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# LYAPUNOV AND NON-LYAPUNOV STABILITY OF LINEAR DISCRETE TIME DELAY SYSTEMS

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**Abstract**. *This paper extends some of the basic results in the area of Lyapunov (asymptotic) and finite time and practical stability to linear, discrete, time invariant time-delay systems. New definitions have been established for the latter concept of stability. Sufficient conditions for this type of stability, concerning the particular class of linear discrete time-delay systems are derived.*
*More over the generalization of some previous results, in the area of Lyapunov stability for the same class of systems has been also established and proved.*
*To the best knowledge of the authors, such results have not yet been reported.*

**Key words**: *Linear discrete time delay systems, Lyapunov stability, finite and practical stability, state space methods*

## 1. INTRODUCTION

The problem of investigation of time delay systems has been exploited over many years. Time delay is very often encountered in various technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc. The existence of pure time lag, regardless if it is present in the control or/and the state, may cause undesirable system transient response, or even instability. Consequently, the problem of stability analysis for this class of systems has been one of the main interests for many researchers. In general, the introduction of time delay factors makes the analysis much more complicated.

In the existing stability criteria, mainly two ways of approach have been adopted.

Namely, one direction is to contrive the stability condition which does not include the information on the delay, and the other is the method which takes it into account. The

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former case is often called the delay - independent criteria and generally provides simple algebraic conditions.

Numerous reports have been published on this matter, with particular emphasis on the application of Lyapunov's second method, or on using the concept of the matrix measure, *Lee and Diant* (1981), *Mori* (1985), *Mori et al*. (1981), *Hmamed* (1986), *Lee et al*. (1986)).

In real-life one is not only interested in system stability (e.g. in the sense of Lyapunov), the bounds of system trajectories are also relevant. A system can be stable but still completely useless because it has undesirable transient performances. Thus, it may be useful to consider the stability of such systems with respect to certain subsets of state-space which are defined *a priori* in a given problem. Besides that, it is of particular significance to concern the behavior of the dynamical systems only over a finite time interval.

These boundedness properties of system responses, i.e. the solution of system models, are very important from the engineering point of view. Realizing this fact, numerous definitions of the so-called *technical and practical stability* were introduced.

Roughly speaking, these definitions are essentially based on the predefined boundaries for the perturbation of initial conditions and allowable perturbation of system response. In the engineering applications of control systems, this fact becomes very important and sometimes crucial, for the purpose of characterizing in advance, in quantitative manner, possible deviations of system response.

Thus, the analysis of these particular boundedness properties of solutions is an important step, which preceedes the design of control signals, when finite time or practical stability control is concern.

Motivated by "the brief discussion" on practical stability in the monograph,*La Salle, Lefchet*, (1961), *Weiss and Infante* (1965, 1967) have introduced various notations of stability over a finite time interval for continuous-time systems and constant set trajectory bounds. Further development of these results was due to many other authors *Michel* (1970), *Grujic* (1971), *Lashirer, Story* (1972)).

Practical stability of simple and interconnected systems with respect to time-varying subsets was considered in *Michel* (1970) and *Grujic* (1975).

A more general type of stability ("practical stability with settling time", practical exponential stability, etc.) which includes many previous definitions of finite stability was introduced and considered in *Grujic* (1971, 1975a, 1975b).

The concept of finite-time stability, called "final stability", was introduced in *Lashirer, Story* (1972) and furtherly developed by *Lam and Weiss* (1974).

The further development of these results and the natural extension to the class of linear continuous time delay systems was due to *Nenadic et al*. (1997), *Debeljkovic et al*. (1997a).

The matrix measure approach was for the first time introduced in *Debeljkovic* et. al. (1997b, c, d), in investigation of this type of systems as well as the approach based on general application of the well known Bellman – Gronwall lemma, *Lazarevic* et al. (2000) and for nonautonomous systems in *Debeljkovic* et al. (2000).

In the context of the practical stability for linear generalized state-space systems, various results were first obtained in *Debeljkovic, Owens* (1985) and *Owens, Debeljkovic* (1986).

Analysis of nonlinear singular and implicit dynamic systems in terms of the generic qualitative and quantitative concepts, which contain technical and practical stability types as special cases, have been introduced and studied in *Bajic* (1988, 1992).

