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A MATHEMATICAL MODEL OF THE TEMPERATURE FIELD IN A FLAT PLATE AT NON-STATIONARY HEATING IN THE FLOW-THROUGH OVEN

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Abstract. *The paper presents a developed two-dimensional model for the calculation of the temperature field in a flat plate of finite thickness at non-stationary heating in the flow-through oven. An exact analytical solution could not be given for the derived differential equation of non-stationary heat transfer, and, therefore, an approximate numerical method of finite differences was applied. The differential equation of non-stationary heat conduction has been transformed into a difference equation, with the two-dimensional plate divided by a net of rectangles. The calculation was performed for the whole set of columns, and the distribution of temperatures was determined and shown for the characteristic values of dimensionless parameters.*

Key Words: *Mathematical Model, Flat Plate, Temperature Field, Numerical Method, Flow-Through Oven*

INTRODUCTION

Working on mathematical modeling of heat calculations for flow-through ovens, the authors of this paper also resolve the problem of non-stationary radiation heating of a finite thickness plate by developing calculation programs. In the calculation of ovens, considering Ref. [1, 2], two simplified mathematical models were used, namely, the »agitator« and »piston-type« flow models. The basic characteristic of these models is that they describe the boundary cases of the process in real ovens. The »agitator« model assumes that the products of combustion in the working area of the oven are ideally mixed up and that the temperature, concentration and thermal and physical properties of effluent gases are uniform in the entire area of the oven. The »piston-type« flow model assumes an oven with flat profiles of flow, and that the gas properties are uniform in lateral directions (planes perpendicular to the direction of flow and to the outer surface of the product), changing only in the longitudinal direction, i.e. in the direction of flow. In this approach the

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mathematical description was simplified by the introduction of effective emissivity in the manner presented in the doctoral dissertation of Mr. W. Schupe [3]. In addition to this basic piston-type flow model, the model of our analysis was also the piston-type flow one with superimposed energy reaction and the piston-type flow model with energy introduced continually. Results of mathematical modeling, graphics of the calculated heat fluxes, temperature distribution, as well as theoretical comparisons and characterization of different oven models are presented.

MATHEMATICAL PROBLEM SETUP

In order to resolve the problem of heat conduction in a solid body, it is necessary to define the equation of the temperature field describing the distribution of temperatures in the body and time. In the case of heating a flat plate of finite thickness and an infinite length in flow-through ovens, we deal with single-dimensional and non-stationary conduction of heat, where temperature of the plate is dependant only on time and the coordinate of the plane perpendicular to the direction of transportation and the plate area.

Therefore, to describe the non-stationary process of heat conduction in a solid body element of dx length, we use the single-dimensional Fourier's differential equation:

$$\frac{\partial T_s}{\partial \tau} = a \frac{\partial^2 T_s}{\partial y^2}, \tag{1}$$

where τ is time, and a temperature conductivity. For time τ at an arbitrary point x and with constant speed of transportation v, we can write:

$$\tau = \frac{x}{v} , \tag{2}$$

so that equation (1) now takes the following form:

$$v \frac{\partial T_s}{\partial x} = a \frac{\partial^2 T_s}{\partial y^2} . \tag{3}$$

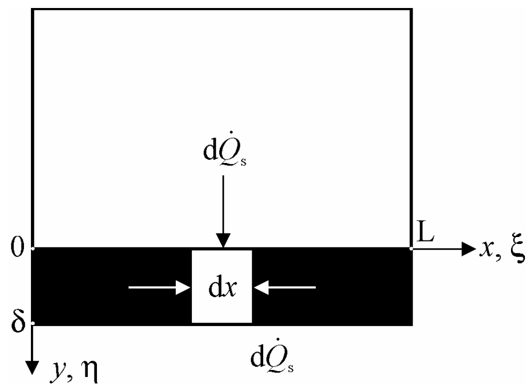

Fig. 1. – Position of Coordinate Axes with the Finite Thickness Plate

The coordinate system x, y is shown in Fig. 1. In addition to this, using oven length L and flat plate thickness δ, we can define dimensionless coordinates $\xi = x / L$ and $\eta = y / \delta$, so that equation (3) can have the following dimensionless form:

$$\frac{\partial \theta_s}{\partial \xi} = \frac{a}{v \cdot \delta} \cdot \frac{L}{\delta} \cdot \frac{\partial^2 \theta_s}{\partial \eta^2}, \tag{4}$$

where $\theta_s = T / T_{gu}$ is the dimensionless temperature of the plate.

