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COUPLED DYNAMICAL SYSTEMS WITH INTERNAL DELAYS AND DELAYED COUPLING: BEHAVIOR AND BIFURCATIONS

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Abstract. *In the two delayed coupled excitable systems with internal delays, each of the isolated units displays excitable, bi-stable, or oscillatory dynamics. Bifurcation relations between the coupling time lag and the coupling constant for different typical values of the internal time –lags and parameter are obtained and analyzed.*

Key words: *Nonlinear Dynamical Systems, Internal Delay, Delayed Coupling, Bifurcation Diagrams*

1. Introduction

Possibility to model oscillatory behavior of complex dynamical systems in Physics and Biology using delayed differential equations (DDE`s) is natural and well known (a sample of references is [1], [2], [3], [4], [5]). Often, in models with more than one variable, several and independent time-lags are justified. However, questions related to stability and bifurcations for systems of DDE`s with more than one fixed and discrete time-lags are comparatively more difficult to analyze than the same questions for systems with one time-lag. Furthermore, complex dynamical units, like, for example, neurons, appear as constitutive elements of more complex systems, and must transmit excitations between them. The transmission of excitations is certainly not instantaneous, and the representation by non-local and instantaneous interactions should be considered only as a very crude approximation. Thus, it is of some interest to study the collective behavior of systems composed of several units which are coupled by time-delayed interaction, and such that each unit if decoupled from the system would have an attractor determined by an intrinsic time-lag.

In particular, we shall be interested in the interplay of oscillations, produced by delayed coupling and by internal time-lags, in a collection of the so-called excitable systems. Excitability is a common property of many complex systems [6]. Although there is no precise definition [7] the intuitive meaning is clear: a small perturbation from the sin-

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gle stable stationary state can result in a large and long lasting excursion away from the stationary state before the system is returned back asymptotically to equilibrium. For example, the excitability is found as the typical behavior of isolated or coupled neurons. Transition from excitable to oscillatory dynamics in common neuronal models, i.e. the bifurcation of the stationary point into a stable periodic orbit, is usually achieved by varying an external parameter. On the other hand, it is known [8] that the same effect can be produced by varying an internal time-lag. However, the bifurcation sequence which leads from the single fixed point attractor to the single limit cycle attractor is more complicated in the case of the transition caused by the internal time-delay. In this case, there is a large region in the parameter domain and nonzero internal time-lags when the isolated single unit has two coexisting attractors, the stable fixed point and the stable limit cycle, i.e. the system is actually bi-stable. For such values of the internal parameters and time-lags the delayed coupling produces interesting effects, which we shall analyze in detail.

As a model of the excitable dynamics we shall use the FitzHugh-Nagumo system in the following form [9]:

$$\begin{aligned} \dot{x} &= -x^3 + (a+1)x^2 - ax - y + I_a \equiv X(x,y) \\ \dot{y} &= bx - \gamma y \equiv Y(x,y) \end{aligned} \tag{1}$$

where a, b ,γ are positive parameters and I_a is an external variable that is dynamically independent of x and y . When $I_a = 0$ and in an appropriate range of a b and γ the system displays typical excitable behavior (see the next section). Parameter I_a is commonly used to induce the Hopf bifurcation, which replaces the stable fixed point by a stable limit cycle, and thus turns the excitable into the oscillatory behavior. Let us stress that in this paper I_a is always equal to zero.

A pair of delayed coupled system (1) is given by:

$$\begin{aligned} \dot{x}_1(t) &= X(x_1(t), y_1(t)) + cf(x_1(t), x_2(t-\tau)) \\ \dot{y}_1(t) &= Y(x_1(t), y_1(t))) \\ \dot{x}_2(t) &= X(x_2(t), y_2(t)) + cf(x_2(t), x_1(t-\tau)) \\ \dot{y}_2(t) &= Y(x_2(t), y_2(t)) \end{aligned} \tag{2}$$

where the field $X(x,y), Y(x,y)$ describes the single isolated unit, and the function $f(x_i(t), x_j(t-\tau))$ describes the time-delayed coupling between the two excitable units with the coupling constant c. The coupled system (2), for such a b, and γ that the single units are excitable, displays transition from excitable to oscillatory behavior as the coupling strength and time-lag are varied, but the transition could be through an intermediate regime when the system (2) is bi-stable, with the coexisting stable fixed point and the limit cycle. This sequence of bifurcations was studied in the references [10] and [11], and the analysis was extended to the case of a chain of N units in [12]. These papers provide an extended list of references to the relevant related results, which we shall not repeat here.

