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ON SOME PRACTICAL ASPECTS OF LINEAR SINGULAR CONTROL THEORY APPLICATION

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D. Lj. Debeljković[1], S. Antonić[1], N. Yi-Yong[2], Q. L. Zhang[3]

[1]University of Belgrade, Faculty of Mechanical Eng,
Department of Control Engineering, 11000 Belgrade, Serbia and Montenegro
[2]Shenyang Institute of Automation, Academia Sinica, Shenyang, 110016 P. R. China
[3]College of Science, Northeastern University, Shenyang, Liaoning, 110004, R. P. China

Abstract. *Singular systems are those the dynamics of which are governed by a mixture of algebraic and differential equations. The complex nature of singular systems causes many difficultes in the analytical and numerical treatment of such systems, particularly when there is a need for their control. In that sense the question of their uniqueness and existence of solution, solvability, question of consistent initial conditions and stability deserves great attention. A brief survey of the results concerning the stability of a particular autonomous class of these systems, in the sense of Lyapunov, are presented as the basis for their high quality dynamical investigation.*

Key words: *Singular systems,Regular singular systems, Irregular singular systems, Solvability, Consistent initial condidtions, Drazin inverse, Lyapunov stability*

1. INTRODUCTION

Singular systems are those the dynamics of which are governed by a mixture of algebraic and differential equations.In that sense the algebrac equations represent the constraints to the solution of the differential part.

These systems are also known as descriptor, semi-state and generalized systems arise naturally as a linear approximation of systems models, or linear system models in many applications such as electrical networks, aircraft dynamics, neutral delay systems, chemical, thermal and diffusion processes, large-scale systems, interconnected systems, economics, optimization problems, feedback systems, robotics, biology, etc.

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2. POSSIBILITIES OF DYNAMICAL ANALYSIS OF LINEAR SINGULAR SYSTEMS

Consider linear singular systems represented, by:

$$\begin{aligned} E\dot{\mathbf{x}}(t) &= A\mathbf{x}(t)\ , \quad \mathbf{x}(t_o) = \mathbf{x}_0 \\ \mathbf{y}(t) &= C\mathbf{x}(t), \end{aligned} \tag{1}$$

$$\begin{aligned} E\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t)\ , \quad \mathbf{x}(t_o) = \mathbf{x}_0 \\ \mathbf{y}(t) &= C\mathbf{x}(t), \end{aligned} \tag{2}$$

with the matrix E possibly singular*,* where $\mathbf{x}(t) \in \mathbf{R}^n$ is a generalized state-space vector and $\mathbf{u}(t) \in \mathbf{R}^m$ is a control variable.

Matrices A, B and C are of the appropriate dimensions and are defined over the field of real numbers.

System given by eq. (1) is operatinig in a free and system given by eq. (2) is operating in a forced regime, i.e. some external force is applied on it. It should be stressed that, in the general case, the initial conditions for an autonomus and a system operating in the forced regime need not be the same.

In order to investigate the stability of irregular singular systems, the folowing suitable canonical form, i.e.:

$$\dot{\mathbf{x}}_1(t) = A_1\mathbf{x}_1(t) + A_2\mathbf{x}_2(t)\ , \tag{3a}$$

$$\mathbf{0} = A_3\mathbf{x}_1(t) + A_4\mathbf{x}_2(t)\ . \tag{3b}$$

can be, also, used.

System models of this form have some important advantages in comparison with models in the *normal form, e.g. when $E = I$* and an appropriate discussion can be found in *Bajic* (1992) and *Debeljkovic et al.* (1996, 1996a, 1998).

The complex nature of singular systems causes many difficultes *in analytical and numerical treatment* that do not appear when systems in the normal form are considered. In this sense questions of existence, solvability, uniqueness, and smothness are present which must be solved in satisfactory manner. A short and concise, acceptable and understandable explanation of all these questions may be found in the paper of *Lazarevic et al.* (2001).

The survey of updated results for singular systems and a broad bibliography can be found in *Bajic*(1992), *Campbell* (1980, 1982), *Lewis* (1986, 1987), *Debeljkovic et al.* (1996.a, 1996.b, 1998) and in the two special issues of the journal *Circuits, Systems and Signal Procesing* (1986, 1989).

The complex nature of singular systems causes many difficulties in numerical analytical treatment of such systems that do not appear when systems in the normal form are concerned.

The existence (solvability), uniqueness and smoothness of solutions of singular systems, as well as their possible canonical forms, are the questions that must be carefully treated.

They significantly differ from those established for the normal systems.

In that sense, our primarily task is, before discussing any questions concerning stability problems for this class of systems, to indicate and demonstrate these problems clearly dispatched on the Fig. 1.1.

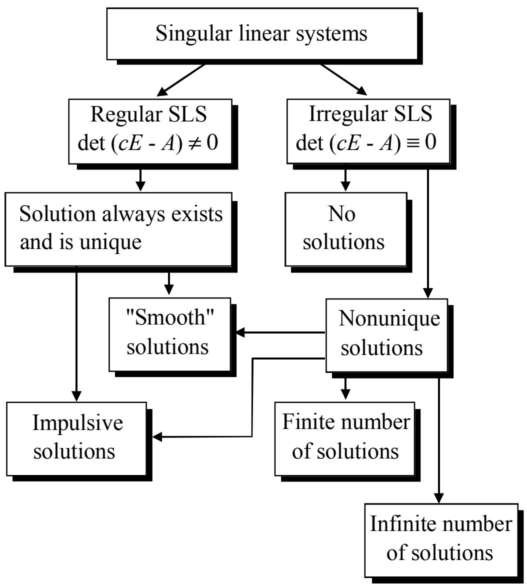

Fig.1 Types of singular systems

2.1 Solvability

According to the Fig. 1., the singular system is *regular*, when the matrix pencil $(cE - A)$ is regular, i.e.

$$\exists c \in \mathrm{R}: \det(cE - A) \neq 0, \tag{4}$$

and then solutions of (1) exist, they are unique and for so-called consistent initial conditions[1] generate smooth solutions.

Moreover, the closed form of these solution is known, *Campbell* (1980, 1982), *Dai* (1989.b).

The regularity condition (4) form the system given by (3) reduces to the following:

$$\det\begin{bmatrix} sI - A_1 & -A_2 \\ -A_3 & -A_4 \end{bmatrix} \neq 0, \tag{5}$$

which is equivalent to:

$$\det(sI - A_1)\det(-A_4 - A_3(sI - A_3)^{-1}A_2) \neq 0, \tag{6}$$

or:

$$\det A_4 \det((sI - A_1) - A_2 A_4^{-1} A_3) \neq 0, \tag{7}$$

Bender, Laub (1987.b).