The following dimensionless parameter appears in equation (4):

$$\frac{v \cdot \delta}{a} = \text{Pe}, \tag{5}$$

representing the Peclet's number. Differential equation (5) is further rearranged by introducing the modified Peclet's number:

$$\text{Pe}^* = \text{Pe} \cdot \frac{\delta}{L} = \frac{v \cdot \delta^2}{a \cdot L}, \tag{6}$$

into the dimensionless form

$$\frac{\partial \theta_s}{\partial \xi} = \frac{1}{\text{Pe}^*} \cdot \frac{\partial^2 \theta_s}{\partial \eta^2}. \tag{7}$$

In order to determine the temperature field, it is necessary to have two boundary conditions. The boundary conditions, in this case, are:

1. The heat flux that is transferred from the gas to the upper surface of the plate is given

$$\mathrm{d}\dot{Q}_{so} = -\lambda \cdot b \cdot dx \left. \frac{\partial T_s}{\partial y} \right|_o \quad \text{and} \tag{8}$$

2. The heat flux that is transferred to the surroundings from the lower side of the plate is given

$$\mathrm{d}\dot{Q}_{s,\,gub} = -\lambda \cdot b \cdot \mathrm{d}x \left. \frac{\partial T_s}{\partial y} \right|_\delta. \tag{9}$$

The radiation heat flux transferred from the gas (combustion product) to the flat plate is determined by the following expression:

$$\mathrm{d}\dot{Q}_{so} = \varepsilon_{ef} \cdot \sigma (T_g^4 - T_{so}^4) \cdot dx, \tag{10}$$

where ε_{ef} – effective emissivity, σ – Stefan-Boltzmann's constant, T_g – temperature of the gas, whereas T_{so} – temperature of the upper plate surface.

By making the right-hand sides of equations (8) and (10) equal to each other, we obtain:

$$\left. \frac{\partial T_s}{\partial y} \right|_o = -\frac{\varepsilon_{ef} \cdot \sigma}{\lambda} (T_g^4 - T_{so}^4), \tag{11}$$

or in the dimensionless form

$$\left.\frac{\partial \theta_s}{\partial \eta}\right|_o = -\mathrm{Th}(\theta_g^4 - \theta_{so}^4)\,, \tag{12}$$

where

$$\mathrm{Th} = \frac{\varepsilon_{ef} \cdot \sigma \cdot T_{gu}^3}{\lambda}\,, \tag{13}$$

The Thring's number, frequently referred to in the literature also as the Sparrow's number [4, 5], representing the relation of the enthalpy transferred by radiation to the product surface and the heat transferred by conduction through the product.

In addition to this, heat losses through the differential wall of the oven are calculated using the expression:

$$d\dot{Q}_{s,\,gub} = k_s \cdot b \cdot (T_{s\delta} - T_o)dx\,. \tag{14}$$

By making the right-hand sides of equations (9) and (14) equal to each other, we obtain:

$$\left.\frac{\partial T_s}{\partial y}\right|_\delta = -\frac{k_s \cdot \delta}{\lambda}(T_{s\delta} - T_o)\,, \tag{15}$$

or in the dimensionless form

$$\left(\frac{\partial \theta_s}{\partial \eta}\right)_\delta = -\mathrm{G}_s(\theta_{s\delta} - \theta_o)\,, \tag{16}$$

where

$$\mathrm{G}_s = \frac{k_s \cdot \delta}{\lambda}\,. \tag{17}$$

The discussed problem is fully defined if the initial condition characterizing the distribution of temperature in the solid body at the initial point of time is also known. It is assumed, in this case, that the solid body initially has a constant temperature. The Fourier's differential equation of the non-stationary heat conduction at these initial and the above defined boundary conditions mathematically describes fully this problem.

NUMERICAL METHOD SETUP

In the cases where it is not possible to determine the analytical solution of differential equation (7), the approximate method of finite differences can be used.