A qualitatively similar transition from excitability to oscillatory dynamics occurs also in the system (1) if we assume that there is a possibility of an internal time-delay between variables x and y pertaining to the single unit. This phenomenon was studied in [8]. Such generalized single unit is described by the following equations:

$$\begin{aligned} \dot{x} &= -x^3 + (a+1)x^2 - ax - [a_1 y + (1-a_1) y(t-\tau_1)], \\ \dot{y} &= b[a_2 x + (1-a_2) x(t-\tau_2)] - \gamma y \end{aligned} \tag{3}$$

with the same limits on the parameters as in (1) with $I_a = 0$.

In this paper we shall study the system (2) but with the fields $X(x,y)$, $Y(x,y)$ given by (3). In what follows the time-lags τ_1 and $\tau_2$will be called internal and τ will be called the coupling time-lag. Such a system could be interpreted as a collection of delayed coupled complex excitable systems where the internal delays could produce the dynamics of the single unit which is akin to that of coupled simple excitable systems.

2. LOCAL BIFURCATION OF THE FIXED POINT

In this section we study bifurcations of the zero stationary point (0,0,0,0) of two diffusively coupled FitzHugh-Nagumo excitable systems with internal delay's and the delay in the diffusive coupling given by the following general form:

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + (a+1)x_1^3 - ax_1 - [a_1 y_1 + (1-a_1) y_1(t-\tau_1)] + c(x_1 - x_2(t-\tau)), \\ \dot{y}_1 &= b[a_2 x_1 + (1-a_2) x_1(t-\tau_2)] - \gamma y_1, \\ \dot{x}_2 &= -x_2^3 + (a+1)x_2^3 - ax_2 - [a_1 y_2 + (1-a_1) y_2(t-\tau_1)] + c(x_2 - x_1(t-\tau)), \\ \dot{y}_2 &= b[a_2 x_2 + (1-a_2) x_2(t-\tau_2)] - \gamma y_2, \end{aligned} \tag{4}$$

In the general form (4) we allow for the possibility that variables x_i, y_i depend on the instantaneous as well as the delayed values of y_i, x_i, respectively. Linearization of the system (4) and substitution

$$x_i(t) = A_i e^{\lambda t}, y_i(t) = B_i e^{\lambda t}, x_i(t-\tau) = A_i e^{\lambda(t-\tau)}, x_i(t-\tau_2) = A_i e^{\lambda(t-\tau_2)}, y_i(t-\tau_1) = B_i e^{\lambda(t-\tau_1)},$$

result in a system of equations for constants A_i and B_i. This system has a nontrivial solution if the following is satisfied:

$$\Delta_1(\lambda) \cdot \Delta_2(\lambda) = 0 \tag{5}$$

where

$$\begin{aligned} \Delta_1(\lambda) &= \lambda^2 + (a+\gamma-c)\lambda + (a-c)\gamma + a_1 a_2 b + a_1(1-a_2) b e^{-\lambda\tau_2} + (1-a_1) a_2 b e^{-\lambda\tau_1} + \\ &+ (1-a_1)(1-a_2) b e^{-\lambda(\tau_1+\tau_2)} - (\lambda+\gamma) c e^{-\lambda\tau} \end{aligned} \tag{6}$$

$$\begin{aligned} \Delta_2(\lambda) &= \lambda^2 + (a+\gamma-c)\lambda + (a-c)\gamma + a_1 a_2 b + a_1(1-a_2) b e^{-\lambda\tau_2} + (1-a_1) a_2 b e^{-\lambda\tau_1} + \\ &+ (1-a_1)(1-a_2) b e^{-\lambda(\tau_1+\tau_2)} + (\lambda+\gamma) c e^{-\lambda\tau} \end{aligned} \tag{7}$$

Equation (5) is the characteristic equation of the system (4). Infinite dimensionality of the system is reflected in the transcendental character of (5). However, the spectrum of the linearization of the equations (4) is discrete and can be divided into infinite dimensional hyperbolic and finite dimensional non-hyperbolic parts [13]. As in the finite dimensional case, the stability of the stationary solution $(x_1, y_1, x_2, y_2) = (0, 0, 0, 0)$ is typically, i.e. in the hyperbolic case, determined by the signs of the roots of (5). Exceptional roots equal to zero or with zero real part, correspond to the finite dimensional center manifold where the qualitative features of the dynamics, such as local stability, depend on the nonlinear terms.