Instead of (4), one can verify the following condition, *Campbell* (1980).

$$\aleph(A) \cap \aleph(E) = \{\mathbf{0}\}, \tag{8}$$

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[1] To be explained in following section.

i.e. $\aleph(A)$ and $\aleph(E)$ have only the trivial intersection where $\aleph(\cdot)$ denotes the null space or kernel of matrix $(\cdot)$.

Owens and Debeljkovic (1985) showed that (8) is equivalent with:

$$W_{k^*} \cap \aleph(E) = \{\mathbf{0}\}, \tag{9}$$

W_{k^*} being subspace of consistent initial conditions.

It should be noted that condition (4) guarantees (8) and (9), but vice versa must not be true.

Alternative characterizations of regularity condition offered by other authors are also presented.

Shuffle algorithm, proposed by Luenberger (1978) is very suitable one and consists only a few steps in order to determine existence and uniqueness of solutions.

If the matrix A_4 is nonsingular (that is, it is regular), the system is solvable.

The familiar conditions were also proposed by *Yip and Sincovec* (1981).

The following expressions are equivalent:

a) The matrix pencil $(E\ A)$ is regular.

b) Let
$$\mathbf{X}_0 = \aleph(A),\ \mathbf{X}_i = \{\mathbf{x}: A\mathbf{x} \in E\mathbf{x}_{i-1}\}, \tag{10}$$
then:
$$\aleph(E) \cap \mathbf{X}_i = \{\mathbf{0}\},\ \forall i = 0, 1, \ldots \tag{11}$$

c) Let
$$\mathbf{Y}_0 = \aleph(A^T)\ \mathrm{Y}_\mathrm{i} = \{\mathbf{x}: A\mathbf{x} \in E\mathbf{y}i{-}1\}, \tag{12}$$
then:
$$\aleph(E^T) \cap \mathbf{Y}_i = \{\mathbf{0}\},\ \forall i = 0, 1, \ldots \tag{13}$$

d) Matrix Matrix

$$F(n) = \underbrace{\begin{bmatrix} E & A & 0 & \cdots & 0 \\ 0 & E & A & \cdots & 0 \\ 0 & 0 & E & \cdots & 0 \\ \vdots & & & & \vdots \\ \vdots & & & & E \\ 0 & & \cdots & & A \end{bmatrix}}_{n+1} \qquad G(n) = \left.\begin{bmatrix} E & 0 & \cdots & 0 \\ A & E & \cdots & 0 \\ 0 & E & \cdots & 0 \\ \vdots & & & \vdots \\ \vdots & & & E \\ 0 & & \cdots & A \end{bmatrix}\right\} n+1 \tag{14}$$

has full row rank for $n = 1, 2, \ldots$. has full column rank for $n = 1, 2, \ldots$

Applying one of the presented criteria, one can easily get an answer to the question whether the singular system is *solvable or not.*

2.2 Consistent initial conditions

Having in mind the possible implicit character of equation (1), with respect to $\dot{\mathbf{x}}(t)$, it is obvious, that *not all* initial conditions are permissible.

The problem of consistent initial conditions is not characteristic for the systems in the normal form, but it is basic one for the singular systems.

We will say that an initial condition $\mathbf{x}_0 \in \mathbb{R}^n$ *is consistent* if there exist a *differentiable, continuous* solution of (1).

The solution $\mathbf{x}(t)$ should be differentiable a finite number of times and it is real analytic on interval $t \geq 0$.

Discussion and generation of consistent initial conditions were treated by several authors. Some of these, most important, results are presented here.

Campbell (1980) showed that $\mathbf{x}_0$ is a consistent initial condition for (2.1) if and only if:

$$(I - \hat{E}\hat{E}^D)\mathbf{x}_0 = \mathbf{0}, \tag{15}$$

or, in equivalent notation:

$$W_{k*} = \aleph(I - \hat{E}\hat{E}^D), \tag{16}$$

where $\hat{E}^D$ is the Drazin inverse of matrix $\hat{E}$ and $\hat{E} = (cE - A)^{-1}E$.

The fundamental geometric tool in the characterization of the subspace of consistent initial conditions is the subspace sequence:

$$\begin{aligned} &W_0 \in \mathbb{R}^n, \\ &\vdots \\ &W_{j+1} = A^{-1}(EW_j), j \geq 0, \end{aligned} \tag{17}$$

where $A^{-1}(\cdot)$ denotes the inverse image of $(\cdot)$ under the operator A, and $\aleph(\cdot)$ and $\Re(\cdot)$ stands for null space and range of any operator $(\cdot)$, respectively.

If k^* is the smallest such integer with this property, then:

$$W_k \cap \aleph(E) = \{\mathbf{0}\}, k \geq k^*, \tag{18}$$

provided that $(cE - A)$ is invertible for some $c \in \mathbb{R}$.

The proof and other detail can be found in ***Lemma,*** *Owens, Debeljkovic* (1985) and are omitted for the sake of brevity.

Consider now the manifold, $\mathtt{M} \subseteq \mathbb{R}^n$, determined by (3) as:

$$\mathtt{M} = \aleph([A_3\ A_4]), \tag{19}$$

For the singular system governed by (3) the set of the consistent initial values is equal to the manifold, or in the other words $\mathbf{x}_0$ has to satisfy:

$$\mathbf{x}_0 \in \mathtt{M} \equiv \aleph([A_3\ A_4]) \equiv W_{k*} \tag{20}$$

Obviously the determination of linear manifold $\mathtt{M}$ requires no additional computations, except those necessary to convert (1) into the form (3).

Assuming that the rank $A_4 = r \leq n_2$, it is clear, on the basis of equation (3), that $(n_1 + n_2 - r)$ components of the vector $\mathbf{x}_0$ can be chosen arbitrarily to active no impulsive solutions of the system governed by (1).

The use of nonconsistent initial conditions leads to impulsive solutions of (1), *Verghese et al* (1981).

3. Stability in the sense of Lyapunov

Stability plays a central role in the theory of systems and control engineering.There are different kinds of stability problems that arise in the study of dynamic systems, such as Lyapunov stability, finite time stabilty, practical stability, technical stabilty and BIBO stability. The first part of this section is concerned with the stability of the equlibrium points in the sense of Lyapunov stability of *linear autonomous singular systems.* As we treat the linear systems this is equivalent to the study of the stabilty of the systems.

The Lyapunov direct method is well exposed in a number of very well known references. Here we present some different and interesting approaches to this problem, including the contributions of the authors of this paper.

Stability definitions

Definition 1. Eq.(1) is exponentially stable if one can find two positive constants α, β such that for every solution of (1), $\|\mathbf{x}(t)\| \le \alpha \|\mathbf{x}_0(t)\| e^{-\beta t}$, *Pandolfi* (1980).