The two-dimensional plate (Fig. 2) is divided by a net of rectangles in $\Delta\xi$ and $\Delta\eta$ steps, the size of which can be selected freely. A larger number of steps results in a higher accuracy, but it requires more calculation. For the current local coordinates, we can write:

$$\xi = n \cdot \Delta\xi \tag{18}$$

$$\eta = k \cdot \Delta\eta\,. \tag{19}$$

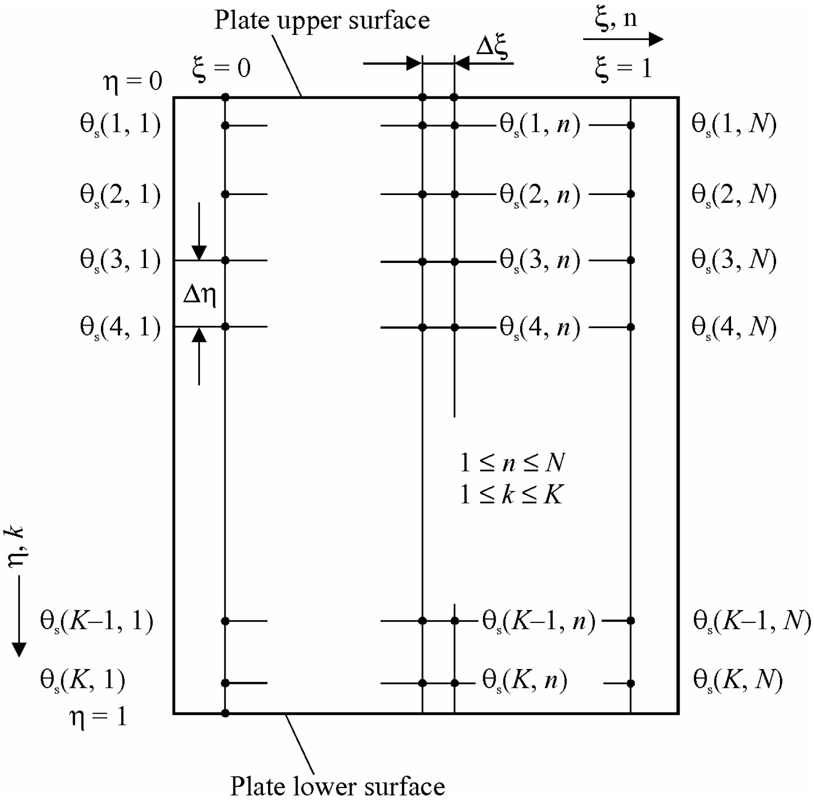

Fig. 2. – The Plate Element that a Net of Points is Established for

Differentials dξ and dη in equation (7) should be replaced with Δξ and Δη intervals. The values of these intervals can be selected independently of each other. In order to have stable mathematical calculation without any senseless temperature oscillations, the criteria of similarity from the literature must be observed [6].

Local coordinates of the oven cross section are divided into a large number of steps (Fig. 2). The solid body (heated plate) is divided by a net of $N \cdot K$ rectangles. Division in N sections is sufficient for the calculation of the gas and oven walls temperature values.

By means of the temperature values of one column (direction η), it is possible to calculate those of the columns that follow. Therefore, the values for the first column must be known in advance. To determine the total temperature field, it will be necessary to perform the calculation for N columns.

It is also necessary to explain briefly the procedure of calculating the values for new (or the following) columns. The temperatures are marked with double indexes, with the first index k relating to the row, and the second index n denoting the column. Transition from one column to another, according to equation (1), is the time interval (direction ξ), and transition to another row is the local step (direction η).

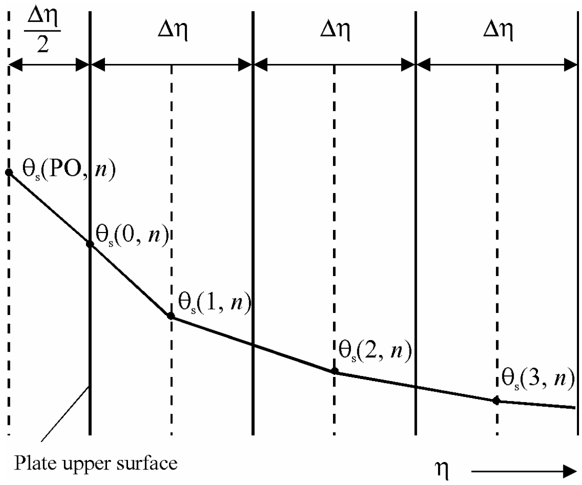

Fig. 3. – Position of the Auxiliary Point for Calculating the Temperature of the Plate Upper Surface