In this short overview, the results in the area of finite and practical stability were only concerned for continuous time systems.

The results concerning Lyapunov stability are well documented in a number of known references, thus for the sake of brevity are omitted here.

Here we examine two problems. The first one is searching for the sufficient conditions that force the system trajectories to stay within the *a priori* given sets for the particular class of linear discrete time-delay systems. The other problem is obtaining sufficient conditions of asymptotic stability for the same class of systems.

## 2. Notation and Preliminaries

A linear, autonomous, multivariable discrete time-delay system can be represented by the difference equation:

$$\mathbf{x}(k+1) = A_0\mathbf{x}(k) + \sum_{i=1}^{N} A_1\mathbf{x}(k-i) \text{ , (1)}$$

and with an associated function of initial state:

$$\mathbf{x}(j) = \varphi(j), \quad j = 0, -1, -2, .... -h, .... -N\,., \tag{2}$$

The equation (1) is referred to as homogenous or the unforced state equation, $\mathbf{x}$(k) is the state vector, $A_0$, and $A_1$ are constant system matrices of appropriate dimensions, and pure time delay is expressed by integers, h = 1, 2,....... *N*.

It is assumed that the equation (1) satisfies the adequate smoothnees requirements so that it's solution exists and is unique and continuous with respect to *k* and initial data and is bounded for all bounded values of its arguments.

Let $\Re^n$ denote the state space of systems given by (1) and $\|(\cdot)\|$ Euclidean norm.

Solutions of (1), in general case, are denoted by:

$$\mathbf{x}(k, k_0, \mathbf{x}_0) \equiv \mathbf{x}(k)\,. \tag{3}$$

With $\mathrm{K}_N$ is denoted discrete-time interval, as a set of non - negative integers:

$$\mathrm{K}_N = \{k : k_0 \le k \le k_0 + k_N\}\,. \tag{4}$$

Quantity $k_N$ can be positive integer or symbol $+\infty$, so that finite time stability and practical stability can be treated simultaneously.

Let $V : \mathrm{K}_N \, x \, \Re^n \to \Re$, so that $V(k,\mathbf{x})$ is bounded for and for which $\|\mathbf{x}\|$ is also bounded.

Define the total difference of *V*/*k*, $\mathbf{x}$(*k*)/ along the trajectory of systems, given by (1), with:

$$V(k, \mathbf{x}(k)) = V(k+1, \mathbf{x}(k+1)) - V(k, \mathbf{x}(k))\,. \tag{5}$$

For time-invariant sets it is assumed: $S_0$ is a bounded, connected and open set. The closure and boundary of $S_0$ are denoted by $\overline{S}_0$ and $\partial S_{()}$, respectively, so: $\partial S_{()} = \overline{S}_{()} \setminus S_{()}$.

Let $S_\beta$ be a given set of all allowable states of system for $\forall k \in \mathrm{K}_N$

Set $S_\alpha$, $S_\alpha \subset S_\beta$ denotes set of all allowable initial states.

Sets $S_\alpha$ and $S_\beta$ are a priori known.

$\lambda(\,)$ denotes the eigenvalues of matrix ( ).

$\Lambda$ is the maximum eigenvalue.

## 3. Main results - Finite and practical stability

**Stability definitions**

**Definition 1**. The linear discrete time delay system (1), is *finite time stable* with respect to $\{\alpha, \beta, k_0, k_M, \|(\cdot)\|\}$, $\alpha < \beta$, $\alpha, \beta \in R^+$, if and only if:

$$\|\mathbf{x}(k)\| < \alpha \,,\ \forall\, k = 0, -1, -2, \cdots, -N \tag{6}$$

implies:

$$\|\mathbf{x}(k)\| < \beta \,,\ \forall\, k \in K_M \,, \tag{7}$$

*Aleksendric (2002).*

**Definition 2.** The linear discrete time delay system (1), is *finite time stable* with respect to $\{\alpha, \beta, M, N, \|(\cdot)\|\}$, $\alpha < \beta$, $\alpha, \beta \in R^+$, if and only if:

$$\|\mathbf{x}(i)\| < \alpha, \quad \forall i = 0, -1, \cdots, \ N \,, \tag{8}$$

implies:

$$\|\mathbf{x}(i)\| < \beta, \quad \forall i = 0, 1, 2, \ \cdots, \ M \,, \tag{9}$$

