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# FINITE ELEMENT DEVELOPMENT FOR GENERALLY SHAPED PIEZOELECTRIC ACTIVE LAMINATES PART I - LINEAR APPROACH

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**Abstract**. *The increasing need for lightweight structures requires development of new kinds of materials with high specific mechanical properties, i.e. ratio between mechanical properties and mass density. Fiber reinforced laminates, which comprise a load-bearing phase in a protective matrix belong to the group of the most advanced structural materials today. They are already used in various applications like space structures, transportation applications, etc. However, the behavior of structures made of composite laminates can be improved by embedding piezoelectric patches in between layers. The generic idea is a synthesis of active structures with controllable dynamic response characteristics. The paper gives the governing equations for active layered laminate and describes 9-node degenerated shell element for modeling of generally shaped piezoelectric active laminates whereby small strains and small displacements are assumed*

**Key words**: *Finite Element Method, Degenerated Shell, Piezoelectric Effect, Active Structure*

## 1. INTRODUCTION

The increasing need for lightweight structures requires development of new kinds of materials. A great number of applications, like space structures, transport applications etc., require high specific mechanical properties, i.e. ratio between mechanical properties and mass density. Fiber reinforced laminates, which have a very high strength and stiffness in desired directions and lower characteristics in other directions offer such a superior combination of characteristics. Also, the material can meet the specific requirements of structures since its properties can be tailored by adjusting the nature of constituents,

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their proportions, orientations of fibers, sequence of layers etc., and this fact makes this material one of the most advanced structural materials today [11]. However, the behavior of the structures made of composite laminates can be further improved. One of the inherent characteristics of natural systems that is easily imposed on the observer is that those systems are capable of recognizing the character and intensity of external stimuli, and to react appropriately in order to give the best performance or protect their integrity. For an artificial system to exhibit a similar behavior it is necessary that it is provided with a smart material – a smart structure is obtained in this manner. In the recent years the study of smart structures has attracted many research projects, due to their potential benefits in a wide range of applications, such as shape control, vibration suppression, noise attenuation, damage detection, and the like [4, 7, 9]. There are several kinds of commercially available smart materials, but the choice of a piezoelectric material in the form of a thin patch is the appropriate one considering required loads and frequency interval of actuating and sensing as well as embedding requirements in the structure made of composite laminates.

The finite element method is the most powerful numerical technique ever developed for solving solid and structural mechanics problems in geometrically complicated regions. The fundamental equations governing the behavior of smart laminates are at first given, and, based on it, the linear finite element formulation of the same problem is described.

## 2. COMPOSITE LAMINATES CONSTITUTIVE EQUATIONS

The most common structural elements made of composite laminates are plates and shells. Hence, two-dimensional theories are appropriate for purposes of their structural analysis. A two-dimensional theory of the third order would allow the satisfaction of all conditions imposed by the theory of elasticity. However, most real world problems, especially those involving composite structures, do not admit exact solutions, requiring one to find an approximate, but representative solution. Also, one should be aware of the objectives of modeling as well as the "numerical effort – achieved accuracy" ratio. If the prior objective is to get a general behavior of the structure, without analyzing the strain and stress fields in localized areas, i.e. in a micro-domain, then a two-dimensional theory of the first order is quite satisfactory in most cases.

Let's consider a generally shaped structure made of composite laminate, Fig. 1. Composite laminates are anisotropic materials and their constitutive equations are to be developed in the local coordinate system (c.s.) considering an infinitesimal cut-out, whereby the local c.s. differs for any two different points of the mid-surface in the case of generally shaped double-curved mid-surface. The model should cover thin as well as moderately thick structures and therefore the first-order shear deformation theory (FSDT) is used.

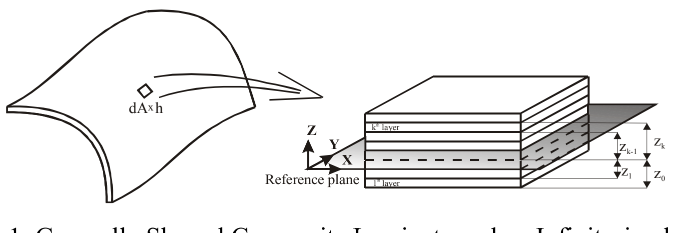



Fig. 1. Generally Shaped Composite Laminate and an Infinitesimal Cut-out

The theory is based on Mindlin-Reissner assumptions [2, 6]: a straight line originally perpendicular to the mid-surface remains straight and inextensible after deformation, but does not remain necessarily perpendicular to the mid-surface. Hence, the displacement field is described in the local c.s. in terms of degrees of freedom of a mid-surface point as:

$$\begin{aligned} u &= u_0 + z\theta_y \\ v &= v_0 - z\theta_x \\ w &= w_0, \end{aligned} \tag{1}$$

where $u_0$ and $v_0$ are the in-plane displacements, $w_0$ is the transverse displacement and $\theta_x$ and $\theta_y$ are the rotations of the transverse normal around the x- and y-axis, respectively, all of them at the mid-surface point and they are functions of the mid-surface point position (most of the papers use positive sign for the rotation in expression that gives displacement v due to differently defined rotations). A linear case is assumed, i.e. small strains and small displacements, so the strain field is defined by kinematical relations, from the displacement field in the following way:

$$\begin{aligned} &\varepsilon_{11} = \frac{\partial u_0}{\partial x} + z\frac{\partial \theta_y}{\partial x} = \varepsilon_{11}{}^0 + z\kappa_{11}{}^f; \quad \varepsilon_{22} = \frac{\partial v_0}{\partial y} - z\frac{\partial \theta_x}{\partial y} = \varepsilon_{22}{}^0 + z\kappa_{22}{}^f; \quad \varepsilon_{13} = \frac{\partial w_0}{\partial x} + \theta_y; \\ &\varepsilon_{12} = (\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}) + z(\frac{\partial \theta_y}{\partial y} - \frac{\partial \theta_x}{\partial x}) = \varepsilon_{12}{}^0 + z\kappa_{12}{}^f; \qquad \varepsilon_{23} = \frac{\partial w_0}{\partial y} - \theta_x \end{aligned} \tag{2}$$

where we can make a distinction between the in-plane strains $\{\varepsilon_m\}^T=\{\varepsilon_{11}{}^0\ \varepsilon_{22}{}^0\ \varepsilon_{12}{}^0\}^T$, the flexural strains $z\{\kappa_f\}^T=z\{\kappa_{11}{}^f\ \kappa_{22}{}^f\ \kappa_{12}{}^f\}^T$, the transverse shear strains $\{\varepsilon_s\}^T=\{\varepsilon_{23}\ \varepsilon_{13}\}^T$ where $\{\kappa_f\}^T$ are the curvatures of deformation. By imposing the Hook's law the stresses in each layer are obtained in terms of reduced material stiffness constants $Q_{ij}'$, which result from the assumption of zero normal strain in thickness direction so that $Q_{ij}'=C_{ij}- C_{i3}\ C_{j3}/C_{33}$, where $C_{ij}$ are the constants of the Hook's matrix. Layerwise integration of stresses over the thickness of the laminate gives the cross-sectional loads, and the laminate constitutive equation is:

$$\begin{Bmatrix} \{N\} \\ \{M\} \\ \{Q\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] & [0] \\ [B] & [D] & [0] \\ [0] & [0] & [F] \end{bmatrix} \begin{Bmatrix} \{\varepsilon_m\} \\ \{\kappa_f\} \\ \{\varepsilon_s\} \end{Bmatrix} \tag{3}$$

where $[N]=[N_x\ N_y\ N_{xy}]$ are the in-plane forces, $[M]=[M_x\ M_y\ M_{xy}]$ are the bending and torsional moments, $[Q]=[Q_{xz}\ Q_{yz}]$ are the transverse shear resultants, [A] is the extensional stiffness matrix, [D] is the bending stiffness matrix, [B] is the bending-extensional coupling stiffness matrix and [F] is the transverse shear stiffness matrix. The coefficients of the stiffness matrices are obtained by appropriate integration of material constants in thickness direction [2]:

$$(A_{ij}, B_{ij}, D_{ij}) = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} Q'^{(k)}_{ij}\,(1, z, z^2)\,dz; \qquad F_{ij} = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} k_{ij} C^{(k)}_{ij}\,dz \tag{4}$$

and $k_{ij}$ are the shear correction factors, here $k_{ij}$=5/6.

## 3. Linear Piezoelectric Constitutive Equations and Piezoelectric Patch Polarized in Thickness Direction

Although the hysteresis is present in the whole range of strain-electric field diagram, the linear piezoelectric constitutive equations can be used as an acceptable approximation for modeling purposes in the case of relatively small values of electric field. The form of the linear piezoelectric equations and governing thermodynamic equation depends on the choice of independent variables [5]. Within the displacement based finite element method, mechanical displacements and electric potentials are independent variables. Hence, it is appropriate to assume mechanical strain {ε} and electric field {E} as independent variables when the form of the piezoelectric constitutive equations is chosen, since they are obtained directly through kinematical relations. The linear piezoelectric equations are then [5, 12]:

$$\begin{aligned} \{\sigma\} &= [C^E]\{\varepsilon\} - \mathbf{[e]}\{E\} \\ \{D\} &= [e]\{\varepsilon\} + [d^\varepsilon]\{E\} \end{aligned} \tag{5}$$

where {σ} is the mechanical stress vector, {D} is the electric displacement vector, $[C^E]$ is the piezoelectric material Hook's matrix at E constant, $[d^\varepsilon]$ is the dielectric permittivity matrix at ε constant, and [e] is the piezoelectric coupling matrix.

Due to the brittleness of the piezoelectric ceramics they are most frequently used in the form of very thin patches for sensing and actuating purposes on thin structures such as those made of composite materials. The patches are added in between layers or are bonded to the surfaces. They are mostly polarized in thickness direction and hence, only the electric field in thickness direction, $E_3$, is considered, so that the constitutive equations in the piezo-layer reference system can be rewritten in the following matrix form:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ D_3 \end{Bmatrix} = \begin{bmatrix} Q_{11}' & Q_{12}' & Q_{16}' & e_{31}' \\ Q_{12}' & Q_{22}' & Q_{26}' & e_{32}' \\ Q_{16}' & Q_{26}' & Q_{66}' & 0 \\ e_{31}' & e_{32}' & 0 & -d_{33}' \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \\ -E_3 \end{Bmatrix} \tag{6}$$

where $\{\sigma_{11}\ \sigma_{22}\ \sigma_{12}\}$ are the in-plane and flexural stresses, $d_{33}'$ is the reduced dielectric permittivity in thickness direction and $e_{3j}'$ are the reduced piezoelectric coupling constants in the reference system of the structure. The transverse shear components are excluded from equation (6) because they are decoupled from other components.