It is desirable that the representative temperature θ_s (k,n) should relate to the middle plane of the plate (Fig. 2). However, problems occur with regard to satisfying the boundary conditions, because these relate to temperature θ_s $(0,n)$ of the plate upper surface, and not to temperature θ_s $(1,n)$ inside the plate. In this case, an auxiliary point of temperature θ_s (PO,n) shown in Fig. 3 is introduced.

The differential equation of non-stationary heat conduction (7) is converted into the deference equation using the general formula of the second derivative according to H. Schuh [6]:

$$\frac{d^2 y(x)}{dx^2} \approx \frac{y(x-h) - 2 \cdot y(x) + y(x+h)}{h^2}, \tag{20}$$

in which h represents the width of step $\Delta\eta$. Then temperature values for the new column are calculated using equation (20).

$$\theta_s(k, n+1) = \frac{\Delta\xi}{\Delta\eta^2} \cdot \frac{1}{\mathrm{Pe}^*} [\theta_s(k-1, n) + 2\theta_s(k, n) + \theta_s(k+1, n)] + \theta_s(k, n)\,, \tag{21}$$

where $1 \le k \le K$.

It is evident that temperatures of boundary surfaces cannot be determined by means of equation (21), but rather by using boundary conditions. The temperature gradient on the body surface can be described by the temperature difference between θ_s (PO,n) and θ_s $(0,n)$. For the boundary condition given by equation (12), having in mind that the step width in direction ξ is small, we can write down with sufficient accuracy:

$$\theta_s(\mathrm{PO}, n+1) - \theta_s(0, n+1) = \frac{\Delta\eta}{2} \cdot \mathrm{Th} \cdot [\theta_g^4(n) - \theta_s^4(0, n)]\,. \tag{22}$$

In this equation, the unknown temperature values are θ_s (PO,n + 1) and θ_s (0,n + 1). However, according to Fig. 3, the following temperature differences are of the same value, i.e.:

$$\theta_s(\mathrm{PO},n)-\theta_s(0,n)=\theta_s(0,n)-\theta_s(1,n)\,. \tag{23}$$

This, of course, is valid for time interval n + 1, and, therefore, the temperature on the plate upper surface can be obtained by means of equations (22) and (23):

$$\theta_s(0,n+1)=\theta_s(1,n+1)+\frac{\Delta\eta}{2}\cdot\mathrm{Th}\cdot[\theta_g^4(n)-\theta_s^4(0,n)]\,, \tag{24}$$

and also the temperature of the auxiliary point

$$\theta_s(\mathrm{PO},n+1)=\theta_s(1,n+1)+\Delta\eta\cdot\mathrm{Th}\cdot[\theta_g^4(n)-\theta_s^4(0,n)]\,, \tag{25}$$

where temperature θ_s (1,n + 1) is determined by equation (21).

The same problem occurs when determining the temperature of the plate lower surface. Taking into account the second boundary condition, and by introducing in the above described manner second auxiliary point (P δ) and index δ for values on the plate lower surface, the following differential equation is obtained:

$$\theta_s(\delta,n+1)-\theta_s(\mathrm{P}\delta,n+1)=\frac{\Delta\eta}{2}\cdot\mathrm{G_s}[\theta_s(\delta,n)-\theta_o]\,. \tag{26}$$

Equality of temperature differences figures here as a condition for determining the auxiliary point:

$$\theta_s(K,n+1)-\theta_s(\delta,n+1)=\theta_s(\delta,n+1)-\theta_s(\mathrm{P}\delta,n+1)\,. \tag{27}$$

Temperature values for the plate lower surface and the auxiliary point are obtained from equations (26) and (27):

$$\theta_s(\delta,n+1)=\theta_s(K,n+1)-\frac{\Delta\eta}{2}\cdot\mathrm{G_s}[\theta_s(\delta,n)-\theta_o]\,, \tag{28}$$

$$\theta_s(\mathrm{P}\delta,n+1)=\theta_s(K,n+1)-\Delta\eta\cdot\mathrm{G_s}\cdot[\theta_s(\delta,n)-\theta_o]\,. \tag{29}$$

All product temperature values in the column that follows are determined in this way. By transforming the differential equation for the gas temperature into the difference equation, it is possible to perform the calculation for the whole set of columns. Care should be paid in this to select the step width properly for calculation stability, as usability of results depends on that.