*Aleksendric (2002).*

These Definitions are analogous to those presented for the first time in *Debeljkovic et al. (1997a)* for *continuous time delay systems.*

**Stability theorems**

**Theorem 1.** A system (1), is *finite time stable* with respect to $\{\alpha, \beta, M, N, \|(\cdot)\|\}$, $\alpha < \beta$, $\alpha, \beta \in R^+$, if the following condition is satisfied:

$$\|\,\Phi(k)\,\| < \frac{\beta / \alpha}{1 + \sum\limits_{i=1}^{N} \|\,A_i\,\|} \,,\ \forall\, k = 0, 1, \cdots, M. \,, \tag{10}$$

$\|\,\Phi(k)\,\|$ being the greatest value on the finite time interval under consideration, *Aleksendrić* (2002).

**Proof.** The solution of (1) with initial function (2) can be expressed in terms of fundamental matrix as it is written below:

$$\begin{aligned}\mathbf{x}(k) = \Phi(k)\mathbf{x}(0) + \Phi(k-1)A_1\mathbf{x}(-1) + \cdots \\ + \Phi(k-N)A_N\mathbf{x}(-N)\end{aligned} . \tag{11}$$

$\Phi(k)$ being the fundamental matrix of linear discrete – time delay system.

In accordance with the property of the norm, one can immediately write:

$$\begin{aligned}\| \mathbf{x}(k) \| &= \| \Phi(k)\mathbf{x}(0) + \Phi(k-1)A_1\mathbf{x}(-1) + \cdots \Phi(k-N)A_N\mathbf{x}(-N) \| \\ &\leq \| \Phi(k) \| \cdot (\| \mathbf{x}(0) \| + \sum_{i=1}^{N} \| A_i \| \cdot \| \mathbf{x}(-i) \|) \\ &< \| \Phi(k) \| \cdot (\alpha + \alpha \sum_{i=1}^{N} \| A_1 \|) \\ &\leq \alpha \cdot \| \Phi(k) \| \cdot (1 + \sum_{i=1}^{N} \| A_i \|),\end{aligned} \tag{12}$$

where the condition given by (8) has been used.

To obtain the final result, one has to use (10), so we can write:

$$\begin{aligned}\| \mathbf{x}(k) \| < \alpha \cdot \frac{\beta / \alpha}{1 + \sum_{i=1}^{N} \| A_i \|} \cdot (1 + \sum_{i=1}^{N} \| A_i \|) < \beta \\ \forall k = 0, 1, 2, \cdots, M.\end{aligned} \tag{13}$$

what has to be proved. ■

These results are analogous to those, for the first time, presented in *Debeljkovic et al.* (1997a) for *continuous time delay systems.*

When $\|A_i\| = 0,\ \forall i = 0, 1, \ldots. N$, the problem is reduced to the case of ordinary linear discrete time systems, Weiss, Lee (1971), *Debeljkovic* (2001, 2002).

Application of this Theorem is in some manner difficult, since one has to find fundamental matrix $\Phi(k)$.

In order to overcome this problem, the following discussion is presented.

The system (1) can be rewritten as:

$$\mathbf{x}_{eq}(k+1) = A_{eq}\mathbf{x}_{eq}(k) . \tag{14}$$

where:

$$A_{eq} = \begin{bmatrix} A_0 & A_1 & ..A_i.. & A_N \\ I_n & 0 & ..0.. & 0 \\ 0 & I_n & ..0.. & 0 \\ 0 & 0 & ..I_n.. & 0 \end{bmatrix} . \tag{15}$$

is new *high-dimenssion system matrix*, with *I* being identity matrix.

Therefore, the necessary and sufficient condition for stabilty of the system (1) is that the solutions of the characteristic equation of system (14):

$$\det(zI - A_{eq}) = 0\,, \tag{16}$$

satisfies:

$$\left| z \right| < 1. \tag{17}$$

But, to check (10), we only need to apply the following relation:

$$\left\| \Phi(k) \right\| = \left\| A_{eq}^{k} \right\| . \tag{18}$$

In the sequel, we present another approach.

