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# DEVELOPMENT OF THE THEORETICAL MODEL FOR DETERMINING THE STRESS ATHORITATIVE FOR CHECKING THE GEAR TOOTH VOLUME STRENGHT

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**Abstract**. *In this paper the effect of the teeth geometry and the load distribution at simultaneously meshed teeth pairs to stress in the tooth root is analyzed, when within the contact period the single and double and double and triple mesh follow each other. On the basis of analytical researches the exact and the approximate mathematical models are formed for determining the operating and critical stress at the tooth root relevant for checking the gear tooth volume strenght are created.*

## 1. INTRODUCTION

Analytical and experimental researches for the purpose of determining the operating and the critical stress of the machine parts and assemblies are always actual, for they represent the basis for evaluating their safety and reliability. The complexity of this problem becomes evident especially at gear teeth pairs due to strong effect of kinematic and geometric conditions and manufacturing accuracy for finding the operating and critical stress which are relevant for checking the gear teeth operating capability from the volume strenght aspect. For checking the tooth volume strenght a series of different methods is developed. With all these methods the gear tooth is approximated by the console shaped mechanical model at the end of which the load is acting. During the gear calculation development the expression for the tooth root stress is widened by different effecting factors according to the number of authors and methods [1,2,3,4]. In the International Standardization Organization (ISO) Standard [10], a detailed calculation flow and the physical meaning for the most of these factors are given. The calculation of the "transverse load distribution factor $K_{F\alpha}$" and the "contact ratio factor $Y_\varepsilon$", is given in the form of algorithm, and the boundary conditions and terms are not based on the fundamental meaning of these factors, but on the appropriate approximations [5,6].

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In this paper the effect of load distribution and the load leading point location onto stress condition at the tooth root is analyzed. The effect of the engagement of the simultaneously meshed teeth pairs in transmitting the gear teeth pair load to stress in the tooth root is analyzed by non-dimensional load distribution factor. For analyzing the geometric and kinematic values the relative stress factor is defined. The difficulties in determining the exact stress relevant for checking the tooth volume strenght, and the possibility of using the adequate approximations are pointed out.

## 2. THE EFFECT OF THE TEETH GEOMETRY AND THE LOAD DISTRIBUTION ONTO TOOTH ROOT STRESS

For analyzing the stress condition at the gear tooth root, the tooth is approximated with the console shaped mechanical model, on the end of which the load acts in the direction of the teeth profile pressure line. On the basis of the analysis carried [5], the general expression for the tooth root stress, for random contact point on meshed teeth profiles, may be written in the form:

$$\sigma_{Fx} = \sigma_0 K_{\alpha x} Y_{Fx} Y_{Sx} \tag{1}$$

where:

$\sigma_0$ – part of stress which is constant during the contact period;

$K_{\alpha x}$ – load distribution factor for simultaneously meshed teeth pairs for any contact point;

$Y_{Fx}$ – tooth form factor for any contact point;

$Y_{S\alpha}$ – stress correction factor for any contact point.

The stress constant part for the tooth root stress of spur gears may be determined on the basis of the expression:

$$\sigma_0 = Y_\alpha \frac{F_t}{bm} K_A K_v K_\beta \tag{2}$$

where:

$F_t$ – nominal tangential load on reference circle;

$b$ – face width;

$m$ – module;

$K_A$ – application factor;

$K_v$ – dynamic factor;

$K_\beta$ – longitudinal load distribution factor;

$Y_\alpha$ – pressure angle on pitch circle factor.

The stress mathematical models at the tooth root are created under assumption that the tangential forces of the meshed gears are located on reference circles. Actually, they are located on pitch circles. On the basis of the introduced assumption, data on the tooth form and stress correction factors at the tooth root for contact at the tooth top (point A on Fig.1) may be presented in the form of tables. To keep the simplicity of determining these factors and at the same time to take into consideration the actual condition, the new factor is introduced the pressure angle on pitch circle factor:

$$Y_{\alpha} = \frac{\cos\alpha}{\cos\alpha_{w}}$$

where: $\alpha$ – flank angle of the basic rack;
$\alpha_w$ – pressure angle on pitch circle factor.