If simplexes of dimensionless coordinates and temperatures are taken into account, it follows that for the mathematical description of the radiation heat transfer in flow-through ovens seven mutually dependent parameters of similarity are used [1]. As the result of transformations performed in the equations, the following dependences are obtained:

$$\omega \cdot \mathrm{Pe}^{*} = \frac{\mathrm{G_s}}{\mathrm{G_{sm}}} , \tag{30}$$

$$\omega \cdot \mathrm{Pe}^{*} = \mathrm{Ko} \cdot \mathrm{Th} . \tag{31}$$

In the above equation, the following dimensionless parameter occurs:

$$\mathrm{Ko} = \frac{\dot{m}_{\mathrm{g}} \cdot c_{\mathrm{g}}}{\varepsilon_{\mathrm{ef}} \cdot \sigma \cdot A_{\mathrm{so}} \cdot T_{\mathrm{gu}}^{3}} , \tag{32}$$

representing the *Konakow's* number, which apparently characterizes physically the relation between the enthalpy of the gas flow and the heat flux transferred by radiation. The other dimensionless parameter in equation (32)

$$\omega = \pm \frac{\dot{m}_{g} \cdot c_{\mathrm{g}}}{\dot{m}_{\mathrm{s}} \cdot c_{\mathrm{s}}} , \tag{33}$$

represents the relation of the heat capacities of the gas and solid body. The sign + denotes the concurrent medium flow, whereas the sign - denotes the countercurrent flow.

If the convection mechanism of heat transfer is also included, then, besides the seven parameters of similarity, the *Biot's* and *Stanton's* numbers come into the picture. These values are connected by the following relation:

$$\omega \cdot \mathrm{Pe}^{*} = \frac{\mathrm{Bi}}{\mathrm{St}} . \tag{34}$$

The set of similarity parameters for the model of the radiation and convection heat transfer is reduced in this way from seven to five parameters.

For further simplification, it is assumed that the observed oven is adiabatic (ideally insulated in relation to the environment) and that the exchange of heat in it develops exclusively by radiation. It is apparent that in this case the number of dimensionless parameters is reduced to the relation of the heat capacities of the gas and of the solid body, and the modified *Peclet's*, *Konakow's* and *Thring's* numbers. The effects of these parameters on the distribution of temperature and exchange of heat fluxes on different oven models are investigated by varying individual parameters. Three of these parameters are selected freely while the fourth is determined from dependence (31).

RESULTS OF TEMPERATURE FIELDS CALCULATION

For the calculation of the temperature field in the flat plate for different oven models from the literature [1], the following numerical values for the dimensionless parameters were used: Ko = 0,75; Th = 1, $|\omega| = 1$ i $|\mathrm{Pe}^{*}| = 0{,}75$.

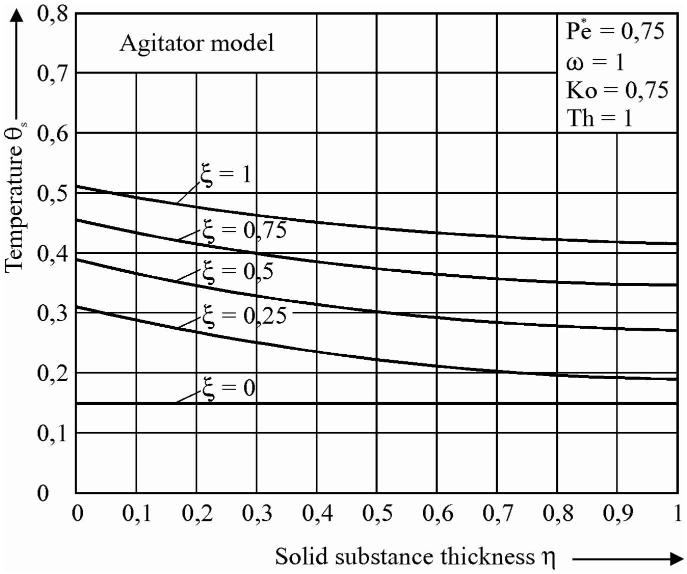

Fig. 4. – Distribution of Temperature in the Flat Plate (Agitator)