The general system (4) apparently has three independent time-lags. Two internal time-lags τ_1 and τ_2 appear independently in the characteristic equation only if $a_1$and a_2 are different from zero and unity. However, possible types of dynamics in this most general case are qualitatively similar to the situation when there is effectively only one internal time lag, which is the case that we shall study further. We shall analyze the roots of (5) in the following two special cases: 1. pure delays $a_1 = a_2 = 0$, and 2. one internal delay $a_2 = 1$; $a_1 \neq 0;1$. In both of these cases equations (4) have two independent time-lags: one for the internal delay and one for the coupling time delay.

2.1. The case of Pure Delays

In this case the system of DDE (4) is reduced to:

$$\begin{aligned}
\dot{x}_1 &= -x_1^3 + (a+1)x_1^3 - ax_1 - y_1(t-\tau_1) + c(x_1 - x_2(t-\tau)),\\
\dot{y}_1 &= bx_1(t-\tau_2) - \gamma y_1,\\
\dot{x}_2 &= -x_2^3 + (a+1)x_2^3 - ax_2 - y_2(t-\tau_1) + c(x_2 - x_1(t-\tau)),\\
\dot{y}_2 &= bx_2(t-\tau_2) - \gamma y_2,
\end{aligned} \tag{8}$$

and the two factors of the characteristic equation (6) and (7) become now:

$$\Delta_1(\lambda) = \lambda^2 + (a+\gamma-c)\lambda + (a-c)\gamma + be^{-\lambda(\tau_1+\tau_2)} - (\lambda+\gamma)ce^{-\lambda\tau} \tag{9}$$

$$\Delta_2(\lambda) = \lambda^2 + (a+\gamma-c)\lambda + (a-c)\gamma + be^{-\lambda(\tau_1+\tau_2)} + (\lambda+\gamma)ce^{-\lambda\tau} \tag{10}$$

As we see the internal delays appear only in the combination $\tau_1 + \tau_2$, which will be denoted by $\tau_1 + \tau_2 = \tau_{12}$.

We shall seek for the relations $\tau = f(a, b, \gamma, a_1, a_2, \tau_{12}, c)$ so that some solutions of the characteristic equation given by (9) and (10) are pure imaginary $\lambda = \pm\iota\omega$ with real and positive ω. Under some additional conditions [13], these relations correspond to the Hopf bifurcation. Substituting $\lambda = \iota\omega$, where ω is real and positive, into first factor, multiplying with $(-\iota\omega + \gamma)$ and separating real and imaginary part gives

$$\begin{aligned}
c(\gamma^2+\omega^2)\cos(\omega\tau) &= (a-c)(\gamma^2+\omega^2) + F\\
c(\gamma^2+\omega^2)\sin(\omega\tau) &= -\omega(\gamma^2+\omega^2) + G
\end{aligned} \tag{11}$$

where

$$\begin{aligned}
F &= b(\gamma\cos(\omega\tau_{12}) - \omega\sin(\omega\tau_{12}))\\
G &= b(\omega\cos(\omega\tau_{12}) + \gamma\sin(\omega\tau_{12}))
\end{aligned}$$

The same manipulations applied with the second factor result in

$$\begin{aligned} c(\gamma^2+\omega^2)\cos(\omega\tau) &= -(a-c)(\gamma^2+\omega^2)-F \\ c(\gamma^2+\omega^2)\sin(\omega\tau) &= \omega(\gamma^2+\omega^2)-G \end{aligned} \tag{12}$$

with F and G given as before.