Definition 2. The system given by (1) will be termed *asymptotically stable* iff, for all consistent initial conditions $\mathbf{x}_0$, $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$, *Owens, Debeljkovic* (1985).

Definition 3. We call system given by (1) *asymptotically stable* if all roots of det *(sE - A)*, *i.e.* all finite eigenvalues of this matrix pencil, are in the open left - half complex plane, and system under consideration is *impulsive free* if there is no $\mathbf{x}_0$ such that $\mathbf{x}$(t) exibits discontinuous behavior in the free regime, *Lewis* (1986).

Definition 4. The system given by (1) is called *asymptotically stable* iff all finite eigenvalues λ_i, $i = 1, \ldots, n_1$, of the matrix pencil (λE −A) have negative parts, *Muller* (1993).

Definition 5. The equilibrium $\boldsymbol{x} = \mathbf{0}$ of system given by (1) is said to be *stable* if for every $\varepsilon > 0$, and for any $t_0 \in J$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$, such that $\|\mathbf{x}(t, t_0, \mathbf{x}_0)\| < \varepsilon$ hold for all $t \ge t_0$, whenever $\mathbf{x}_0 \in W_k$ and $\|\mathbf{x}_0\| < \delta$, where J denotes time interval such that $J = [t_0, +\infty)$, $t_0 \ge 0$, *Chen, Liu* (1997).

Definition 6. The equilibrium $\mathbf{x} = \mathbf{0}$ of a system given by (1) is said to be *unstable* if there exist a $\varepsilon > 0$, and $t_0 \in J$, for any $\delta > 0$, such that there exists a $t^* \ge t_0$, for which $\|\mathbf{x}(t^*, t_0, \mathbf{x}_0)\| \ge \varepsilon$ holds, although $\mathbf{x}_0 \in W_k$ and $\|\mathbf{x}_0\| < \delta$, *Chen, Liu* (1997).

Definition 7. The equilibrium $\mathbf{x} = \mathbf{0}$ of a system given by (1) is said to be *attractive* if for every $t_0 \in J$, there exists an $\eta = \eta(t_0) > 0$, such that $\lim_{t\to\infty} \mathbf{x}(t, t_0, \mathbf{x}_0) = \mathbf{0}$, whenever $\mathbf{x}_0 \in W_k$ and $\|\mathbf{x}_0\| < \eta$, *Chen, Liu* (1997).

Definition 8. The equilibrium $\mathbf{x} = \mathbf{0}$ of a singular system given by (1) is said to be *asymptotically stable* if it is stable and attractive*, Chen, Liu* (1997).

Lemma 1. The equilibrium $\mathbf{x} = \mathbf{0}$ of a linear singular system given by (1) is *asymptotically stable* if and only if it is *impulsive-free*, and $\sigma(E,A) \subset C^{-}$, *Chen, Liu* (1997).

Lemma 2. The equilibrium $\mathbf{x} = \mathbf{0}$ of a system given by (1) is *asymptotically stable* if and only if it is *impulsive-free*, *and* $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$, *Chen, Liu* (1997).

Stability theorems

Theorem 1. System (1), with $A = I$, I being the identity matrix, is *exponentially stable* if and only if the eigenvalues of E have non positive real parts, *Pandolfi* (1980). ◆

Theorem 2. Let I_Ω be the matrix which represents the operator on $\mathbf{R}^n$ which is the identity on Ω and the zero operator on Λ.

System (1), with $A = I$, is stable if an $n\times n$ matrix P exist, which is the solution of the matrix equation:

$$E^T P + PE = -I_\Omega \,, \quad (21)$$

with the following properties:

$$P = P^T \,, \quad (22)$$

$$P\mathbf{q} = \mathbf{0},\ \boldsymbol{q} \in \Lambda, \quad (23)$$

$$\mathbf{q}^T \mathrm{P}\mathbf{q} > 0,\ \mathbf{q} \neq \mathbf{0},\ \mathbf{q} \in \Omega, \quad (24)$$

where:

$$\Omega = W_k = \aleph(I - EE^D)\,, \quad (25)$$

$$\Lambda = \aleph(EE^D)\,, \quad (26)$$

where W_k is the subspace of consistent intial conditions.

$\aleph$ denotes the kerrnel or null space of the matrix ( ), *Pandolfi* (1980). ◆

Theorem 3. The system (1) is *asymptotically stable* if and only if:

a) A is invertible and

b) a positive-definite, self-adjoint operator P on $\mathbf{R}^n$ exist, such that:

c) $$A^T PE + E^T PA = -Q \quad (27)$$

where Q is self-adjoint and positive in the sense that:

$$\mathbf{x}^T(t) Q \mathbf{x}(t) > 0 \text{ for all } \boldsymbol{x} \in W_{k^*} \setminus \{\mathbf{0}\}, \quad (28)$$

Owens, Debeljkovic (1985). ◆

Theorem 4. The system given by Eq. (1) is *asymptotically stable* if and only if:

a) A is invertible and

b) a positive-definite, self-adjoint operator P exist, such that:

$$\mathbf{x}^T(t)\,(A^T PE + E^T PA)\,\mathbf{x}(t) = -\mathbf{x}^T(t) I \mathbf{x}(t)\,, \text{ for all } \boldsymbol{x} \in W_{k^*} \quad (29)$$

Owens, Debeljkovic (1985). ◆

Theorem 5. Let (E,A) be regular and (E,A,C) be observable. Then (E,A) is *impulsive free* and *asymptotically stable* if and only if a positive definite solution P to:

$$A^T PE + E^T PA + E^T C^T CE = 0 , \tag{30}$$

exist and if P_1 and P_2 are two such solutions, then $E^T P_1 E = E^T P_2 E$, *Lewis* (1986). ◆

Theorem 6. If there are symmetric matrices P, Q satisfying:

$$A^T PE + E^T PA = -Q, \tag{31}$$

and if:

$$\mathrm{x}^T E^T PE\,\mathrm{x} > 0\ \forall \mathrm{x} = S_1\mathbf{y}_1 \neq 0, \ (32)$$

$$\mathrm{x}^T Q\mathrm{x} \geq 0\ \forall \mathrm{x} = S_1\mathbf{y}_1 , \ (33)$$

then the system described by (1) is *asymptotically stable* if:

$$rank\begin{bmatrix} sE - A \\ S_1^T Q \end{bmatrix} = n \quad \forall s \in \mathbf{C}, \tag{34}$$

and marginally stable if the condition given by (34) does not hold, *Muller* (1993). ◆

Theorem 7. The equilibrium $\boldsymbol{x} = \mathbf{0}$ of a system (1) is *asymptotically stable,* if an $n \times n$ symmetric positive definite matrix P exist, such that along the solutions of system, (1), the derivative of function $V(E\boldsymbol{x}) = (E\boldsymbol{x})^T P(E\boldsymbol{x})$, is a negative definite for the variates of $E\boldsymbol{x}$, *Chen, Liu* (1997). ◆