## 4. MAIN RESULTS – LYAPUNOV STABILITY

**Theorem 2.** A system (1) is *asymptotically stable* if the following condition is satisfied:

$$\sum_{i=0}^{N} \left\| A_i \right\| < 1\,, \tag{19}$$

*Aleksendrić* (2002).

**Proof.** Following the basic result, given in *Januševski* (1978).

The solution of (1) with the initial function (2) can be expressed in terms of fundamental matrix as it is written below:

$$\mathbf{x}(k) = \Phi(k)\mathbf{x}(0) + \sum_{i=1}^{N} \Phi(k-i) A_i \mathbf{x}(-i)\,, \tag{20}$$

If it is possible to establish the stability of (20) in the sense of *Koepcke* (1965), *e.g.* that we have a progressively decreasing function $\|\mathbf{x}(k)\|$ as $k$ tends infinity, then the rest of the proof is straightforward.

Therefor, one can use the crucial fundamental matrix property:

$$\begin{aligned} \Phi(k+1) &= A_0\Phi(k) + A_1\Phi(k-1) + \cdots + A_N\Phi(k-N) \\ &= \sum_{i=0}^{N} A_i \Phi(k-i) \end{aligned} \tag{21}$$

In accordance with the property of the norm, one can immediately write:

$$\left\| \Phi(k+1) \right\| \le \sum_{i=0}^{N} \left\| A_i \right\| \cdot \left\| \Phi(k-i) \right\| . \tag{22}$$

Let us take $\delta = \varepsilon$, such that $\left\| \Phi(k-i) \right\| \le \varepsilon \ \ \forall k$, and taking into consideration (19), it is obvious that :

$$\left\| \Phi(k+1) \right\| < \sum_{i=0}^{N} \left\| \Phi(k-i) \right\| \le \varepsilon\,. \tag{23}$$

Simple mathematical induction shows that the preceeding equation is satisfied for any $\mathbf{x}(k+j), \forall j > 1$, so the stabilty in the very well known sense "$\varepsilon - \delta$" is proved.

The norm of $\Phi(k)$ uniformly decreases as k increases. ■

In the sequel, we shall present the further extension of the last result derived.

Namely, the matrix measure has been widely used in the literature when dealing with stability of time delay systems,

The matrix measure $\mu$ for any matrix $A \in \mathbb{C}^{n\times n}$ is defined as follows:

$$\mu(A) \stackrel{\Delta}{=} \lim_{\varepsilon\to 0}\frac{\| I+\varepsilon A\| -1}{\varepsilon} . \tag{24}$$

The matrix measure defined in (1) can be defined in three different ways, depending on the norm utilized in its definitions.

$$\mu_1(A) = \max_k\left( \operatorname{Re}(a_{kk}) + \sum_{\substack{i=1\\ i\neq k}}^{n} | a_{ik} | \right), \tag{25}$$

$$\mu_2(A) = \frac{1}{2}\max_i \lambda_i (A^* + A) , \tag{26}$$

*Desoer, Vidyasagar* (1975).

If $A$ is real matrix, then:

$$\mu_2(A) = \operatorname{Re}(\lambda_{\max}(A)) , \tag{27}$$

follows from (26) and when $\lambda = \lambda_{\max}(A)$ being, in general, complex if and only if $|\lambda| \geq |\lambda_i|$, $i = 1, 2 \ldots n$.

$$\mu_\infty(A) = \max_i\left( \operatorname{Re}(a_{ii}) + \sum_{\substack{k=1\\ k\neq i}}^{n} | a_{ki} | \right), \tag{28}$$

*Alstruey, De La Sen (1996).*

Some other important features of matrix measure is cited below:

$$\mu(A) \leq \| A \| , \tag{29}$$

$$\operatorname{Re}\lambda_i(A) \leq \mu(A) , \tag{30}$$

*Desoer, Vidyasagar* (1975).

Moreover, it is very well known, that:

$$\rho(A) = \max_i |\lambda_i(A)|, \; i = 1,2,\ldots n. \tag{31}$$

$\rho(A)$ being spectral radius of matrix $A$.

$$\rho(A) \leq \| A \| , \tag{32}$$

*Barnett and Story* (1970), ($A$) and $\|A\|$ being any matrix and any matrix norm, respectively.

Let us, again, consider the system described by the linear difference equation, (1-2).