Squaring and adding the previous two pairs of equations (11) or (12) result in the same parametric equation for the coupling strength

$$c=\frac{(a^2+\omega^2)(\gamma^2+\omega^2)-2G\omega+2aF+b^2}{2a(\gamma^2+\omega^2)+2F} \tag{13}$$

The corresponding critical time lag follows by dividing the pair of equations (11) for the first factor:

$$\tau^1_{c,k}=\tfrac{1}{\omega}[\arctan(\frac{-\omega(\gamma^2+\omega^2)+G}{(a-c)(\gamma^2+\omega^2)+F})+k\pi], \tag{14}$$

and the critical time lag from the second factor is obtained by dividing the pair of equations (12):

$$\tau^2_{c,k}=\tfrac{1}{\omega}[\arctan(\frac{\omega(\gamma^2+\omega^2)-G}{-(a-c)(\gamma^2+\omega^2)-F})+k\pi], \tag{15}$$

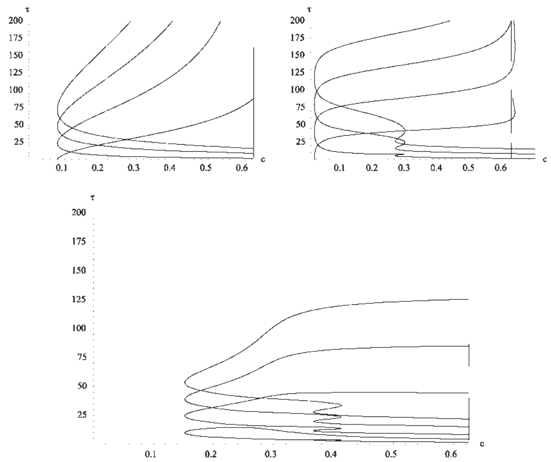


Fig.1 a, b, c for (a) $\tau_1=\tau_2=3$ (case α);(b) $\tau_1=\tau_2=9$ (case β); (c) $\tau_1=\tau_2=15$ (case γ) for $a=0.25, b=\gamma=0.02$.

The bifurcation curves are illustrated in Figs. 1a, b, c. The figures correspond to three typical situations that occur in a single isolated unit for different values of $\tau_1+\tau_2$. The internal delays are chosen so that in the case α (Fig. 1a) the internal delays are small and the isolated unit has the stable fixed point as the only attractor; in the case β for medium internal time-lags (Fig. 1b) the isolated unit has the stable fixed point and the stable limit cycle as the only two attractor and in the case γ (Fig. 1c) for large $\tau_1 + \tau_2$ the isolated unit has the stable limit cycle as the only attractor.

2.2 One Internal Time Lag

General equation (4) in the case of one internal time-lag with $a_2 = 1$; $a_1 \neq 0;1$ reduces to

$$\begin{aligned}
\dot{x}_1 &= -x_1^{\,3} + (a+1)x_1^{\,3} - ax_1 - [a_1 y_1 + (1-a_1)y_1(t-\tau_1)] + c(x_1 - x_2(t-\tau)),\\
\dot{y}_1 &= bx_1 - \gamma y_1,\\
\dot{x}_2 &= -x_2^{\,3} + (a+1)x_2^{\,3} - ax_2 - [a_1 y_2 + (1-a_1)y_2(t-\tau_1)] + c(x_2 - x_1(t-\tau)),\\
\dot{y}_2 &= bx_2 - \gamma y_2,
\end{aligned} \tag{16}$$

with corresponding factors of the characteristic equation:

$$\Delta_1(\lambda) = \lambda^2 + (a+\gamma-c)\lambda + (a-c)\gamma + a_1 b + (1-a_1)be^{-\lambda\tau_1} - (\lambda+\gamma)ce^{-\lambda\tau} \tag{17}$$

$$\Delta_2(\lambda) = \lambda^2 + (a+\gamma-c)\lambda + (a-c)\gamma + a_1 b + (1-a_1)be^{-\lambda\tau_1} + (\lambda+\gamma)ce^{-\lambda\tau} \tag{18}$$