Theorem 8. If an *nxn* symmetric, positive definite matrix P exists*,* such that along with the solutions of system, (1), the derivative of the function $V(E\boldsymbol{x}) = (E\boldsymbol{x})^T P(E\boldsymbol{x})$ i.e. $\dot{V}(E\boldsymbol{x})$ is a positive definite for all variates of $E\boldsymbol{x}$, then the equilibrium $\boldsymbol{x} = \mathbf{0}$ of the system given by (1) is *unstable, Chen, Liu* (1997). ◆

Theorem 9. If an *nxn* symmetric, positive definite matrix P exists, such that along with the solutions of system, (1), the derivative of the function $V(E\boldsymbol{x}) = (E\boldsymbol{x})^T P(E\boldsymbol{x})$ i.e. $\dot{V}(E\boldsymbol{x})$ is negative semi definite for all variates of $E\boldsymbol{x}$, then the equilibrium $\boldsymbol{x} = \mathbf{0}$ of the system, given by (1), is *stable, Chen, Liu* (1997). ◆

Theorem 10. Let (E,A) be regular and (E,A,C) be impulse observable and finite dynamics detectable. Then (E,A) is stable and impulse-free if and only if a solution (P,H) to the generalized *Lyapunov equations (GLE)* exists.

$$A^T P + H^T A + C^T C = 0 , \tag{37}$$

$$H^T E = E^T P \geq 0 , \tag{38}$$

Takaba et al. (1995). ◆

Some assumptions and preliminaries are needed for further exposures.

Suppose that matrices E and A commute that is: $EA = AE$.

Then a real number λ exists such that $\lambda E - I = A$, otherwise, from the property of regularity, one may multiply (2) by $(\lambda E - A)^{-1}$ so one can obtain the system that satisfy the above assumption and keep the stability the same as the original system.

It is well known that there always exists linear nonsingular transformation, with invertible matrix T, such that system (2), can be put in the following form:

$$[TET^{-1} \quad TAT^{-1}] = \{diag[E_1 \quad E_2] \ \ diag[A_1 \quad A_2]\}, \tag{39}$$

where E_1 is of full rank and E_2 is a nilpotent matrix, satisfying:

$$E_2^h \neq 0, \ E_2^{h+1} = 0, \ h \geq 0. \tag{40}$$

In addition, it is evident:

$$A_1 = \lambda E_1 - I_1, \ A_2 = \lambda E_2 - I_2. \tag{41}$$

The system, (2), is equivalent to:

$$E_1 \dot{\mathbf{x}}_1(t) = A_1 \mathbf{x}_1(t) + B_1 \mathbf{u}(t), \tag{42a}$$

$$E_2 \dot{\mathbf{x}}_2 = A_2 \mathbf{x}_2(t) + B_2 \mathbf{u}(t), \tag{42b}$$

where $\mathbf{x}^T = [\mathbf{x}_1^T \quad \mathbf{x}_2^T]$.

Lemma 3. The system, (1), is *asymptotically stable* if and only if the *"slow" sub - system*, (42a) is asymptotically stable, *Zhang et al.* (1998a)

Lemma 4. $\mathbf{x}_1 \neq \mathbf{0}$ is equivalent to $E^{h+1}\mathbf{x} \neq \mathbf{0}$, *Zhang et al.* (1998a).

Define Lyapunov function as:

$$V(E^{h+1}\mathbf{x}) = \mathbf{x}^T (E^{h+1})^T P E^{h+1} \mathbf{x}, \tag{43}$$

where: $P > 0, \ P \in \mathbf{R}^{nxn}$ satisfying: $V(E^{h+1}\mathbf{x}) > 0$ if $E^{h+1}\mathbf{x} \neq \mathbf{0}$, when $V(0) = 0$.

Bearing in mind that $EA = AE$, one can obtain:

$$(E^h)^T A^T P E^{h+1} + (E^{h+1})^T P A E^h = -(E^{h+1})^T W E^{h+1} \tag{44}$$

where $W > 0, \ W \in \mathbf{R}^{nxn}$.

(44) is said to be Lyapunov equation for a system given by (42).

Denote with:

$$\deg\det(sE - A) = rankE_1 = r. \tag{45}$$

Theorem 11. The system, given by (1), is *asymptotically stable* if and only if for any matrix $W > 0$, (44) has a solution $P \geq 0$ with a positive external exponent r, *Zhang et al.* (1998a). ◆

Theorem 12. The system, given by (1), is *asymptotically stable* if and only if for any given $W > 0$ the Lyapunov (44) has the solution $P > 0$, *Zhang et al* (1998a). ◆

It should be noted that the results of the preceeding theorems are very similar in some way and are derived only for *regular linear singular systems*.

In order to investigate the stability of irregular singular systems, the folowing results can be used, *Bajic at al.*(1992).

For this case, the linear singular system is used in the form (3) i.e.:

Herewith, we examine the problem of the existence of solutions which converge toward the origin of the systems phase-space for *regular and irregular singular linear systems.* By a suitable nonsingular transformation, the original system is transformed to a convenient form. This form of system equations enables development and easy application of *Lyapunov's diect method* for the intended existence analysis for a subclass of solutions.

In the case when the existence of such solutions is established, an understimation of the weak domain of the attraction of the origin is obtained on the basis of *symmetric positive definite solutions of a reduced order matrix Lyapunov equation.*

The estimated weak domain of attraction consists of points of the phase space, which generate at least one solution convergent to the origin.

Let as, before, the set of the consistent initial values of (3) be denoted by W_{k^*}. Also, consider the manifold $\mathrm{M} \subseteq \mathbf{R}^{nxn}$ (3b) as $\mathrm{M} = \aleph([A_3 \ \ A_4])$.

For the system governed by (3) the set W_{k^*} of the consistent initial values is equal to the manifold $\boldsymbol{m}$, that is $W_{k^*} = \mathrm{M}$.

It is easy to see, that the convergence of the solutions of system (1) and system (3), toward the origin is an equivalent problem, since the proposed transformation is nonsingular.

Thus, for the null solution of (3), the weak domain of attraction is going to be estimated.