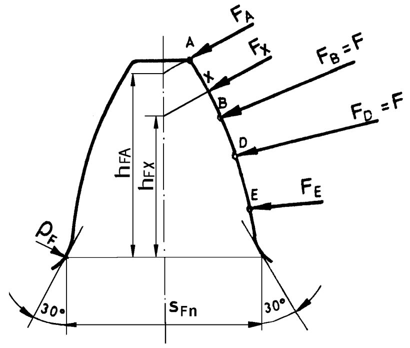



Fig. 1. Critical section, bending moment arm and contact points within which the load transfer

**2.1 The load distribution onto simultaneously meshed teeth pairs**

The load distribution analysis at the simultaneously meshed teeth pairs may be carried on the basis of the load distribution non-dimensional factor.

The load distribution factor at the simultaneously meshed teeth pairs is defined with the load ratio transmitted by the observed teeth pair at the certain contact moment ($F_x$) and the gear teeth pair total load ($F$):

$$K_{\alpha x} = \frac{F_x}{F} \tag{3}$$

The characteristic profile contact points of the simultaneously meshed teeth pairs in which the load transfer is carried out when during the contact period double and single mesh follow each other, ($1<\varepsilon_\alpha<2$) are shown on Fig.1.

From the load distribution aspect at the simultaneously meshed teeth pairs two characteristic cases may appear: the even load distribution and the explicitly uneven distribution.

At even load distribution all the simultaneously meshed teeth pairs participate evenly in teeth pair load transmission. The load distribution factor change flow, when during the contact period double and single mesh follow each other, is shown on Fig. 2. In the area of single mesh (area between B and D points on Fig. 2) the gear teeth pair load is transmitted through one teeth pair. The load distribution factor, in accordance with (3) equals one

$$K_{\alpha x} = 1{,}0 \qquad x \in (\mathrm{B...D})\,.$$

In the area of double mesh (area between A and B points and D and E points on Fig. 2) the gear teeth pair load is evenly transmitted through two teeth pairs. In accordance with (3), the load distribution factor in these areas, in case of even load distribution is:

$$K_{\alpha x} = 0{,}5 \qquad x \in (\mathrm{A...B}) \wedge x \in (\mathrm{D...E})\ .$$

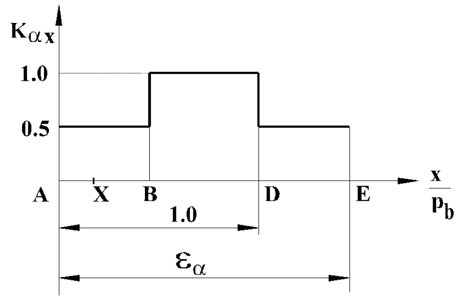



Fig. 2. Tooth load distribution at the single and duble mesh

The load distribution factor change flow, when during the contact period double and triple mesh follow each other, i.e. when $2<\varepsilon_\alpha<3$, is shown on Fig. 3. On the basis of this change the following analysis may be made

$$K_{\alpha x} = 1/2 \qquad x \in (\mathrm{B...D}) \wedge x \in (\mathrm{G...H})$$
$$K_{\alpha x} = 1/3 \quad x \in (\mathrm{A...B}) \wedge x \in (\mathrm{D...G}) \wedge x \in (\mathrm{H...E})$$

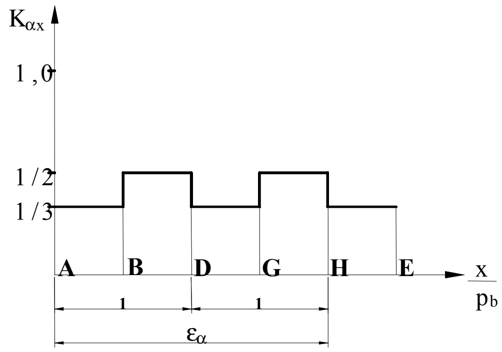



Fig. 3. Tooth load distribution at the duble and triple mesh

At explicitly uneven load distribution and in area of multiple meshes, the gear teeth pair load is transmitted only through one teeth pair. The other teeth pairs remain unloaded, due to impossibility of canceling the basic pitch differences at simultaneously meshed teeth pairs by flexible deformations. In that case the load distribution factor is

