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DISCRIPTOR TIME DELAYED SYSTEM STABILITY THEORY IN THE SENSE OF LYAPUNOV: NEW RESULTS

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D. Lj. Debeljković[1], S. B. Stojanović[2], M. B. Jovanović[3], S. A. Milinković[3]

[1]University of Belgrade, Faculty of Mechanical Eng, Dept. of Control Eng., 11000 Belgrade, Serbia and Montenegro
[2]University of Niš, Faculty of Technology, Bulevar Oslobodjenja 124, 16000 Leskovac, Serbia and Montenegro
[3]University of Belgrade, Faculty of Metallurgy and Technology, 11000 Belgrade, Serbia and Montenegro
E-mail: ddebeljkovic@mas.bg.ac.yu

Abstract. *This paper extends one of the basic results in the area of Lyapunov (asymptotic) to linear, continuous singular time invariant time-delay systems. This result is given in the form of only sufficient conditions and is related to the following class of systems* $E\dot{x}(t) = A_0 x(t) + A_1(t-\tau)$.
Our aim is to derive new and efficient results concerning asymptotic stability of a particular class of linear continuous singular time delay systems.
To the best knowledge of the authors, such result have not yet been reported.

1. INTRODUCTION

It should be noticed that in some systems we must consider their character of dynamic and static state at the same time. Singular systems are those the dynamics of which are governed by a mixture of algebraic and differential equations. Recently many scholars have paid much attention to singular systems and have obtained many good consequences. The complex nature of singular systems causes many difficulties in the analytical and numerical treatment of such systems, particularly when there is a need for their control.

The problem of investigation of time delay systems has been exploited over many years. Time delay is very often encountered in various technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc. The existence of pure time lag, regardless if it is present in the control or/and the state, may cause undesirable system transient response, or even instability. Consequently, the

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problem of stability analysis for this class of systems has been one of the main interests for many researchers. In general, the introduction of time delay factors makes the analysis much more complicated.

We must emphasize that there are a lot of systems that have the phenomena of *time delay* and *singular* simultaneously, we call such systems as *the singular differential systems with time delay*. These systems have many special characters. If we want to describe them more exactly, to design them more accurately and to control them more effectively, we must paid tremendous endeavor to investigate them, but that is obviously very difficult work.

In recent references authors had discussed such systems and got some consequences, *Xu*, *Yang* (2000.a, 2000.b), *Xu et al.* (2001, 2002, 2003).

But in the study of such systems, there are still many problems to be considered.

Generally, *the singular differential control systems with time delay* can be written as:

$$\begin{aligned} E(t)\dot{\mathbf{x}}(t) &= \mathbf{f}(t,\mathbf{x}(t),\mathbf{x}(t-\tau),\mathbf{u}(t)), \quad t \geq 0 \\ \mathbf{x}(t) &= \varphi(t), \quad -\tau \leq t \leq 0\,, \end{aligned} \tag{1}$$

where $\mathbf{x}(t) \in \mathrm{R}^n$ is a state vector, $\mathbf{u}(t) \in \mathrm{R}^l$ is a control vector, $E(t) \in \mathrm{R}^{n\times n}$ is a singular matrix, $\varphi \in C = C\,([-\tau,0], \mathrm{R}^n)$ is an admissible initial state functional, $C = C\,([-\tau,0], \mathrm{R}^n)$ is the Banach space of continuous functions mapping the interval $[-\tau,0]$ into R^n with topology of uniform convergence.

When the general time delay systems are considered, in the existing stability criteria, mainly two ways of approach have been adopted.

Namely, one direction is to contrive the stability condition which does not include the information on the delay, and the other is the method which takes it into account.

The former case is often called the delay - independent criteria and generally provides simple algebraic conditions.

In that sense the question of their stability deserves great attention.

Some attempts have been made in the area of Lyapunov stability of this class of systems, *Xu et al*. (2002) but their approach follows the idea of canonical decomposition of basic system description and offers a complex equations to be solved.

2. MAIN RESULTS

Consider a linear autonomous singular time invariant time delay system:

$$E\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1(t-\tau)\,, \tag{2.a}$$

with associated compatible initial vector valued continuous function:

$$\mathrm{x}(t) = \varphi(t), \quad -\tau \leq t \leq 0\,. \tag{2.b}$$

Definition 1. Matrix par (E,A_0) is said to be regular if $\det(sE - A)$ is not identically zero.
Definition 2. Matrix par (E,A_0) is said to be impulse free if:

$$\deg(\det(sE - A)) = rang\ E. \tag{3}$$

The singular time delay system, *given* (2), may have no impulsive solutions, however, the regularity and absence of impulses of the matrix pair (E, A_0) quarantee the existence and uniqueness of an impulse free solution to the system under consideration.