The electric field in thickness direction is defined by $E_3 = -\partial\varphi/\partial z$, φ being the electric potential. Piezoelectric materials belong to the group of dielectrics and the 4th Maxwell's equation yields in that case $\partial D_3/\partial z=0$, since there is no free electric charge in the volume of dielectrics. Using the 4th Maxwell's equation with $D_3$ from relation (6) where the above relation for $E_3$ is introduced, we obtain the following differential equation for φ:

$$\frac{\partial^2\varphi}{\partial z^2} = -\frac{e_{31}'_k}{d_{33}'_k}\left(\frac{\partial\varepsilon_{11}}{\partial z} + \frac{\partial\varepsilon_{22}}{\partial z}\right) = -\frac{e_{31}'_k}{d_{33}'_k}(\kappa_{11_k} + \kappa_{22_k}) \tag{7}$$

and this value is constant with respect to the thickness coordinate, z.

Integrating equation (7) over the thickness coordinate, z, and imposing the boundary conditions, φ=0 on the lower surface electrode and φ=$\Delta\Phi_k$ on the upper surface electrode, we obtain the following distribution of the electric potential and electric field over the thickness of the $k^{th}$ piezoelectric layer:

$$\varphi_k = \frac{1}{2}\frac{e_{31}'_k}{d_{33}'_k}(\kappa_{11_k} + \kappa_{22_k})\left((z - z_{mk})^2 - \left(\frac{h_k}{2}\right)^2\right) + \Delta\Phi_k\left(\frac{z - z_{mk}}{h_k} + \frac{1}{2}\right) \tag{8}$$

$$E_{3k} = -\frac{\Delta\Phi_k}{h_k} - \frac{e'_{31k}}{\varepsilon'_{33k}}(\kappa_{11_k} + \kappa_{22_k})(z - z_{mk}) \tag{9}$$

where $\Delta\Phi_k$ is the difference of electric potential between electrodes of the $k^{th}$ piezo-layer, $z_{mk}$ is the distance of the mid-surface of the $k^{th}$ piezo-layer to the mid-surface of the laminate and $h_k$ is the thickness of the $k^{th}$ piezo-layer.

The piezoelectric laminate constitutive equations are now:

$$\begin{Bmatrix} \{N\} \\ \{M\} \\ \{Q\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] & [0] \\ [B] & [D]+[D^e] & [0] \\ [0] & [0] & [F] \end{bmatrix} \begin{Bmatrix} \{\varepsilon_m\} \\ \{\kappa_f\} \\ \{\varepsilon_s\} \end{Bmatrix} + \sum_{k=1}^{n_e} \begin{bmatrix} [I_3] \\ z_{mk}[I_3] \\ [0] \end{bmatrix} [T_\varepsilon]\{e_k'\}\Delta\Phi_k \tag{10}$$

where $[D^e]$ is the electric-bending coupling stiffness matrix and this is the additional mechanical stiffening effect due to the electric field dependency on the curvature of deformation (see (9)), and $[T_\varepsilon]$ rotates the strain field from the structure reference system to the reference system of the $k^{th}$ piezo-layer. Constants of matrix $[D^e]$ are given as:

$$D_{ij}^e = \begin{cases} 0, \text{ if } i = 3 \text{ or } j = 3 \\ \sum_{k=1}^{n_e} \frac{e_{3i}'_k \, e_{3j}'_k}{d_{33}'_k} \int_{z_k}^{z_{k+1}} z(z - z_{mk})\,dz, \text{ otherwise} \end{cases} \tag{11}$$

A small remark should be added considering relation (10). On the right hand-side of equation (10) the matrix $[T_\varepsilon]$ is present. This matrix can be interpreted in two ways. One of them is already mentioned and it is related to the strain field rotation. Alternatively, it could be understood as a matrix that rotates the piezoelectric properties from the piezo-layer reference system to the reference system of the structure. Most of the piezoceramics belong to the group of transversely isotropic materials considering mechanical as well as electrical properties, and in that case this matrix is not needed.

## 4. DEGENERATED SHELL ELEMENT

The degenerated shell element is obtained from the 3D solid element through the degeneration process, whereby the assumptions fully compatible with the Mindlin-Reissner kinematical assumptions are made [3]. The final result of the degeneration process is that the displacement of any point in the volume of the element can be described in terms of six global degrees of freedom of the appropriate mid-surface point, three translations and three rotations. Furthermore, the degrees of freedom of any mid-surface point can be obtained through the interpolation in terms of degrees of freedom of mid-surface nodes. Hence, it is enough to consider the element mid-surface only, whereby the thickness of the element is defined by user for each node.

Three different coordinate systems (c.s.) are used in the formulation of the element: the global c.s (x, y, z), natural c.s. (r, s, t) and local-running c.s. (x', y', z'), Fig. 2b. The global c.s. is the basic one, all user defined data and results of the analysis are given in this coordinate system. The natural c.s. facilitates the definition of the shape functions and 2D numerical integration. The local-running c.s. is of crucial importance since it is used for the description of the element geometry and because of the anisotropic properties of composite laminates. It will be defined in the sequel.

Taking advantage of the isoparametric approach to develop the 9-noded element, the geometry of the mid-surface in the global coordinate system is given by parametric relations:

$$x_{t=0} = \sum_{i=1}^{9} N_i x_i, \quad y_{t=0} = \sum_{i=1}^{9} N_i y_i, \quad z_{t=0} = \sum_{i=1}^{9} N_i z_i \tag{12}$$

where $N_i=N_i(r,s)$ are the full biquadratic shape functions defined in the natural c.s. and $x_i$, $y_i$ and $z_i$ are the global coordinates of the nine nodes. Having described the mid-surface geometry, we can define the local-running c.s. so as to have two axes lying in the mid-surface tangential plane and the third one is, of course, perpendicular to it:

$$\vec{t}_{v1} = \begin{Bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \end{Bmatrix}^T, \quad \vec{t}_{v2}' = \begin{Bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \end{Bmatrix}^T, \quad \begin{aligned} \vec{e}_{v3} &= \vec{t}_{v1} \times \vec{t}_{v2}', \\ \vec{t}_{v2} &= \vec{e}_{v3} \times \vec{t}_{v1}. \end{aligned} \tag{13}$$