$$K_{\alpha x} = 1{,}0\ .$$

The actual load distribution is between this extreme, boundary load distribution. At this load distribution all the simultaneously meshed teeth pairs are engaged in transmitting the gear teeth pair load. In this, their engagement level depends on the gear teeth pair load intensity, rigidity and the meshed teeth basic pitch difference. The functional relation between these values and the load distribution factors may be established by observing the simultaneously meshed teeth pairs as the statically undetermined system. A sample of analytical expression for the load distribution factor in the area of double mesh, for the contact in A point (when $1<\varepsilon_\alpha<2$) is given in the form of [5]:

$$K_{\alpha A}=\frac{1}{1+\frac{C_{\mathrm{BD}}}{C_{\mathrm{AE}}}}\left(1+\frac{C_{\mathrm{BD}}\Delta p_b}{\frac{F_t}{b}}\right) \tag{4}$$

For the area of triple mesh ($2<\varepsilon_\alpha<3$), the analytical expression for the load distribution factor for the contact in A point (when the basic pitch absolute effective difference at the simultaneously meshed teeth pairs is not the same $\Delta p_b' \neq \Delta p_b''$), may be written in the form:

$$K_{\alpha A}=\frac{1}{1+\frac{C_{GB}+C_{DH}}{C_{AE}}}\left[1+\left(\frac{C_{GB}+C_{DH}}{F_t\,/\,b}\Delta p_b' - \frac{C_{GB}}{F_t\,/\,b}\Delta p_b''\right)\right] \tag{4a}$$

If we assume that it is $\Delta p_b' = \Delta p_b'' = \Delta p_b$, the expression (4a) gets the form:

$$K_{\alpha A}=\frac{1}{1+\frac{C_{GB}+C_{DH}}{C_{AE}}}\left(1+\frac{C_{DH}}{F_t\,/\,b}\Delta p_b\right) \tag{4b}$$

For the contact in B point the expression for the load distribution factor is given in the form:

$$K_{\alpha B}=\frac{1}{1+\frac{C_{DG}}{C_{BH}}}\left(1+\frac{C_{DG}}{F_t\,/\,b}\Delta p_b\right) \tag{4c}$$

where:

$\Delta p_b$ – diference of base pitch for simultaneously meshed pair of teeth;
$C_{GB}$; $C_{DH}$; $C_{BD}$ and $C_{AE}$ – stiffness for the meshed gear teeth for the contact points within which the load transfer.

When using the load distribution factor, take care about its maximum – physical actual value. Namely, if by the calculation we get that

$$K_{\alpha x}>1,\ \text{we have to use}\ K_{\alpha x}=1{,}0. \tag{5}$$

### 2.2 The teeth geometry analysis

The teeth profile shape is determined by the tool geometry, addendum modification coefficient and the gear teeth number. The effect of the teeth profile shape onto teeth root stress shall be taken into consideration through the tooth form factor. The analytical expression of the tooth form factor for the tooth profile random contact point is given in the form:

$$Y_{Fx}=\frac{6\left(\frac{h_{Fx}}{m}\right)\cos\alpha_x'}{\left(\frac{S_F}{m}\right)^2\cos\alpha}$$

where: $h_{Fx}$ – bending moment arm;
$\alpha_x$' – angle of application of load;
$S_x$ – tooth root chord at the critical section.

The gradual tooth profile fillet radius towards the gear body makes it difficult to find the location of the tooth root critical section. There are several suggestions for finding the tooth root critical section location. According to the American authors, the critical section location is determined with the contact points of tangents drown from the console tip to the tooth profile fillet radius. By approaching the console tip to the tooth root, the critical section location, changes so that the critical section surface increases. This procedure complicates the calculation a lot, as to each tooth profile contact point, to each console tip corresponds the critical section new location. The same problem exists with Russian authors as well, who instead of tangents draw the second order parabola with top on the console tip. The German authors determine the critical section location independently from the console tip location. According to them, the critical section location is determined by contact points of the tooth profile fillet radius and the tangents that with the tooth symmetry make an angle of 30º, Fig.1. In this case the analytical expression for determining the tooth thickness in the critical section is:

$$\frac{S_F}{m}=z\sin\left(\frac{\pi}{3}-\theta\right)+\left(\frac{G}{\cos\theta}-\frac{\rho_{a0}}{m}\right)\sqrt{3} \tag{6}$$

where:

$$G=\frac{\rho_{a0}}{m}+x-\frac{h_{a0}}{m}; \tag{6a}$$

$$\theta=H-\frac{2G}{z}\mathrm{tg}\theta; \tag{6b}$$

$$H=\frac{\pi}{3}-\frac{2}{z}\left(\frac{E}{m}-\frac{\pi}{2}\right); \tag{6c}$$

$$E=\frac{m\pi}{4}-h_{a0}\mathrm{tg}\alpha-\frac{\rho_{a0}}{\cos\alpha}(1-\sin\alpha); \tag{6d}$$