Now we are in position to present the following results.

**Theorem 3.** The system (1-2) is asymptotically stable if the following condition is satisfied:

(i) Both matrices *$A_0$ and $A_1$ are discrete stable*

(ii) $\max_i \mathrm{Re}\,\lambda_i(A_0) + \rho(A_1) < 1$, (33)

*Debeljkovic, Zhang, Nie* (2002).

**Theorem 4.** The system (1-2) is asymptotically stable if the following condition is satisfied:

(i) Both matrices *$A_0$ and $A_1$ are discrete stable*

(ii)
$$\mu_2(A_0) + \mu_2(A_1) < 1, \tag{34}$$

*Debeljkovic, Zhang, Nie* (2002).

**Theorem 5.** The system (1-2) is asymptotically stable if the following condition is satisfied:

(i) Both matrices *$A_0$ and $A_1$ are discrete stable*

(ii)
$$\max_i \mathrm{Re}\,\lambda_i(A_0) + \mu_2(A_1) < 1, \tag{35}$$

*Debeljkovic, Zhang, Nie* (2002).

**Proof.** The proof can be obtained for all of given *Theorems* simultaneously.

In order to guarantee asymptotic stability, it is necessary, for both matrices *$A_0$ and $A_1$ to be discrete stable matrices*.

For this one should clarify (19).

It is obvious that:

$$\mu_2(A_0) + \mu_2(A_1) < \|A_0\| + \|A_1\| < 1, \tag{36}$$

having in mind (29).

Using (27) and (30-32) conditions of *Theorems 4 and 5* can be proved in the similar way.

For the sake of brevity these proofs are omitted here.

First term of inequality (33) is a special case of matrix measure, since $A(\cdot) \in \Re^{n\times n}$, and therefore (27) is valid.

If we rewrite (1-2) in the particular form:

$$\mathbf{x}(k+1) = A_0\mathbf{x}(k) + \gamma A_1(k-i), \tag{37a}$$

$$\mathbf{x}(j) = \varphi(j), \quad j = 0, -1, -2, .... -h, .... -N, \tag{37b}$$

where $\gamma > 0$ is a small positive parameter, it is easy to show that results given, using *Theorems 3,4,5* gives less restrictive results in compare with the basic one, *Theorem 2.* his can be easily seen from the given examples.

## Conclusion

New definitions for the particular class of *linear discrete time delay systems* have been established.

Delay - independent criteria in the form of sufficient conditions of finite time and practical stability, were also derived. It should be pointed out the presence of pure time delay in this systems can not endanger their absolute stability but it can influence only their relative stability characteristics. A suitable example has been worked out to show this effect, see *Appendix A*.

To the best knowledge of authors such results have not been yet reported.

Moreover the generalization of some previous results, in the area of Lyapunov stability for the same class of systems have been also established and proved.

## APPENDIX A

**Example A1.** Assume that we have two scalar linear systems, given by:

a) $x(k+1) = -0{,}1x(k)$,

b) $x(k+1) = -0{,}1x(k-h)$ .

We are looking for the distribution of zeroes of theirs characteristic equations, increasing the time delay h of system **b** from zero to $N = 34$.

Details are given below.

System **a** is linear discrete time system with eigenvalue $\lambda = -0{,}1$ and is obviously stable.

System **b** is linear discrete time delay system :

$$x(k+1) = -0{,}1x(k-1),$$

with the equivalent matrix:

$$A_{eq1} = \begin{bmatrix} 0 & -0{,}1 \\ 1 & 0 \end{bmatrix}.$$

and corresponding eigenvalues

$$\lambda(A_{eq1}) = \{+0{,}3162j, -0{,}3162j\} .$$

If $h = 2$, then we have:

$$x(k+1) = -0{,}1x(k-2).$$

with:

$$A_{eq2} = \begin{bmatrix} 0 & 0 & -0{,}1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

and:

$$\lambda(A_{eq2}) = \{-0{,}4642,\ 0{,}2321 \pm 0{,}4020j\}.$$

At the end lets take: $h = N = 34$.

$$\dim A_{eq} = n \cdot (N+1) = 35 .$$

The results are shown in Table No. A1 and Fig. A1.