The weak domain of attraction of the null solution $\mathbf{x}(t) \equiv \mathbf{0}$ of system (3) is defined by:

$$\boldsymbol{D} \overset{\Delta}{=} \{ \mathbf{x}_0 \in \Re^n : \mathbf{x}_0 \in \boldsymbol{m}, \exists \mathbf{x}(t, \mathbf{x}_0), \lim_{t \to \infty} \| \mathbf{x}(t, \mathbf{x}_0) \| \to \mathbf{0} \}. \tag{46}$$

The term weak is used because solutions of (3) need not to be unique, and thus for every $\mathbf{x}_0 \in \boldsymbol{D}$ there may also exist solutions which do not converge toward the origin. In our case $\boldsymbol{D} = \mathrm{M} = W_{k^*}$, and the weak domain of attraction may be thought of as the weak global domain of attraction

Our task is to estimate the set $\boldsymbol{D}$.

We will use LDM to obtain the underestimate $\boldsymbol{D}_e$ of the set $\boldsymbol{D}$ (i.e. $\boldsymbol{D}_e \subseteq \boldsymbol{D}$).

Our development will not require the regularity condition of the matrix pencil $(sE - A)$.

For the systems in the form of (3) the Lyapunov-like function can be selected as:

$$V(\mathbf{x}(t)) = \mathbf{x}_1^T(t) P \mathbf{x}_1(t), P = P^T, \tag{47}$$

where P will be assumed to be a positive definite and real matrix.

The total time derivative of V along the solutions of (3) is then:

$$\dot{V}(\mathbf{x}(t)) = \mathbf{x}_1^T(t)(A_1^T P + P A_1)\mathbf{x}_1(t) + \mathbf{x}_1^T(t) P A_2 \mathbf{x}_2(t) + \mathbf{x}_2^T(t) A_2^T P \mathbf{x}_1(t). \tag{48}$$

A brief consideration of the attraction problem shows that if (48) is negative definite, then for every $\mathbf{x}_0 \in W_{k^*}$ we have $\| \mathbf{x}_1(t) \| \to 0$ as $t \to \infty$.

Then $\| \mathbf{x}_2(t) \| \to 0$ as $t \to \infty$, for all those solutions for which the following connection between $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ holds:

$$\mathbf{x}_2(t) = L\,\mathbf{x}_1(t)\,,\ \forall t \in \mathbf{R}. \tag{49}$$

The main question is if the relation (49) can be established in a way so as not to contradict the constraints. Since it is not possible for irregular singular linear system, then we have to reformulate our task to establish the relation (49) so that it does not pose to many addtional novel constraints to (3b).

In order to efficiently use this fact for the analysis of the attraction problem, we introduce the following consideration that also proposes a construction of the matrix L.

Let (49) hold.

Assume that the rank condition:

$$\text{rank}\,[A_3\ \ A_4] = \text{rank}\,A_4 = r \le n_2, \tag{50}$$

is satisfied.

Then a matrix L exist, Tseng and Kokotovic (1988), being any solution of the matrix equation:

$$0 = A_3 + A_4 L\,, \tag{51}$$

where 0 is a null matrix of dimensions the same as A_3.

One can investigate in more detail the implications of the introduced equations. When they hold, then all solutions of the system (3), which satisfy (49), are subject to algebraic constraints:

$$F\mathbf{x}(t) = \begin{bmatrix} A_3 & A_4 \\ L & -I \end{bmatrix} \mathbf{x}(t) = 0. \tag{52}$$

Now (48) and (49) are employed to obtain:

$$\dot{V}\,(\mathbf{x}(t)) = \mathbf{x}_1^T(t)((A_1 + A_2 L)^T P + P(A_1 + A_2 L))\mathbf{x}_1(t)\,, \tag{53}$$

which is a negative definite with respect to $\mathbf{x}_1(t)$ if and only if:

$$\Omega^T P + P\Omega = -Q, \quad \Omega = A_1 + A_2 L\,, \tag{54}$$

where Q is real a symmetric positive definite matrix.

Theorem 13. Let the rank condition (50) hold and let rank $F < n$, where the matrix F is defined in (52). Then, the underestimate $\boldsymbol{D}_e$ of the weak domain $\boldsymbol{D}$ of the attraction of the null solution of system given by (3), is determinated by (46), providing $(A_1 + A_2L)$ is a ***Hurwitz matrix***.

If $\boldsymbol{D}_e$ is not a singleton, then there are solutions of (3) different form null solution, $\mathbf{x}(t) \equiv \mathbf{0}$, which converge toward the origin as time $t \to +\infty$, Bajic et al. (1992). ◆

4. NUMERICAL EXAMPLE

Example 4.1. For the given system:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{x}(t) \cdot$$

dynamical analysis leads to:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot$$

and $\det E = 0$ with $\det A = 1$.

Solvability

Since matrix A is nonsingular one, pair (E, A) is regular, so system under consideration is solvable.

Let us check this using other approaches.

Gantmacher (1977):

$$\det(sE - A) = \begin{vmatrix} s & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & s+1 \end{vmatrix} = -(s+1) \neq 0, \ \forall s \in R \wedge s \neq -1,$$

what means that (E, A) is regular. ■

Campbell (1980)

Let $\lambda = 0$.

Matrix pair (E, A) is regular since $\det(\lambda E + A) = \det(A) = 1 \neq 0$. ■

Yip, Sincovec (1981)

a) Null space of matrices E and A is given with:

$$\aleph(E) = span\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\} = \alpha e_2, \quad \alpha \in R,$$

$$\aleph(A) = \{\mathbf{0}\},$$

$$X_0 = \aleph(A) = \{\mathbf{0}\},$$

$$EX_0 = \mathbf{0}$$

$$X_i = \{\mathbf{x} : \ A\mathbf{x} \in EX_{i-1}\}, \quad \forall i = 1, 2, \dots,$$

$$X_i = \{\mathbf{0}\}, \quad \forall i = 1, 2, \dots .$$

since the condition:

$$\aleph(E) \cap X_i = \{\mathbf{0}\}, \quad \forall i = 0,1,2,...,$$

is satisfied, system is solvable. ■

c) Since:

$$E^T = E \wedge A^T = A$$

so it follows:

$$\aleph(E^T) = \aleph(E),\ \aleph(A^T) = \aleph(A),$$

and

$$Y_i = X_i,$$

so it is obvious that:

$$\aleph(E^T) \cap Y_i = \aleph(E) \cap X_i = \{\mathbf{0}\}, \quad \forall i = 0,1,2,...,$$

and system is solvable.

d) Matrix:
$$G(1) = \begin{bmatrix} E \\ A \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

has a full column rank.

e) Matrix:
$$F(1) = \begin{bmatrix} E & A \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

has a full row rank.

»Shuffle« algoriham:

$$\begin{matrix} & E & & & A & \\ \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{matrix}$$

interchanging of rows, leads to:

$$\begin{matrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{matrix},$$

what should be:

$$\begin{matrix} T & A_1 \\ 0 & A_2 \end{matrix},$$

where matrix:

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

has full row rank, so the »shuffle« may be applied:

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{matrix} \qquad \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{matrix}.$$

Since the matrix on left is singular:

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 0,$$

algorithm should be continued:

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix} \qquad \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{matrix},$$

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{matrix} \qquad \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{matrix}$$

since, now matrix on the left is regular:

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix} = 1,$$

what ends the procedure and system is solvable.