Boundary condition: $1-\theta_g = \int_0^1 [\frac{1}{Ko}(\theta_g^4 - \theta_{so}^4) + G_g(\theta_z - \theta_o)]d\xi$ [1]

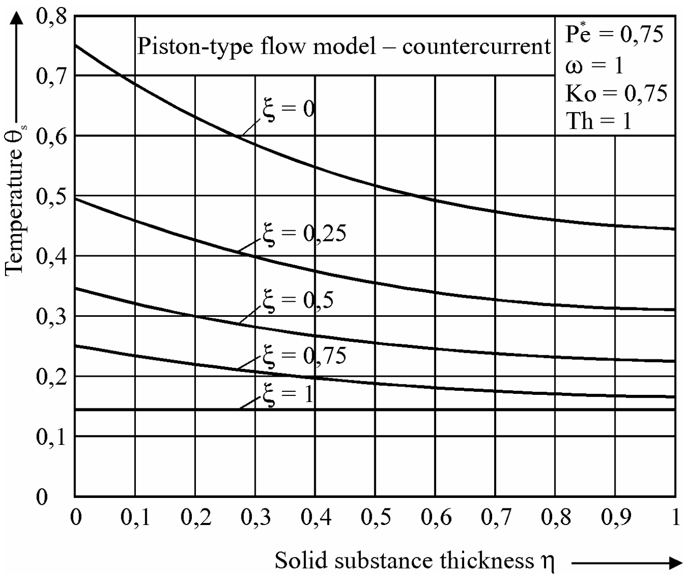

Fig. 5. – Distribution of Temperature in the Flat Plate (Piston-type Flow Model – Countercurrent)

Boundary condition: $\frac{d\theta_g}{d\xi} + \frac{1}{Ko}(\theta_g^4 - \theta_{so}^4) + G_g(\theta_z - \theta_o) = 0$ [1]

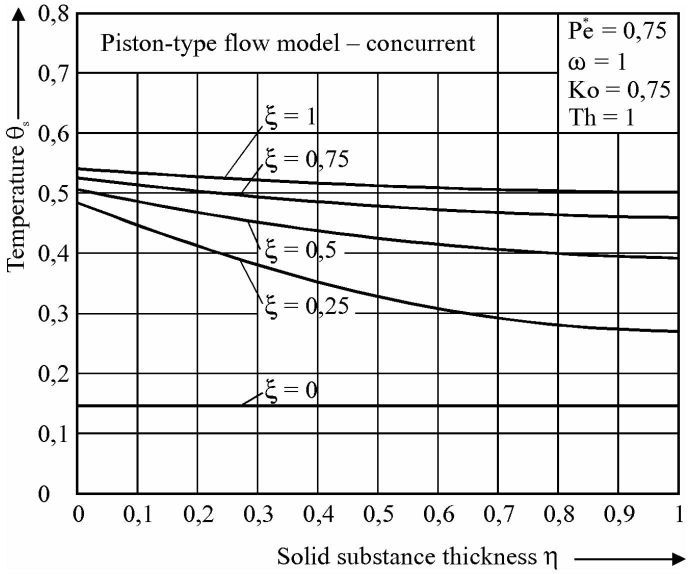

Fig. 6. – Distribution of Temperature in the Flat Plate (Piston-type Flow Model – Concurrent)

Boundary condition: $$\frac{d\theta_g}{d\xi}+\frac{1}{Ko}(\theta_g^4-\theta_{so}^4)+G_g(\theta_z-\theta_o)=0 \qquad [1]$$