Table A1.

| System | Symbol |
|---|---|
| Without delay | (+) |
| Delay h = 1 | (◊) |
| Delay h = 2 | (□) |
| Dealy h = N=34 | (*) |

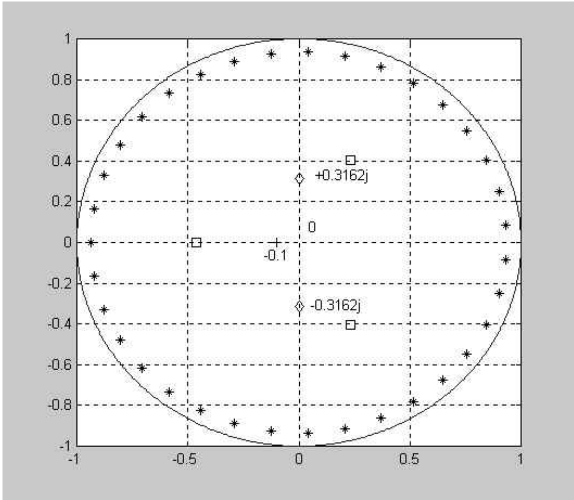


Fig. A1.

**Example A2.**

Let us consider a linear discrete time-delay system, given by the equation:

$$\mathbf{x}(k+1) = A_0\mathbf{x}(k) + A_1\mathbf{x}(k-1) \,,$$

where tha matrices are :

$$A_0 = \begin{bmatrix} 0{,}2 & 0 \\ 0 & 0{,}4 \end{bmatrix}, \; A_1 = \begin{bmatrix} 0{,}1 & 0{,}1 \\ 0{,}12 & 0{,}4 \end{bmatrix}.$$

For testing the asymptotic stability of the system, first we denote the eigenvalues:

$$\| A_0 \| = 0{,}400 \quad \| A_1 \| = 0{,}436$$

$$\lambda(A_0) = \{0{,}2;\; 0{,}4\} \implies \max \operatorname{Re} \lambda_i(A_0) = 0{,}4$$

$$\lambda(A_1) = \{0{,}0643;\; 0{,}4357\} \Rightarrow \rho(A_1) = 0{,}4357$$

$$\max_i \operatorname{Re} \lambda_i(A_0) + \rho(A_1) = 0{,}4 + 0{,}4357 = 0{,}8357 < 1$$

The matrices are obviously discrete-stable, and since the second condition is also satisfied the system is asymptoticly stable.

To prove the stability, we can examine the equivalent non-delayed system i.e. check the eigenvalues of the matrix of the given discrete system:

$$A_{eq} = \begin{bmatrix} A_0 & A_1 \\ I & 0 \end{bmatrix} = \begin{bmatrix} 0{,}2 & 0 & 0{,}1 & 0{,}1 \\ 0 & 0{,}4 & 0{,}12 & 0{,}4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The eigenvalues are: $\lambda(A_{eq}) = \{0{,}383;\ -0{,}166;\ 0{,}881;\ -0{,}499\}$, so the system is asymptotically stable.

**Example A3.**
Let us consider a linear discrete time-delay system, given by the equation:

$$\mathbf{x}(k+1) = A_0\mathbf{x}(k) + A_1\mathbf{x}(k-1)\,,$$

where the matrices are

$$A_0 = \begin{bmatrix} -0{,}3 & -0{,}2 \\ 0 & -0{,}4 \end{bmatrix},\quad A_1 = \begin{bmatrix} -0{,}6 & -0{,}25 \\ 0{,}22 & -0{,}01 \end{bmatrix}$$

For testing the asymptotic stability of the system, first we denote the eigenvalues:

$$\|A_0\| = 0{,}476 \quad \|A_1\| = 0{,}682\,,$$

$$\lambda(A_0) = \{-0{,}3; -0{,}4\} \ \Rightarrow \mu_2(A_0) = -0{,}300\,,$$

$$\lambda(A_1) = \{-0{,}6819; -0{,}0719\} \ \Rightarrow \mu_2(A_1) = -0{,}072\,,$$

$$\mu(A_0) + \mu(A_1) = -0{,}3 + (-0{,}0719) = -0{,}3719 < 1\,.$$

The matrices $A_0$,$A_1$ are obviously discrete-stable, and since the second condition, Theorem 3, is also satisfied the system is asymptotically stable.