Now, we transfer the system under consideration to *normal canonical form*:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\mathbf{y}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{y},$$

where:

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \\ x_2 \end{bmatrix}.$$

where:

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ A_3 = \begin{bmatrix} 1 & 0 \end{bmatrix},\ \text{i}\ A_4 = 0.$$

Since the condition:

$$\det(sI - A_1)\det(-A_4 - A_3(sI - A_1)^{-1}A_2) = -(s+1) \neq 0\ ,\ ,$$

is fullfiled such that: $\exists s \in \mathrm{R} \rightarrow \det(sI - A_1) \neq 0$, system is solvable. ■

Consistent initial conditions

Campbell (1980):
Let $\lambda = 0$:

$$\hat{A} = \left(\lambda E + A\right)^{-1} A = A^{-1} A = I \ ,$$

$$\hat{E} = \left(\lambda E + A\right)^{-1} E = A^{-1} E = AE = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

since:

$$A^{-1} = A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Eigenvalues of matrix $\hat{E}$ are:

$$\sigma(\hat{E}) = \{0,0,-1\} \ ,$$

so:

$$\hat{E}^D = -\hat{E}^2 = -\begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \ \hat{E}\hat{E}^D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\left(I - \hat{E}\hat{E}^D\right)\mathbf{x}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0} \rightarrow x_1 = 0, \ x_2 = 0, \ x_3 \in \mathbb{R},$$

E^D being Drazin inverse of matrix E. ■

The set of consistent initial conditions is given with:

$$W_k = \aleph(I - \hat{E}\hat{E}^D) = \text{span}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

or:

$$W_k = \{\mathbf{x} \in \mathrm{R}^n : \ x_1 = 0, \ x_2 = 0, \ x_3 = \mathrm{R}\}.$$

Owens, Debeljković (1985):

$$W_0 = \mathrm{R}^3$$

$$W_{j+1} = A^{-1}(EW_j), \ \ j \geq 1,$$

$$EW_0 = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

$$W_1 = A^{-1}(EW_0) = \{\mathbf{x} \in \mathrm{R}^3 : \; A\mathbf{x} \in EW_0\}$$
$$= \left\{ \mathbf{x} \in \mathrm{R}^3 : \; A\mathbf{x} = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \right\}$$
$$= \left\{ \mathbf{x} \in \mathrm{R}^3 : \; x_{10} = 0, \; x_{20} = x_{10}, \; x_{30} \in \mathrm{R} \right\}$$
$$W_1 = \mathrm{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\},$$
$$W_2 = A^{-1}(EW_1),$$
$$EW_1 = span \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

Since: $$EW_1 = EW_0,$$
it is obvious that:
$$W_{k^*} = \{\mathbf{x} \in \mathrm{R}^n : \; x_1 = 0, \; x_2 = 0, \; x_3 = \mathrm{R}\}, \; k^* = 1. \blacksquare$$

If in any way the subset of initial conditions $W_k{}^*$ is known, *solvabilty can be chaked* using the following expression:

$$W_k \cap \aleph(E) = \{\mathbf{0}\}, \quad \forall k \geq k^*,$$
$$W_k \cap \aleph(E) = \mathrm{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \cap \mathrm{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} = \{\mathbf{0}\}, \quad \forall k \geq k^* = 1,$$

with positive conclussion, *Owens, Debeljkovic* (1985). ■

When the system is in its normal canonical form, consistent initial conditions can be achived using:

$$\mathbf{0} = A_3\mathbf{y}_{10} + A_4\mathbf{y}_{20},$$

under condition that the following relation is fullfiled:

$$\mathrm{rang}\,[A_3 \;\; A_4] = \mathrm{rang}\,[A_4].$$

Since:
rang $[A_3 \;\; A_4] = [1 \;\; 0 \;\; 0] = 1 \neq$
$$\mathrm{rang}\big[A_3 \quad A_4\big] = \mathrm{rang}\big[1 \quad 0 \quad 0\big] = 1 \neq \mathrm{rang}\big[A_4\big] = \mathrm{rang}\big[0\big] = 0,$$
the before mentioned equation can not be used for this purpose. ■

Equilibrium points

Since matrix A is nonsingular and the system under consideration is *regular* equlibrium point $\mathbf{x} = \mathbf{0}$, or $\aleph(A) = \{\mathbf{0}\}$ is unique one. ■

State space solution

Campbell (1980):

Let λ = 0.

Then:

$$\hat{A} = (\lambda E + A)^{-1} A = I, \qquad E = (\lambda E + A)^{-1} E = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\hat{E}^D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot -\hat{E}^D \hat{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so:

$$e^{\hat{E}^D \hat{A} t} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} h(t),$$

$$\hat{E}\hat{E}^D \mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \\ x_{30} \end{bmatrix}, \quad \mathbf{x}_0 \in W_k,$$

and state space solution, for consistent initial conditions, is given with:

$$\mathbf{x}(t) = e^{\hat{E}^D \hat{A} t} \hat{E}\hat{E}^D \mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \\ e^{-t} x_{30} \end{bmatrix}, \quad \mathbf{x}_0 \in W_k \cdot$$

The same result is achived by direct integration:

$$\begin{aligned} \dot{x}_3 &= -x_3 \rightarrow x_3(t) = e^{-t} x_{30} h(t), \\ x_1 &= 0 \quad \rightarrow x_1(t) = (1 - h(t)) x_{10}, \cdot \blacksquare \\ \dot{x}_1 &= x_2 \quad \rightarrow x_2(t) = -\delta(t) x_{10}, \end{aligned}$$

Using norms, one can get:

$$\|\mathbf{x}(t)\| = \|x_3(t)\| = |e^{-t} x_{30}| = e^{-t} |x_{30}|, \quad \mathbf{x}_0 \in W_k,$$

$$\|\mathbf{x}(t)\| = \begin{cases} \|x_3(t)\| = e^{-t} |x_{30}|, & t > 0, \\ +\infty, & t = 0, \end{cases} \quad \mathbf{x}_0 \notin W_k,$$

what is dispitched on **Fig. 1.** under condition: $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \\ x_{30} \end{bmatrix}, \ x_{30} \in \mathrm{R} \cdot \blacksquare$

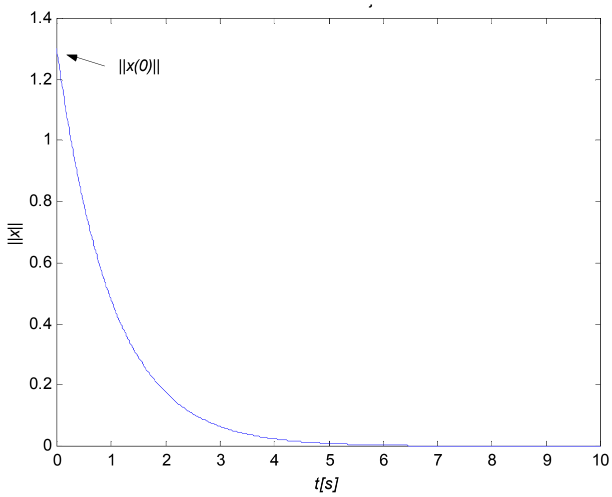

Fig. 1.