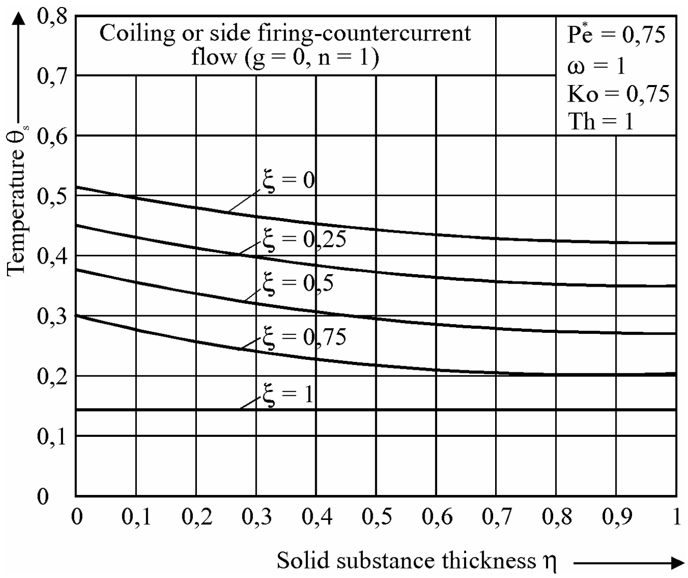

Fig.7. – Distribution of Temperature in the Flat Plate (Piston-type Flow Model with Energy Addition).

Boundary Condition:

$$\frac{d\theta_g}{d\xi} = \frac{1}{g + (1-g)\xi^n}\{(1-\theta_g)n(1-g)\xi^{n-1} - [\frac{1}{\mathrm{Ko}}(\theta_g^4 - \theta_{so}^4) + G_g(\theta_z - \theta_o)]\} \qquad [1]$$

Temperature distribution in a finite thickness plate (direction η) for different oven models is shown in Figs. 4 – 7. With piston-type flow models having the concurrent flow, high densities of the heat flux occur, quite naturally, on the inlet side of the solid plate, whereas with the piston-type flow models having the countercurrent flow they occur on the outlet side of the solid plate. This is the reason why the temperature difference between the surface of the plate and its middle at the concurrent flow is the lowest, and the highest with the countercurrent flow.

The agitator and piston-type flow models with ceiling and side firing are between these two boundary cases.

In addition to this, temperature distribution depends both on the method of controlling the process and on the heat conductivity and thickness of the plate. These values are contained in the *Thring's* number, and, therefore, effects of this number on the distribution of temperature in the flat plate should be investigated.

List of Designations

a – temperature conductivity
A – area
b – oven width
ef – effective
g – gas, mass portion of the fuel
gu – gas at the oven inlet
gub. – loss
h – step size
k – heat transfer coefficient, row index
K – number of rows
L – oven length
n – measure of energy distribution along the oven, column index
N – number of columns
o – environment
PO – auxiliary point on the plate upper surface
Pδ – auxiliary point on the plate lower surface
$\dot{Q}$ – heat flux
s – solid body (plate)
so – plate upper surface
sδ – plate lower surface
T – thermodynamic temperature
v – velocity
z – wall
x – coordinate
y – coordinate

Greek Letters:

δ – plate thickness
ε – emissivity
η – dimensionless coordinate
θ – dimensionless temperature
λ – heat conductivity
σ – *Stefan-Boltzmann's* constant
ξ – dimensionless coordinate
τ – time
Δ – finite difference of two values

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MATEMATIČKI MODEL TEMPERATURNOG POLJA U RAVNOJ PLOČI PRI NESTACIONARNOM ZAGREVANJU U PROTOČNOJ PEĆI

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U radu je razvijen dvodimenzionalni model za proračun temperaturnog polja u ravnoj ploči konačne debljine pri nestacionarnom zagrevanju u protočnoj peći. Za izvedenu diferencijalnu jednačinu nestacionarnog provođenja toplote nije se moglo dati tačno analitičko rešenje, pa je korišćena približna numerička metoda konačnih razlika. Transformisana je diferencijalna jednačina nestacionarnog provođenja toplote u diferencnu, a dvodimenzionalna ploča izdeljena je mrežom pravougaonika. Proračun je izveden za ceo set kolona, a utvrđena je i grafički prikazana raspodela temperatura za karakteristične vrednosti bezdimenzionalnih kompleksa.

Ključne reči: *matematički model, ravna ploča, temperaturno polje, numerička metoda, protočna peć*