Pure imaginary solutions of (17) and (18) are analyzed is the same way as in the first case. The real and imaginary parts for the first factor (17) are:

$$\begin{aligned}
c(\gamma^2+\omega^2)\cos(\omega\tau) &= (a-c)(\gamma^2+\omega^2) + ba_1\gamma + FF\\
c(\gamma^2+\omega^2)\sin(\omega\tau) &= -\omega(\gamma^2+\omega^2) + ba_1\omega + GG
\end{aligned} \tag{19}$$

and for the second factor

$$\begin{aligned}
c(\gamma^2+\omega^2)\cos(\omega\tau) &= -(a-c)(\gamma^2+\omega^2) - ba_1\gamma - FF\\
c(\gamma^2+\omega^2)\sin(\omega\tau) &= \omega(\gamma^2+\omega^2) - ba_1\omega - GG
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
FF &= (1-a_1)b(\gamma\cos(\omega\tau_1) - \omega\sin(\omega\tau_1))\\
GG &= (1-a_1)b(\omega\cos(\omega\tau_1) + \tau\sin(\omega\tau_1))
\end{aligned}$$

From these equations, in the same manner as in the first case, we obtain equations for the coupling strength:

$$c = \frac{(a^2+\omega^2)(\gamma^2+\omega^2) - 2H + b^2 I}{2a(\gamma^2+\omega^2) + 2a_1 b\gamma + 2FF} \tag{21}$$

where

$$\begin{aligned}
H &= aa_1 b\gamma + aFF - \omega^2 a_1 b - \omega GG\\
I &= a_1^2 + 2a_1(1-a_1)\cos(\omega\tau_1) + (1-a_1)^2
\end{aligned}$$

The critical time lags from the first factor are given by

$$\tau^1_{c,k} = \tfrac{1}{\omega}[\arctan(\frac{-\omega(\gamma^2+\omega^2)+ba_1\omega+GG}{(a-c)(\gamma^2+\omega^2)+ba_1\gamma+FF})+k\pi], \tag{22}$$

and from the second factor by

$$\tau^2_{c,k} = \tfrac{1}{\omega}[\arctan(\frac{\omega(\gamma^2+\omega^2)-ba_1\omega-GG}{-(a-c)(\gamma^2+\omega^2)-ba_1\gamma-FF})+k\pi], \tag{23}$$

The previous formulas (13-15) for the case of pure delay and for the case with one delayed argument (21-23) give parametric representations of the bifurcation curves in the plane (c, τ) for fixed values of internal parameters.

The type of the Hopf bifurcation can be seen by calculation of the variations of the real parts ***Re***λ as the time lag is changed true to the critical values. This is given by the sign of d**Re**λ/dτ for $\tau = \tau^c$. Using the factorized characteristic equations, we obtain

$$\left[\frac{d\,\mathrm{Re}\,\lambda}{d\tau}\right] = \frac{2\omega^3+(\gamma^2+a^2-2ac)\omega-A_1-B_1}{c^2\omega(\gamma^2+\omega^2)} \tag{24}$$

where

$$A_1 = b\omega\cos(\omega\tau_{12})+G+b(a-c)\sin(\omega\tau_{12})$$
$$B_1 = \tau_{12}((a-c)G+\omega F)$$

for the first case, and for the second case:

$$A_2 = b\omega\cos(\omega\tau_1)+G+b(a-c)\sin(\omega\tau_1)$$
$$B_2 = \tau_1((a-c)G+b^2a_1\sin(\omega\tau_1)+\omega F)$$

Substitution of particular values in this formulas gives the sign of d**Re**λ/dτ and it determines the type of the considered Hopf bifurcation.

3. Influence of Changing Different Parametars on the Bifurcation Diagrams

In this section we illustrate the effects of different parameters on the bifurcation curves (diagrams).

- Firstly, bifurcation curves for different values of the internal delay ($\tau_{12} = \tau_1 + \tau_2$), and for the case of pure delay are shown in Fig. 3.1-20. The range of the illustrated values is $\tau_{12} = 0$ –20, where $\tau = 20$ is roughly one fifth of the refractory period.