The normalization yields the basis ($t_{n1}$, $t_{n2}$, $e_{n3}$). The only condition used so far for the in-plane axes is that they have to be in the mid-surface tangential plane. No additional condition was imposed. The so-obtained axes can be further used in the case of isotropic material [1]. For an anisotropic material, such as composite laminate, the local-running c.s. shall be fixed with respect to a structure reference system. The structure reference system is a user defined system, with respect to which the sequence of layers of the laminate is determined. Any fixation, i.e. any fixed angle $\alpha_f$ between the structure reference direction and one of the in-plane axes is equally acceptable. For the sake of simplicity we will simply overlap one of the in-plane axis and the structure reference direction, i.e. $\alpha_f = 0$ (Fig. 2 c).

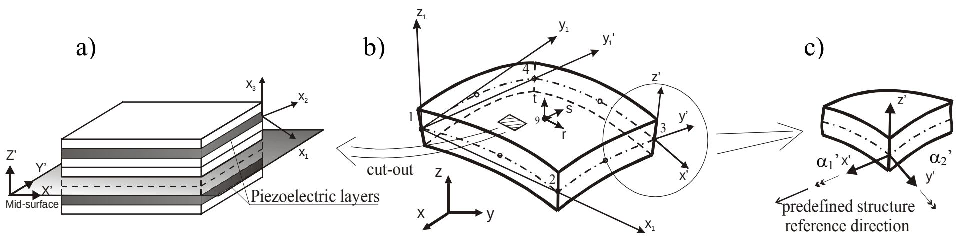



Fig. 2. Degenerated Shell Element, Cross Sectional Cut-out and Local c.s.

The local-running c.s. is important for the description of the element geometry. Furthermore, the laminate constitutive equation is valid in the local-running c.s. Therefore, the strain field will have to be developed in the same c.s. The thickness of the shell is assumed to be in direction normal to the mid-surface of the element, with nodal values $h_i$. Let's denote the local-running c.s. basis as $\{l_a, m_a, n_a\}$ where $a$ determines the unity vector of the basis, i.e. $a$=1, 2, 3, and l, m, and n are the direction cosines of the unity vector with respect to the global c.s. basis vectors. For the description of the element geometry in the natural c.s., we need the geometry of the mid-surface (12), nodal thickness of the element, $h_i$, and the thickness direction vector, which is the third vector of the local-running c.s. basis:

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \sum_{i=1}^{9} N_i \left\{ \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix} + \frac{1}{2} t h_i \begin{Bmatrix} l_{3i} \\ m_{3i} \\ n_{3i} \end{Bmatrix} \right\} \tag{14}$$

According to the assumptions made in the degeneration process (and Mindlin-Reissner assumptions as well), each node has in the local-running c.s. two rotational degrees of freedom, $_i$ and $\alpha'_{2i}$, around the axes lying in the mid-surface tangential plane. Upon transformation of rotational degrees of freedom from local to global c.s. the displacement field is obtained in terms of global nodal degrees of freedom:

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \sum_{i=1}^{9} N_i \left\{ \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} + \frac{1}{2} t h_i \begin{Bmatrix} \alpha_{2i}' \\ -\alpha_{1i}' \\ 0 \end{Bmatrix} \right\} \Rightarrow \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \sum_{i=1}^{9} N_i \left\{ \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} + \frac{1}{2} t h_i \begin{Bmatrix} n_{3i}\theta_{yi} - m_{3i}\theta_{zi} \\ l_{3i}\theta_{zi} - n_{3i}\theta_{xi} \\ m_{3i}\theta x_{yi} - l_{3i}\theta_{yi} \end{Bmatrix} \right\} \tag{15}$$

In order to develop the strain field, the partial derivatives of displacements in the local-running c.s, u', v' and w', with respect to the local-running coordinates are needed. At first, the partial derivatives of displacements in the global c.s. (15) with respect to the natural coordinates are obtained:

$$\begin{bmatrix} \frac{\partial u}{\partial r} & \frac{\partial v}{\partial r} & \frac{\partial w}{\partial r} \\ \frac{\partial u}{\partial s} & \frac{\partial v}{\partial s} & \frac{\partial w}{\partial s} \\ \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t} & \frac{\partial w}{\partial t} \end{bmatrix} = \sum_{i=1}^{9} \begin{bmatrix} \frac{\partial N_i}{\partial r} u_i & \frac{\partial N_i}{\partial r} v_i & \frac{\partial N_i}{\partial r} w_i \\ \frac{\partial N_i}{\partial s} u_i & \frac{\partial N_i}{\partial s} v_i & \frac{\partial N_i}{\partial s} w_i \\ 0 & 0 & 0 \end{bmatrix} +$$

$$+ \sum_{i=1}^{9} \frac{h_i}{2} \begin{bmatrix} t\frac{\partial N_i}{\partial r}(n_{3i}\theta_{yi} - m_{3i}\theta_{zi}) & t\frac{\partial N_i}{\partial r}(l_{3i}\theta_{zi} - n_{3i}\theta_{zi}) & t\frac{\partial N_i}{\partial r}(m_{3i}\theta_{xi} - l_{3i}\theta_{yi}) \\ t\frac{\partial N_i}{\partial s}(n_{3i}\theta_{yi} - m_{3i}\theta_{zi}) & t\frac{\partial N_i}{\partial s}(l_{3i}\theta_{zi} - n_{3i}\theta_{xi}) & t\frac{\partial N_i}{\partial s}(m_{3i}\theta_{xi} - l_{3i}\theta_{yi}) \\ N_i(n_{3i}\theta_{yi} - m_{3i}\theta_{zi}) & N_i(l_{3i}\theta_{zi} - n_{3i}\theta_{xi}) & N_i(m_{3i}\theta_{xi} - l_{3i}\theta_{yi}) \end{bmatrix} \tag{16}$$