System stabilty

Pandoli (1980)*:*

Since: $\det A = 1$,

system can be transfered o the form:

$$E_0\dot{\mathbf{x}}(t) = I\mathbf{x}(t)\,,$$

where:

$$E_0 = A^{-1}E = E_0 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

$$\sigma(E_0) = \{0,0,-1\},$$

so, the sysytem is exponentialy stable. ■

Another approach is also used:

$$\Omega = \aleph\left(I - EE^D\right) = \text{span}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\Lambda = \aleph\left(EE^D\right) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\},$$

$$I_\Omega = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$E_0^T P + PE_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 & p_3 \\ p_2 & p_4 & p_5 \\ p_3 & p_5 & p_6 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 & p_3 \\ p_2 & p_4 & p_5 \\ p_3 & p_5 & p_6 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} p_2 & p_4 & p_5 \\ 0 & 0 & 0 \\ -p_3 & -p_5 & -p_6 \end{bmatrix} + \begin{bmatrix} p_2 & 0 & -p_3 \\ p_4 & 0 & -p_5 \\ p_5 & 0 & -p_6 \end{bmatrix}$$

$$= \begin{bmatrix} 2p_2 & p_4 & p_5 - p_3 \\ p_4 & 0 & -p_5 \\ p_5 - p_3 & -p_5 & -2p_6 \end{bmatrix} = -I_\Omega = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Matrix P which satisfie this equation is given with:

$$P = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/2 \end{bmatrix},$$

where p_1 is arbitry.

Obviously $p_1 = 0$, since then the following conditions are fullfiled:

$$P = P^T$$
$$P\mathbf{x} = 0, \ \ \mathbf{x} \in \Lambda,$$
$$\mathbf{x}^T P\mathbf{x} > 0 \ \mathbf{x} \neq \mathbf{0}, \ \mathbf{x} \in \Omega,$$

and the system is stable, *Pandolfi* (1980). ■

Owens, Debeljković (1985)*:*

Matrix A is nonsingular, $\det A = 1$.

Let:

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} = P^T, \quad G = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{12} & g_{22} & g_{23} \\ g_{13} & g_{23} & g_{33} \end{bmatrix} = G^T,$$

$$A^T PE + E^T PA = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2p_{12} & p_{11} & p_{23} - p_{13} \\ p_{11} & 0 & p_{13} \\ p_{23} - p_{13} & p_{13} & -2p_{33} \end{bmatrix} = -G.$$

It is obvious that $g_{22} = 0$.

Let $p_{12}= p_{23}= p_{13}= 0$, then matrix G is in the form:

$$G=\begin{bmatrix} 0 & g_{12} & 0 \\ g_{12} & 0 & 0 \\ 0 & 0 & g_{33} \end{bmatrix},$$

where: $g_{22} = -p_{11} \neq 0$, $g_{33} = 2p_{33} \neq 0$.

$$\mathbf{x}^T G\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 0 & g_{12} & 0 \\ g_{12} & 0 & 0 \\ 0 & 0 & g_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2g_{12}x_1x_2 + g_{33}x_3^2 .$$

Since:

$$W_k = \{\mathbf{x} \in \mathrm{R}^n : x_1 = 0,\, x_2 = 0,\, x_3 \in \mathrm{R}\},$$

it follows:

$$\mathbf{x}^T G\mathbf{x} = 2g_{12}x_1x_2 + g_{33}x_3^2 > 0, \quad \forall \mathbf{x} \in W_k \setminus \{\mathbf{0}\}.$$

so system is asymptotically stable on the subspace of consistent initial conditions. ■

Lewis (1986):

$$f_E(s) = \det(sE - A) = -(s+1) \rightarrow s_1 = -1 < 0 .$$

Since $\det A_4 = 0$, there is an impulsive solutions and conclussion is not evident. ■

Chen, Liu (1997)

$$A^T PE + E^T PA = -E^T WE.$$

Let:

$$P=\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix}, \quad W=\begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{12} & w_{22} & w_{23} \\ w_{13} & w_{23} & w_{33} \end{bmatrix},$$

and:

$$A^T PE + E^T PA = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$+\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} =$$

$$= \begin{bmatrix} p_{12} & 0 & p_{23} \\ p_{11} & 0 & p_{13} \\ -p_{13} & 0 & -p_{33} \end{bmatrix} + \begin{bmatrix} p_{12} & p_{11} & -p_{13} \\ 0 & 0 & 0 \\ p_{23} & p_{13} & -p_{33} \end{bmatrix} = \begin{bmatrix} 2p_{12} & p_{11} & p_{23} - p_{13} \\ p_{11} & 0 & p_{13} \\ p_{23} - p_{13} & p_{13} & -2p_{33} \end{bmatrix},$$

$$-E^T WE = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{12} & w_{22} & w_{23} \\ w_{13} & w_{23} & w_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -w_{11} & 0 & -w_{13} \\ 0 & 0 & 0 \\ -w_{13} & 0 & -w_{33} \end{bmatrix}.$$

If one addopt matrix W as:

$$W = \begin{bmatrix} w_{11} & 0 & w_{13} \\ 0 & w_{22} & 0 \\ w_{13} & 0 & w_{33} \end{bmatrix}.$$

It is obvious taht matrix P should be:

$$P = \begin{bmatrix} 0 & p_{12} & 0 \\ p_{12} & p_{22} & p_{23} \\ 0 & p_{23} & p_{33} \end{bmatrix}.$$

Principal minors of matrix P are given with:

$$\Delta_1 = 0,\ \Delta_2 = -p_{12}^2,\ \Delta_3 = -p_{33}p_{12}^2 \ .$$

It means ***that there no exist*** a positive definite symmetric matrix P, such that the sign of time derivative of Lyapunov function $V(E\mathbf{x}) = (E\mathbf{x})^T P(E\mathbf{x})$ can be determined for all variates $E\mathbf{x}$.