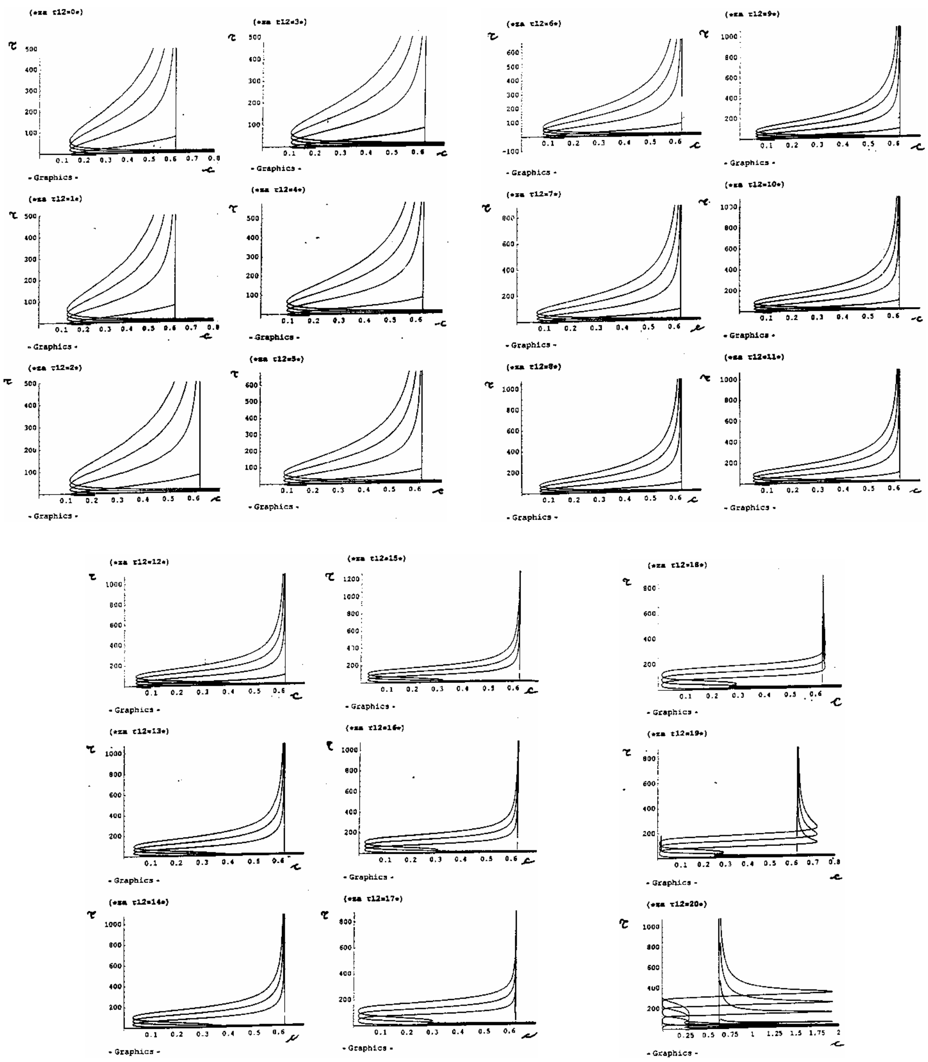


Fig. 3 1-20

From the figure we can conclude that: Increasing the internal delay above $\tau_{12} = 12$ (roughly one sixth of the refractory period [8]) leads to a noticeable change of the bifurcation curves. For $\tau_{12} \geq 18$ the bifurcation curves are quite different, developing structures that are completely absent at low values of $.c = 0$ Bifurcation curves bunch near x-axes and the stability domain shrinks.

- Secondly, bifurcation diagrams for the cases with pure delay and with one internal delay are compared for different values of the internal delay. The relevant parameters are: $\tau_{12} = \tau_1 = 0;1;4;8;16$, and the diagrams are presented in Fig. 3.21-25