They are afterwards mapped from the natural c.s. to the global c.s by means of the Jacobian inverse:

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} J_{11}^* & J_{12}^* & J_{13}^* \\ J_{21}^* & J_{22}^* & J_{23}^* \\ J_{31}^* & J_{32}^* & J_{33}^* \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial r} & \frac{\partial v}{\partial r} & \frac{\partial w}{\partial r} \\ \frac{\partial u}{\partial s} & \frac{\partial v}{\partial s} & \frac{\partial w}{\partial s} \\ \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t} & \frac{\partial w}{\partial t} \end{bmatrix} \tag{17}$$

whereby the Jacobian is obtained in the following manner:

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{bmatrix} = \sum_{i=1}^{9} \begin{bmatrix} \frac{\partial N_i}{\partial r} x_i & \frac{\partial N_i}{\partial r} y_i & \frac{\partial N_i}{\partial r} z_i \\ \frac{\partial N_i}{\partial s} x_i & \frac{\partial N_i}{\partial s} y_i & \frac{\partial N_i}{\partial s} z_i \\ \frac{1}{2} N_i h_i l_{3i} & \frac{1}{2} N_i h_i m_{3i} & \frac{1}{2} N_i h_i n_{3i} \end{bmatrix} \tag{18}$$

Finally, the derivatives within the local-running c.s. are easily obtained by means of the transformation matrix that comprises the components of the local-running c.s. basis:

$$\begin{bmatrix} \frac{\partial u'}{\partial x'} & \frac{\partial v'}{\partial x'} & \frac{\partial w'}{\partial x'} \\ \frac{\partial u'}{\partial y'} & \frac{\partial v'}{\partial y'} & \frac{\partial w'}{\partial y'} \\ \frac{\partial u'}{\partial z'} & \frac{\partial v'}{\partial z'} & \frac{\partial w'}{\partial z'} \end{bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \tag{19}$$

The derivatives (19) can be divided into the part relating translations and part relating rotations only. For the sake of brevity, we shall introduce the following notation: $u'=u_1'$, $v'=u_2'$, $w'=u_3'$, $x'=x_1'$, $y'=x_2'$, $z'=x_3'$, and we can write now:

$$\begin{gathered} \left(\frac{\partial u_\alpha'}{\partial x_\beta'}\right)_T = \sum_{i=1}^{9}\left(l_\alpha B'(\beta,i)u_{1i} + m_\alpha B'(\beta,i)u_{2i} + n_\alpha B'(\beta,i)u_{3i}\right) \\ \left(\frac{\partial u_\alpha'}{\partial x_\beta'}\right)_R = \sum_{i=1}^{9}\left(\delta_{3\beta}\frac{1}{2}h_i N_i B'' + \frac{1}{2}h_i\, t\, B'(\beta,i)\right)\left(C_x(\alpha,i)\theta_{xi} + C_y(\alpha,i)\theta_{yi} + C_z(\alpha,i)\theta_{zi}\right)(1-\delta_{3\alpha}\delta_{3\beta}) \end{gathered} \tag{20}$$

where $\alpha,\beta=1,2,3$, $\delta_{\alpha\beta}$ is the Kronecker delta, and the following notations were introduced:

$$B'(a,i) = \frac{\partial N_i}{\partial x} l_a + \frac{\partial N_i}{\partial y} m_a + \frac{\partial N_i}{\partial z} n_a \qquad B'' = l_3 J_{13}^* + m_3 J_{23}^* + n_3 J_{33}^* \tag{21}$$

$$C_{x'}(a,i) = m_{3i} n_a - n_{3i} m_a; \quad C_{y'}(a,i) = n_{3i} l_a - l_{3i} n_a; \quad C_{z'}(a,i) = l_{3i} m_a - m_{3i} l_a; \tag{22}$$

where $J_{\alpha\beta}^*$ are the elements of the Jacobian inverse.

It is now a straight-forward task to develop the linear strain-displacement relations by means of (20). Also, a part depending on thickness coordinate can be distinguished from the part independent from thickness coordinate, so that we can write:

$$\begin{Bmatrix} \{\varepsilon'_{mf}\} \\ -- \\ \{\varepsilon'_s\} \end{Bmatrix} = \sum_{i=1}^{9} \begin{bmatrix} [B_{Tmi}] & \vdots & t[B_{R1mi}] \\ --- & + & ---------- \\ [B_{Tsi}] & \vdots & [B_{R0mi}] + t[B_{R1mi}] \end{bmatrix} \begin{Bmatrix} \{u_i^T\} \\ -- \\ \{u_i^R\} \end{Bmatrix} = [B_u]\{u_i\} \tag{23}$$

where $\{\varepsilon_{mf}'\}$ comprises membrane and flexural strains, and $\{\varepsilon_s'\}$ transverse shear strains, both in the local-running c.s, $\{u_i^T\}$ are nodal translations, $\{u_i^T\}^T = \{u_i\ v_i\ w_i\}^T$, and $\{u_i^R\}$ nodal rotations, $\{u_i^R\}^T = \{\theta_{xi}\ \theta_{yi}\ \theta_{zi}\}^T$, both in the global c.s. The submatrices on the right hand side of equation (23) result from relations (20) and according to the dependence on the natural thickness coordinate t. They are all parts of the linear strain-displacement matrix, $[B_u]$. This representation facilitates the analytical layerwise integration in thickness direction.

