efikasan ako se proračunavanja vrše samo po jednom parametru. Međutim, ako se proračun vrši prema većem broju parametara lokalne geometrije površine zubaca, ovaj pristup daje bolje rezultate u onosu na druge metode.