Function $V(E\mathbf{x})$ can not be used for investigation of this system since the system has an impulsive behaviour . ■

Müller (1994):

Let us take matrices R and S *like this choice*:

$$R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$S = \begin{bmatrix} S_1 & S_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right],$$

so one can get:

$$RES = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$RAS = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ \hline y_2 \\ y_3 \end{bmatrix},$$

$$\mathbf{x} = S\mathbf{y} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_2 \\ y_3 \\ y_1 \end{bmatrix} = \begin{bmatrix} \mathbf{y}_2 \\ \mathbf{y}_1 \end{bmatrix},$$

so it is:

$$A_1 = -1, \ \ I_{n1} = 1, \ \ I_{n2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \ N_k = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \ \ k = 2 \cdot$$

Let:

$$P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and:

$$A^T PE + E^T PA = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} = -G,$$

$$E^T PE = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ S_1\mathbf{y}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} y_1 = \begin{bmatrix} 0 \\ 0 \\ y_1 \end{bmatrix},$$

$$\mathbf{x}^T E^T PE\mathbf{x} = \left(S_1\mathbf{y}_1\right)^T E^T PE\left(S_1\mathbf{y}_1\right) = \begin{bmatrix} 0 & 0 & y_1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ y_1 \end{bmatrix} = y_1^2 > 0, \ \ \forall \mathbf{x} = S_1\mathbf{y}_1 \neq 0,$$

$$\mathbf{x}^T G\mathbf{x} = \left(S_1\mathbf{y}_1\right)^T G\left(S_1\mathbf{y}_1\right) = 2y_1^2 \geq 0, \qquad \forall \mathbf{x} = S_1\mathbf{y}_1,$$

so system is stable. ■

Furthemore:

$$\text{rang}\begin{bmatrix} sE - A \\ S_1^T G \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & s+1 \\ 0 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

and matrix:

$$\begin{bmatrix} sE - A \\ S_1^T G \end{bmatrix},$$

has full row rank $\forall s \in \mathrm{R}$, so system is asymptotically stable. ■

Bajićet al (1992)*:*

Since rank condition is not satisfied:

$$\text{rang}\begin{bmatrix} A_3 & A_4 \end{bmatrix} = \text{rang}\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = 1 \neq \text{rang}\begin{bmatrix} A_4 \end{bmatrix} = \text{rang}\begin{bmatrix} 0 \end{bmatrix} = 0,$$

nothing can be done. ■

5. CONCLUSSION

To assure *asymptotical stability for linear singular systems* it is not enough only to have the eigenvalues of matrix pair (E,A) in the left half complex plane, but also to provide an impulse-free motion of the system under consideration.

Some different approaches have been shown in order to construct Lyapunov stability theory for a particular class of linear singular systems operating in free and forced regimes.

A numerical example has been worked out in order to show all difficulties in applying high level control system theory for all aspects of dynamical analysis of singular systems.

APPENDIX A - *Ussual notations*

With $\aleph(F)$ and $\Re(F)$ we will denote the kernel (null space) and range on any operator F, respectively, i.e.:

$$\aleph(F) = \{\mathbf{x}: F\mathbf{x} = \mathbf{0},\ \forall\ \mathbf{x} \in \mathbf{R}^n\}, \tag{A1}$$

$$\Re(F) = \{\mathbf{y} \in \Re^m,\ \mathbf{y} = F\mathbf{x},\ \mathbf{x} \in \mathbf{R}^n\}, \tag{A2}$$

with:

$$\dim \aleph(F) + \dim \Re(F) = n\ . \tag{A3}$$

APPENDIX B – *Equilibrium points*

Definition B1. Equilibrium $\mathbf{x}_r \in R^n$, of (1) is any state of the system under consideration, for which the following condition is fulfilled:

$$\mathbf{x}(t;\, t_0,\, E\mathbf{x}_r) = \mathbf{x}_r\ , \quad t \geq t_0\ , \tag{E1}$$

Antonic (2001).

Theorem B1. The state $\boldsymbol{x} = \mathbf{0}$ is equilibrium point of system given by (1) if :

$$A\mathbf{x}_r = \mathbf{0}, \tag{E2}$$

Antonic (2001).

Theorem B2. If the linear singular system (1) is *regular e.g.* det $(sE - A) \neq 0$, then:

$$\mathbf{x} = \mathbf{x}_r\ , \quad \forall \mathrm{x} \in\ \aleph(\mathrm{A}) \in\ W_k \tag{E3}$$

W_k being the subspace of consistent initial conditions, *Antonic* (2001).

Corollary B1. Under the conditions of *Theorem B2*, (E2) is then necessary and sufficient condition in order that some state should be its equilibrium point, *Antonic,* (2001).

So the null space of matrix A represents the set of all equilibrium points of (1).

It is obvious that there is no difference between conditions for linear singular and *normal* systems, when the question of existence and uniqueness of equilibrium points is discussed.

Theorem B3. If the matrix A is nonsingular (det $A \neq 0$) then the singular system (1) is regular and its equilibrium $\mathbf{x}_r = \mathbf{0}$, is unique, *Antonic* (2001).

Theorem B4. State $\mathbf{x} = \mathbf{0}$, is asymptotically stable if it is unique equilibrium point of system (1), *Antonic* (2001).

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NEKI PRAKTIČNI ASPEKTI PRIMENE TEORIJE LINEARNIH SINGULARNIH SISTEMA

D. Lj. Debeljković, S. Antonić, N. Yi-Yong, Q.L. Zhang

Dinamika linearnih singularnih sistema opisana je, u matematičkom smislu, kombinacijom diferencijalnih i algebarskih jednačina. U tom smislu ove druge predstavljaju ograničenje na rešavanje prvog dela sistema jednačina. Imajući tu činjenicu u vidu veliki broj novih, dodatnih poteškoća u primeni postojećih rezultata prirodno se nameću. To se prvenstveno odnosi na pitanja postojanja i jedinstvenosti rešenja ovako hidbridnog sistema jednačina, pitanja konzistentnih (dozvoljenih) početnih uslova, imulsnog ponašanja i stabilnosti.

U radu je dat prikaz baz̆ičnih pitanja kao i njihovih rešenja koja daju odgovor po pitanju glavnih dinamičkih performansi ovih sistema sa posebnim osvrtom i akcentom na pitanja stabilnosti u smislu Ljapunova. Prikazani su referentni rezultati savremene publicističke delatnosti na tom planu a koji uključuju i rezultata samih autora.

Izloženi teorijski rezultati propraćeni su jednim eklatantnim primerom koji ukazuje i ujedno objašnjava sve detalje vezane za njihovu neposrednu primenu.

Ključne reči: Singularni sistemi, regularni singularni sistemi, iregularni singularni sistemi, rešljivost, konzistentni početni uslovi, Drazinova inverzija, stabilnost u smislu Ljapunova.