Lemma 1. Suppose the matrix pair (E, A_0) is regular and impulse free, the solution to the (2) exist and is impulse free and unique on $[0,\infty)$, *Xu et al* (2002).

Definition 3. *The singular time delay system*, *given* (2) is regular and impulse free if the matrix pair (E, A_0) is regular and impuls free.

Defintion 4. *The singular time delay system*, *given* (2), is said to be stable if for any $\varepsilon > 0$ there exist scalar $\delta(\varepsilon) > 0$, such that for any compatible initial conditions $\varphi(t)$, satisfying: $\sup\limits_{-\tau \le t \le 0} \|\varphi(t)\| \le \delta(\varepsilon)$, the solution $\mathbf{x}(t)$, system given (2), satisfies:

$$\|\mathbf{X}(t)\| \le \varepsilon,\ \forall t \ge 0.$$

Furthermore if $\lim\limits_{t\to\infty} \|\mathrm{x(t)}\| \to 0$ system (2)is asymptotically stable, *Xu et al* (2002).

Theorem 1. Suppose the matrix pair (E, A_0) is regular and impulse free and system matrix A_0 is *regular* as well, e.g. $\det A_0 \ne 0$.

The system, given (2) is asymptotically stable, independent of delay, if :

$$\|A_1\| < \frac{\sigma_{\min}(Q^{1/2})}{\sigma_{\max}(Q^{-1/2}E^T P)}, \tag{4}$$

and if there exist symmetric positive definite matrix P on W, such that:

$$A_0^T PE + E^T PA_0 = -2(S+Q), \tag{5}$$

where matrix S is symmetric and positive in the sense that:

$$\mathbf{x}^T(t)(S+Q)\mathbf{x}(t) > 0,\ \forall\ \mathbf{x}(t) \in W_{k^*} \setminus \{0\}, \tag{6}$$

W_{k^*} being subspace of consistent initial conditions and matrix $Q > 0$, *Owens*, *Debeljković* (1985).

Here $\sigma_{\max}(\cdot)$ and $\sigma_{\min}(\cdot)$ are the maximum and minimum singular values of matrix $(\cdot)$, respectively.

Proof. Let us consider Lyapunov 's functional:

$$V(\mathbf{x}(t)) = \mathbf{x}^T(t)E^T PE\mathbf{x}(t) + \int_{t-\tau}^{t} \mathbf{x}^T(\kappa)Q\mathbf{x}(\kappa)d\kappa. \tag{7}$$

It should be noted, that from *Theorem* 2.1, *Debeljković* (1985) quadratic form:

$$V(\mathrm{x}(t)) = \mathrm{x}^T(t)E^T PE\mathrm{x}(t), \tag{8}$$

is positive – definite on W_{k^*}. It is obvious that all smooth solutions $\mathbf{x}(t)$ evolve in W_{k^*} so $V(\mathbf{x}(t))$ can be used as a "Lyapunov function", *Owens*, *Debeljković* (1985).

Clearly, using the equation of system motion (2), the first time derivative along the trajectories in the state space is given by:

$$\begin{aligned}\dot{V}(\mathbf{x}(t)) &= \mathbf{x}^T(t)(A_0^T PE + E^T PA_0 + Q)\mathbf{x}(t) + \\ &+ 2\mathbf{x}^T(t)(E^T PA_1)\mathbf{x}(t-\tau) - \mathbf{x}^T(t-\tau)Q\mathbf{x}(t-\tau),\end{aligned} \tag{9}$$

and after a little rearrangements:

$$\begin{aligned}\dot{V}(\mathbf{x}(t)) &= \mathbf{x}^T(t)(A_0^T PE + E^T PA_0 + 2Q + 2S)\mathbf{x}(t) + 2\mathbf{x}^T(t)(E^T PA_1) - \\ &- \mathbf{x}^T(t)Q\mathrm{x}(t) - \mathbf{x}^T(t)S\mathbf{x}(t) - \mathbf{x}^T(t-\tau)Q\mathbf{x}(t-\tau).\end{aligned} \tag{10}$$