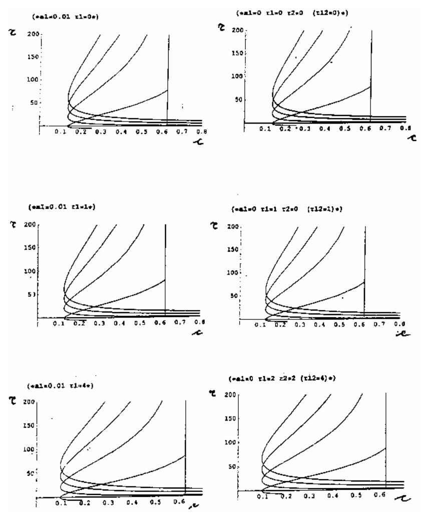
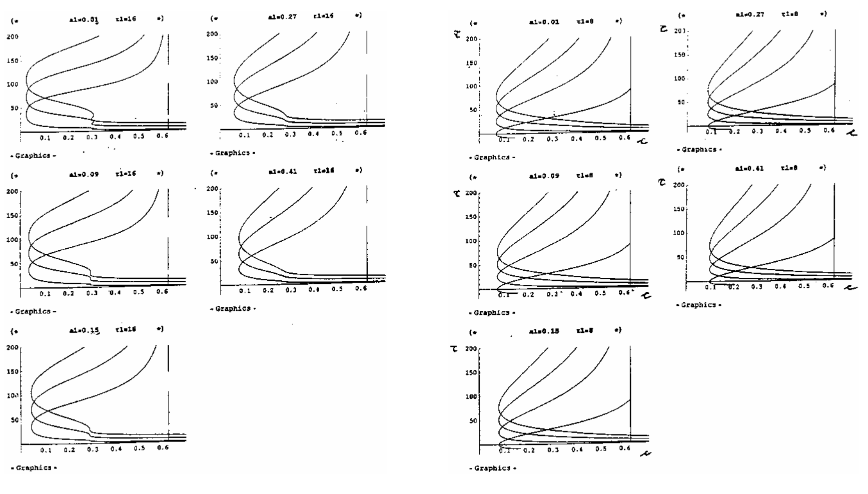


Fig. 3 21-25

Comparison of the cases with pure delay and with one internal delay confirms the expected results that there are no qualitative differences between the two cases for moderate values of the coupling delays. Some differences might only occur for quite large values of the coupling delays.

- Finally, we illustrate the influence on bifurcation curves of different values of the parameter $a_1 = 0.01;0.09;0.15;0.27;0.41$ for two values of the internal delay. Diagrams are shown in Figs. 3.25-29 and 3.29-33



Figs. 3 25-29 and 3.29-33

4. CONCLUSION

We have studied a pair of identical FitzHugh-Nagumo systems with internal and coupling time delays. The parameters of each of the isolated units are such that for the zero values of the internal delays the units are excitable, i.e. each has only one attractor in the form of the stable fixed point. However, there is a large interval of values of the internal delays such that each unit is bi-stable, with the stable fixed point and the stable limit cycle. In this case, each of the units could be considered as modeling dynamics that could occur in a collection of delayed coupled simple excitable systems without internal delays.

We have analyzed the stability of the fixed point of the coupled system. Bifurcation curves in the plain of coupling constant and the coupling delay, ((c ,τ) plain), are obtained for various fixed values of the internal delays. This indicates that three cases should be distinguished, depending on the values of the internal delays. The case when the units are excitable, the case when units are bi-stable and the case when units are oscillatory. Dependence of the global dynamics on the coupling and coupling delay in these three case is studied numerically.

The following picture emerged from our analyzes. Increasing the internal delay above $\tau_{12} = 12$ leads to noticable change of the bifurcation curves. For $\tau_{12} \geq 18$ the bifurcation curves are quite different, developing structures that are completely absent at low values of . $c = 0$Bifurcation curves bunch near x-axes and the stability domain shrinks. Comparison of the cases with pure delay and with one internal delay confirms the expected results that there are no qualitative differences between the two cases for moderate values of the coupling delays. Some differences might only occur for quite large values of the coupling delays which are of no interest to us.

There are several directions in which our analyses should be extended. It would be interesting to study systems like (4) but with more that two, identical or no identical, units and with local or global coupling. Further more, the influence of multiplicative or additive noise on the synchronization properties should be studied.

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VEZANI DINAMIČKI SISTEMI SA UNUTRAŠNJIM KAŠNJENJEM I KAŠNJENJEM U VEZI: PONAŠANJE I BIFURKACIJE

Ines Grozdanović

Proučavana su dva vezana dinamička sistema sa unutrašnjim kašnjenjem i kašnjenjem u vezi. Za različite vrednosti unutrašnjih kašnjenja svaki od izolovanih sistema pokazuje: ekscitabilnu, bistabilnu, ili oscilatornu dinamiku. Dobijene su bifurkacione relacije između kašnjenja u vezi i jačine veze, i međusobno poređene za različite tipične vrednosti unutrašnjeg kašnjenja, kao i za različite vrednosti parametra , za slučaj jednog unutrašnjeg kašnjenja.

Ključne reči: *nelinearni dinamički sistemi, unutrašnje kašnjenje, kašnjenje u vezi, bifurkacione krive*