A special attention is needed when the drilling degree of freedom is considered. The element has five degrees of freedom in the local-running c.s. (only two rotations), but due to their transformation there are in general six degrees of freedom in the global c.s. (all three rotations). Consequently, for a special position of the element, when the z'-axis of the local-running c.s. coincides with one of the global coordinate axes, the stiffness constant corresponding to the rotational degree of freedom around that axis is equal to zero. The element will not be constraint enough, and any slight disturbance in the load corresponding to this degree of freedom could cause an erratic behavior of the element. The solution of the problem can be achieved through the introduction of additional torsional stiffness by defining the governing torsional strain energy that behaves as a penalty function forcing the local rotation $\alpha_3'$ be approximately equal to $0.5\cdot(\partial v'/\partial x' - \partial u'/\partial y')$ at integration points:

$$E_t = \frac{1}{2}\alpha_t Gh \iint_A \left[ \alpha_3' - \frac{1}{2}\left( \frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'} \right) \right]^2_{(r,s,o)} dA \tag{24}$$

where $\alpha_t Gh$ has to be relatively large with respect to the factor $Eh^3$ appearing in the bending energy calculations, G being the shear module, E the Young's elasticity module and $\alpha_t$ the so called torsional coefficient. It is enough to have $\alpha_t \geq 0.1$ [3].

As noticed before, the electric field depends on the mechanical strain field, besides the difference of electric potentials between electrodes $\Delta\Phi$. The difference of electric potentials is assumed to be constant for each layer and it is introduced as electrical degree of freedom, $\phi_i$. Since the discretized strain field is already defined, the electric field can now be written as:

$$\{E\} = \frac{\partial}{\partial z}[B_{\phi u}]\{u_i\} - [B_\phi]\{\phi_i\} \tag{25}$$

where $[B_{\phi u}]$ defines the part of the mechanical strain that contributes in the definition of electric field (equations (7) and (9)) and $[B_\phi]$ is a diagonal matrix with elements equal to $1/h_k$ on the diagonal (equation (9)). Practically, $[B_{\phi u}]$ is a part of the strain-displacement matrix giving the membrane and flexural strains $\{\varepsilon_{mf}'\}$, and the partial derivation with respect to thickness coordinate leaves only the matrix $[B_{R1m}]$ (see (23)) contributing in the definition of the discretized electric field. It was seen in the previous section and will also be seen in the next section that this dependency of the electric field on rotational degrees of freedom results in an additional mechanical stiffening effect.

The represented element is of Mindlin type and requires only $C^0$-continuity from the shape functions, which accounts for its advantage. However, it is susceptible to numerical problems well-known as locking and this is certainly a disadvantage. The element should asymptotically recover the Kirchhoff-Love assumptions of zero transverse shear strains as the thickness of the element tends to zero. The absence of this condition leads to shear locking. Similarly, the element should be capable of representing the case of inextensional bending for rather small values of the element thickness. If this condition is not fulfilled, we speak about membrane locking. The full biquadratic element, i.e. the 9-noded element, does not suffer severe locking problems, but it suffers a sub-optimal convergence with the increasing slenderness of the element. One of the simplest and still satisfactory techniques to avoid this problem is a selective and uniformly reduced integration. The selective integration rule is optimal for a rectangular planform of the element. On the other hand, it is shown that the uniform reduced integration technique is optimal in the case of distorted meshes [8]. However, this technique leads to zero-energy ("hour-glass") modes and additional techniques, like the so-called b-treatment, are to be used for their stabilization.

## 5. Finite element equations

The governing equation of the structure's dynamical behavior is given by the Hamilton's principle [7], i.e. the system takes the path of the least action:

$$\delta\int_{t_1}^{t_2}(L+W)dt = 0 \tag{26}$$

where L represents the Lagrangian, and W is the virtual work of external forces. The Lagrangian is to be properly adapted in order to include the contribution from electrical field besides the contribution from mechanical field. Since the mechanical strain and electric field are independent variables, the governing thermodynamic equation for piezoelectric material is electric enthalpy (electric Gibbs energy), H [12]:

$$H = \frac{1}{2}c^E_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - e_{mij}E_m\varepsilon_{ij} - \frac{1}{2}d^\varepsilon_{mn}E_mE_n = \frac{1}{2}[\{\varepsilon\}^T\{\sigma\} - \{E\}^T\{D\}] \tag{27}$$

and the Lagrangian is defined in terms of mechanical kinetic energy $E_k$ and electric enthalpy as $L=E_k-H$. Upon integration by parts of kinetic energy, the Hamilton's principle for the piezoelectric continuum can be rewritten in a developed form:

$$\int_V [\rho\{\delta u\}^T\{\ddot{u}\} + \{\delta\varepsilon\}^T[C^E]\{\varepsilon\} - \{\delta\varepsilon\}^T[e]^T\{E\} - \{\delta E\}^T[e]\{\varepsilon\} - \{\delta E\}^T[d^\varepsilon]\{E\}]dV =$$

$$= \int_V \{\delta u\}^T\{\delta F_V\}dV + \int_{S_1}\{\delta u\}^T\{\delta F_{S_1}\}dS_1 + \{\delta u\}^T\{\delta F_P\} - \int_{S_2}\delta\phi\, q\, dS_2 - \delta\phi Q \tag{28}$$

where $F_v$, $F_{S1}$ and $F_P$ are the external volume, surface (acting on surface $S_1$) and point loads, respectively, q and Q are the surface (acting on surface $S_2$) and point electric charges, respectively. Due to the assumption of constant $\Delta\Phi$ for each piezoelectric layer, only the corresponding electrical loads are considered and those are the uniformly distributed surface electric charges.