From (5) and fact that matrix S can be chosen such that:

$$\mathbf{x}^T(t)S\mathbf{x}(t) \geq 0,\ \forall\ \mathbf{x}(t) \in W_{k*} \setminus \{0\}, \tag{11}$$

equation (10) is reduced to:

$$\dot{V}(\mathbf{x}(t)) \leq 2\mathbf{x}^T(t)(E^T PA_1)\mathbf{x}(t-\tau) - \mathbf{x}^T(t)Q\mathbf{x}(t) - \mathbf{x}^T(t-\tau)Q\mathbf{x}(t-\tau), \tag{12}$$

Based on very well known inequality[1]:

$$\begin{aligned}&2\mathbf{x}^T(t)E^T PA_1\mathbf{x}(t-\tau) = 2\mathbf{x}^T(t)(E^T PA_1 Q^{-1/2}Q^{1/2})\mathbf{x}(t-\tau) \\ &\leq \mathbf{x}^T(t)E^T PA_1 Q^{-1}A_1^T PE^T\mathbf{x}(t) + \mathbf{x}^T(t-\tau)Q\mathbf{x}(t-\tau),\end{aligned} \tag{13}$$

one can easily get:

$$\dot{V}(\mathbf{x}(t)) \leq -\mathbf{x}^T(t)\Omega\mathbf{x}(t) + \mathbf{x}^T(t)E^T PA_1\Omega^{-1}A_1^T PE\mathbf{x}(t), \tag{14}$$

or:

$$\dot{V}(\mathbf{x}(t)) \leq -\mathbf{x}^T(t)\Omega^{1/2}(I - Q^{-1/2}E^T PA_1 Q^{-1/2}Q^{-1/2}A_1^T PEQ^{-1/2})\Omega^{1/2}\mathbf{x}(t). \tag{15}$$

$\dot{V}(\mathbf{x}(t))$ is negative – definite if:

$$1 - \lambda_{\max}(\Omega^{-1/2}E^T PA_1\Omega^{-1/2}\Omega^{-1/2^{-1}}A_1^T PE\Omega^{-1/2}) > 0, \tag{16}$$

which is satisfied if:

$$1 - \sigma_{\max}^2(\Omega^{-1/2}E^T PA_1\Omega^{-1/2}) > 0. \tag{17}$$

Using the properties of singular values of matrices, *Amir - Moez* (1956), the condition (17) holds if:

$$1 - \sigma_{\max}^2(\Omega^{-1/2}E^T P)\sigma_{\max}^2(A_1\Omega^{-1/2}) > 0, \tag{18}$$

which is satisfied if:

$$1 - \frac{\| A_1 \|^2\ \sigma_{\max}^2(\Omega^{-1/2}E^T P)}{\sigma_{\min}^2(\Omega^{1/2})} > 0. \tag{19}$$

This ends proof.

Remark 1. If one treat linear ordinary (non-singular) system, then $E = I$, and results derived are reduced to that given in *Tissir*, *Hmamed* (1996).

[1] $2\mathbf{u}^T(t)\mathbf{v}(t) \leq \mathbf{u}^T(t)P\mathbf{u}(t) + \mathbf{v}^T(t)P^{-1}\mathbf{v}(t),\ \ P > 0$

3. Conclusion

A new result in the area of Lyapunov (asymptotic) stability to linear, *continuous singular time invariant time-delay systems* is derived.

This sufficient condition is derived under the minimal number of assumptions and is given in the form of so called the delay - independent criteria.

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TEORIJA STABILNOSTI DESKRIPTIVNIH SISTEMA SA KAŠNJENJEM U SMISLU LJAPUNOVA: NOVI REZULTATI

D. Lj. Debeljković S. B. Stojanović,
M. B. Jovanović, S. A. Milinković

Ovaj rad proširejuje bazične rezultate iz asimptotske teorije stabilnosti u smislu Ljapunova na posebnu klasu vremenski neprekidnih deskriptivnih sistema sa čistim vremenskim kašnjenjem. Izvedeni rezultati dati su u formi samo dovoljnih uslova i odnose se na klasu sistema opisanih u prostoru stanja svojim modelom tipa: $E\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau)$ *. Cilj rada je bio da se izvedu novi i efikasni rezultati koji se tiču asimptotske stabilnosti ove klase sistema. Po saznanjima autora, ovakvi rezultati, nisu do sada objavljeni.*