But it should be noted that the basic condition of Theorem 2,(19) is not satisfied. So the simple conclusion is evident. Condition given by (34) is less restrictive than (19).

To prove the stability, we can examine the equivalent non-delayed system i.e. check the eigenvalues of the matrix of given discrete system in this example:

$$A_{eq} = \begin{bmatrix} A_0 & A_1 \\ I & 0 \end{bmatrix} = \begin{bmatrix} -0{,}3 & -0{,}2 & -0{,}6 & 0{,}25 \\ 0 & -0{,}4 & 0{,}22 & -0{,}01 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The eigenvalues are: $\lambda(A_{eq}) = \{-0{,}128 \pm 0.819\,j;\ -0{,}5692,\ 0{,}1252\}$, so the system is asymptotically stable.

**Example A4.**
Let us consider a linear discrete time-delay system, given by the equation:

$$\mathbf{x}(k+1) = A_0\mathbf{x}(k) + A_1\mathbf{x}(k-1)\,,$$

with the matrices:

$$A_0 = \begin{bmatrix} 0{,}2 & -0{,}1 \\ 0{,}37 & 0{,}09 \end{bmatrix},\quad A_1 = \begin{bmatrix} 0{,}17 & -0{,}22 \\ -0{,}5 & 0{,}34 \end{bmatrix}$$

For testing the asymptotic stability of the system, first we denote the eigenvalues:

$$\|A_0\| = 0{,}422 \quad \|A_1\| = 0{,}661\,,$$

$$\lambda(A_0) = \{0{,}145 \pm 0{,}1843j\}\ .$$

$$\lambda(A_1) = \{-0{,}0874; 0{,}5974\} \Rightarrow \mu(A_1) = 0{,}5974\ .$$

$$\max \mathrm{Re}(\lambda_i(A_0)) + \mu(A_1) = 0{,}145 + 0{,}5974 < 1\ .$$

The matrices $A_0, A_1$ are obviously discrete-stable, and since the second condition is also satisfied the system is asymptotically stable.

But it should be noted that the basic condition of Theorem 2,(19) is not satisfied. So the simple conclusion is evident. Condition given by (35) is less restrictive than (19).

To prove the stability, we can examine the equivalent non-delayed system i.e. check the eigenvalues of the matrix of given discrete system:

$$A_{eq} = \begin{bmatrix} A_0 & A_1 \\ I & 0 \end{bmatrix} = \begin{bmatrix} 0{,}2 & -0{,}1 & 0{,}17 & -0{,}22 \\ 0{,}37 & 0{,}09 & -0{,}5 & 0{,}34 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The eigenvalues are: $\lambda(A_{eq}) = \{-0{,}358 \pm 0.101j;\ 0{,}613 \pm 0{,}146j\}$, so the system is asymptotically stable.

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# LJAPUNOVSKA I NELJAPUNOVSKA STABILNOST LINEARNIH DISKRETNIH SISTEMA SA ČISTIM VREMENSKIM KAŠNJENJEM

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*Prvi deo rada bavi se daljim proširenjem koncepata neljapunovske stabilnosti na klasu linearnih diskretnih sistema sa čistim vremenskim kašnjenjem u stanju a za njihov rad u slobodnom radnom režimu. Izvedeni su samo dovoljni uslovi stabilnosti, koji su sa inženjersko – tehničke tačke gledišta, sasvim prihvatljivi i u analizi i sintezi razmatranih sistema.*

*U drugom delu radu iznosi se uopštenje poznatog kriterijuma za ispitivanje asimptotske stabilnosti posebne klase linearnih autonomnih diskretnih sistema sa čistim vremenskim kašnjenjem u stanju. Dovoljni uslovi stabilnosti izvedeni su u vremenskom domenu za razliku od bazičnog rezultata koji je ovu problematiku razmatrao u kompleksnom domenu i pružaju mogućnost za brzo i efikasno testiranje ova važne osobine svakog sistema automatskog upravljanja.*

*Pažljivo odabranim numeričkim primerom, ilustrovan je glavni i originalni doprinos ovog rada.*

*Ključne reči: linearni diskretni sistemi sa kašnjenjem, ljapunovska stabilnost, neljapunovska stabilnost, metode prostora stanja*