After the discretization of the continuum the following system of semi-discrete equations is obtained:

$$[M_u]\{\ddot{u}_i\}+[K_{uu}]\{u_i\}+[K_{u\phi}]\{\phi_i\}=\{f_{ext_i}\} \tag{29}$$

$$[K_{\phi u}]\{u_i\}+[K_{\phi\phi}]\{\phi_i\}=\{q_{ext\,i}\} \tag{30}$$

where the following matrices and vectors are defined on the element level:

- mass matrix: $$[M_u]=\int_V [N_u]^T \rho [N_u]\, dV \tag{31}$$

- mechanical stiffness matrix:

$$[K_{uu}]=\int_V\left([B_u]^T[C^E][B_u]+[B_u]^T[e]^T\frac{\partial}{\partial z}[B_{\phi u}]+\frac{\partial}{\partial z}[B_{\phi u}]^T[e][B_u]-\frac{\partial}{\partial z}[B_{\phi u}]^T[d^\varepsilon]\frac{\partial}{\partial z}[B_{\phi u}]\right)dV \tag{32}$$

- piezoelectric coupling matrix: $$[K_{u\phi}]=\int_V\left([B_u]^T[e][B_\phi]-\frac{\partial}{\partial x}[B_{\phi u}]^T[d^\varepsilon][B_\phi]\right)dV=[K_{\phi u}]^T \tag{33}$$

- dielectric stiffness matrix: $$[K_{\phi\phi}]=\int_V [B_\phi]^T[d^\varepsilon][B_\phi]\, dV \tag{34}$$

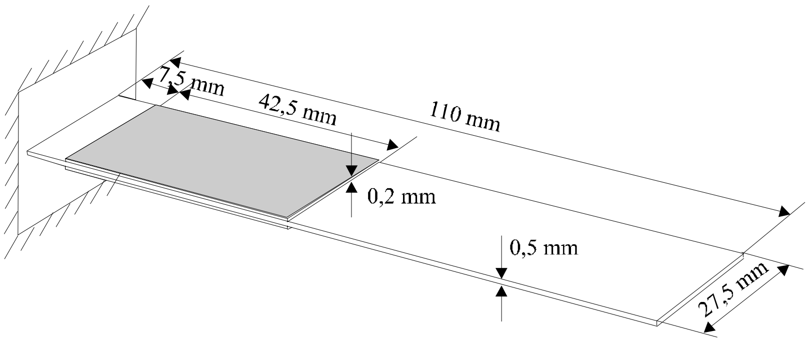
- mechanical loads: $$\{f_{ext_i}\}=\int_V [N_u]^T\{F_V\}\, dV+\int_{S_1}[N_u]^T\{F_{S_i}\}\, dS_1+[N_u]^T\{F_P\} \tag{35}$$

- electric charge: $$\{q_{ext\,i}\}=-\int_{S_2} q\, dS_2 \tag{36}$$

where [ $N_u$] are the nodal displacement interpolation matrices. The remaining terms after the first one on the right-hand side of (32) represent the already mentioned additional mechanical stiffening effect. It should be pointed out that this stiffening effect is quite negligible for the present class of piezoelectric materials. Nevertheless, expecting higher values of piezoelectric coupling constants from the future class of piezoelectric materials, it was decided to keep it in the formulation, because this effect will play a more important role.

## 6. EXAMPLE

The following example offers a simple demonstration of the behavior of the 9-node degenerated shell element for piezoelectric active laminates introduced in this paper. The example gives a comparison of the displacement solution for a clamped beam structure with piezoelectric actuators at the upper and lower side of the structure obtained from different FE models. Figure 3 shows the geometry, position of the actuators and mechanical and piezoelectric properties (IEEE standard [12] is used for notation of constants). For the sake of simplicity, aluminium is chosen as a structural material. The piezoelectric actuators have opposite polarization, which results in bending moments when the same voltage is applied to both of them.




| | | | |
|---|---|---|---|
| Material properties: | | Piezoelectric layer : | |
| beam: Young's modulus | $E = 7.03\ 10^4$ N/mm$^2$ | Elastic compliance | $S_{11}=15.0\ 10^{-6}$ mm$^2$/N |
| | | | $S_{33}=19.0\ 10^{-6}$ mm$^2$/N |
| Poisson's ratio | $\nu=9.345$ | Density | $\rho=7.80$ g/cm$^3$ |
| | | Piezoelectric constants | $d_{31}= -21.0\ 10^{-8}$ mm/V |
| | | | $d_{33}=45.0\ 10^{-8}$ mm/V |
| | | | $d_{15}=58.0\ 10^{-8}$ mm/V |
| | | Dielectric constants | $\varepsilon_{33}/\varepsilon_0=2100$ |
| | | | $\varepsilon_{11}/\varepsilon_0=1980$ |

Fig. 3. Model of a Clamped Beam with Piezoelectric Actuators and Material Properties

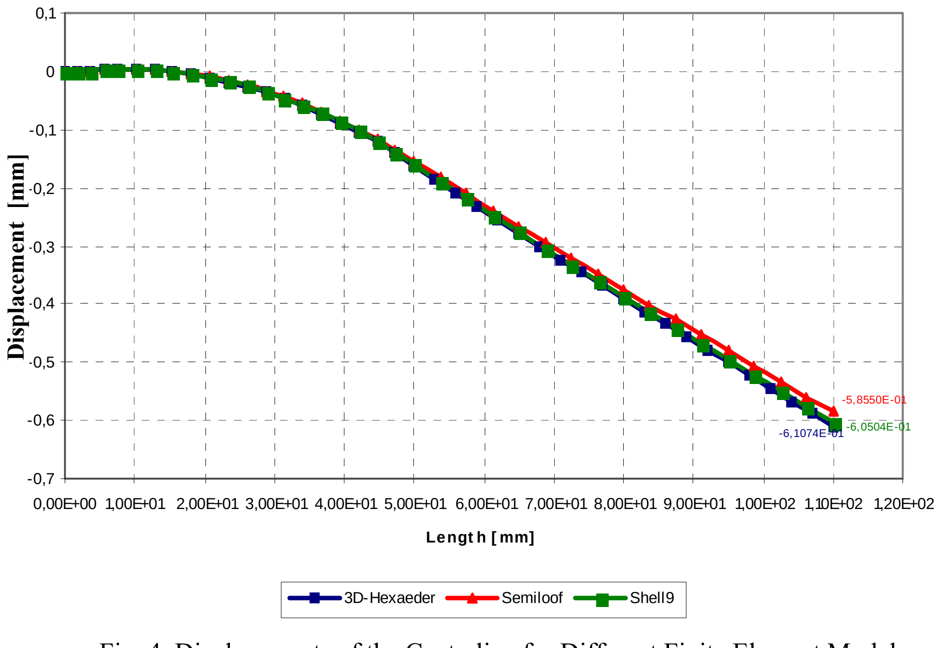



Fig. 4. Displacements of the Centerline for Different Finite Element Models

For this example a control voltage of 100 V is used. The piezoceramic layers of 0.2 mm thickness are made of PZT material. The static deflections induced by externally applied voltages are calculated for the 3D hexahedron finite elements with 20 nodes per element and for two models with shell elements. The 3D elements use a fine mesh and it gives a referent solution for the comparison. The Semi-Loof element and degenerated shell element both use the same, relatively rough mesh (2 x 9 elements). The diagram in Fig. 4 compares the FE solutions of the different models for the static deflection along the beam length (centerline). The curve with triangle symbols represents the solution for the model with the 8-node Semi-Loof [13] shell elements. The line with square symbols shows the displacement results for the introduced 9-node degenerated shell element and the reference solution is given by the blanc curve. The 3D element and Semi-Loof shell element both use a direct solver. The degenerated shell element uses an iterative solver for dynamic analysis, with artificial mass and critical damping so that the performed motion of the structure is an aperiodic oscillation converging very slowly to the static solution. The results obtained with the shell elements show a good agreement with the reference solution.

## 7. CONCLUSIONS

The generic idea represented in the paper is a synthesis of active structures with controllable dynamic response characteristics. Composite laminate as one of the most advanced structural materials today is chosen as a load-bearing material. Furthermore, this structural material is given the capability of sensing and actuating by introducing layers made of piezoelectric material in between original layers, or bonding them to the surfaces of the composite laminate. The aim hereby is to provide a model for this type of structure.

The finite element method represents currently the most powerful method for structural analysis. Therefore, the objective of the paper is to describe the development of a finite element designed for modeling of generally shaped composite laminates with embedded piezoelectric active components. The 9-node degenerated shell element is chosen since it can model arbitrarily shaped thin as well as moderately thick structures – general double curved mid-surface and variable thickness of the structure. Although it is designed for modeling of multi-layered structures the element utilizes the individual layer approach with analytical layerwise integration in thickness direction, which provides a higher numerical effectiveness. For the very same reason, the element is aimed for obtaining the general overall behavior of the structure and not for conducting the analysis in the localized areas of the structure. A fact that both composite laminates and piezoelectric materials belong to the group of anisotropic materials required certain specific steps in development of the element and they are described in the paper. The element has six mechanical degrees of freedom at each node and additionally $n_e$ electrical degrees of freedom, each one corresponding to a piezoelectric layer. The drilling degree of freedom requires a special attention. Considering electric field, a more accurate quadratic distribution of electric potential over the thickness of the piezoelectric patch is taken into account. The uniformly reduced integration offers simple and still satisfactory solution for the locking problems present at this type of element. The element is integrated in the general purpose finite element package COSAR (www.femcos.de), originally developed at the Institute of Mechanics of the Otto-von-Guericke University in Magdeburg.

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# RAZVOJ KONAČNOG ELEMENTA ZA KONSTRUKCIJE OPŠTEG OBLIKA OD KOMPOZITNIH LAMINATA SA PIEZOELEKTRIČNIM KOMPONENTAMA DEO I – LINEARNI PRISTUP

**Dragan Marinković, Heinz Köppe, Ulrich Gabbert**

*Sve veća potreba za lakim konstrukcijama zahteva razvoj novih vrsta materijala sa visokim specifičnim svojstvima, tj. visokim odnosom mehaničkih karakteristika i specifične gustine. Laminarni kompozitni materijali, koji sadrže ojačanja u vidu vlakana smeštenih u zaštitnoj matrici, spadaju u grupu najsavremenijih konstrukcionih materijala. Ovi materijali se već uveliko koriste u različite svrhe, npr. u konstrukcijama za svemirska istraživanja, transportoj tehnici itd. Viši kvalitet u ponašanju konstrukcija od ovih materijala može se postići integrisanjem piezolektričnih pločica između slojeva materijala. Osnovna ideja je sinteza aktivnih konstrukcija sa upravljivim dinamičkim odgovorom. U radu su date osnovne jednačine kompozitnog laminata sa aktivnim komponentama i opisan je konačni element tipa "degenerisane ljuske" sa 9 čvorova za modeliranje konstrukcija opšteg oblika od aktivnih laminata, pri čemu su predpostavljene male deformacije i mala pomeranja.*

Ključne reči: *metoda konačnih elemenata, degenerisana ljuska, piezoelektrični efekat, aktivna konstrukcija*