$h_{a0}$ – addendum of basic rack;
$\rho_{a0}$ – root fillet radius of the basic rack;
$x$ – addendum modification coefficient.

The console tip, by which the tooth is approximated, is determined by the bending moment arm. This analytical expression for the bending moment arm in any contact point is:

$$\frac{h_{Fx}}{m}=\frac{1}{2}\left[z\left(\frac{\cos\alpha}{\cos\alpha_x}-\cos(\frac{\pi}{3}-\theta)\right)+\frac{G}{\cos\theta}+\frac{\rho_{a0}}{m}\right]. \tag{7}$$

The effect of the tooth root stress concentration, normal stresses due to pressing, and tangential stresses due to shear to the tooth root total stress, shall be taken into consideration through the stress correction factor, The factor values may be get by theoretical – experimental procedure. According to the International Standardization Organization (ISO) Standard the stress correction factor is determined with expression:

$$Y_{Sx} = \left(1{,}2 + 0{,}13\frac{S_F}{h_{Fx}}\right)q_S^p \tag{8}$$

where:

$$p = \frac{1}{1{,}21 + 2{,}3\frac{h_{Fx}}{S_F}};$$

$$q_S = \frac{1}{2}\frac{S_F}{\rho_F};$$

$$\rho_F = \frac{\rho_{a0}}{m} + \frac{2G^2}{\cos\theta(z\cos^2\theta - 2G)}$$ - tooth-root fillet radius at the critical section.

**2.3 Relative stress factor**

To understand the effect of the teeth geometry and the teeth profile force point location onto stress at the tooth roots, the following assumption is introduced:

$$K_{\alpha A} = K_{\alpha B} = .... = K_{\alpha x} = ... K_{\alpha E}$$

i.e. let us suppose that all the meshed teeth profile points, during the contact period participate equally in transmitting the gear teeth pair load. By this assumption the load distribution effect onto stress at the tooth root is eliminated.

On the basis of the introduced assumption and the expression (1), the stress ratio at the tooth root for the contact in any "x" point and for the contact in A point, may be written in the form:

$$Y_{\varepsilon x} = \frac{\sigma_{Fx}}{\sigma_{FA}} = \frac{Y_{Fx}Y_{Sx}}{Y_{FA}Y_{SA}} \tag{9}$$

With these stress ratios the relative stress factor is defined. This factor achieves the maximum value for the contact in the A point (top of the tooth):

$$Y_{\varepsilon\max} = Y_{\varepsilon A} = 1{,}0 \tag{10}$$

The relative stress factor depends on the meshed gear teeth profile shapes. According to definition (9), the following exact expressions for the relative stress factor for the meshed gear teeth, for the contact points within which the load transfer is carried on may be written

$$Y_{\varepsilon xi} = \frac{Y_{Fxi}Y_{Sxi}}{Y_{FAi}Y_{SAi}} \tag{11}$$

where: $i$ = 1;2, usual asignment of small gear is "1" and of great gear – "2"; $x$ = A, B, D,…, E.

To determine the relative stress factors for the characteristic contact points on the meshed teeth profiles according to (11), the knowledge of the $Y_{FAi}$ $Y_{SAi}$ , $Y_{Fxi}$ and $Y_{Sxi}$ factor values in the same points is required, by solving the equation (6), (6a), (6c), (6d), (7), (8) and transcendental equation (6b). The numerical values of the $Y_{FAi}$, $Y_{SAi}$ factor values can be found in the literature in the form of tables or diagrams. For the other characteristic contact points ($x$) due to a number of varying values (teeth number, addendum modification coefficient and the transverse contact ratio) on which the $Y_{Fxi}$ and $Y_{Sxi}$ factors depend, their numerical values cannot be presented in the form of tables and diagrams, but should be determined by solving the mentioned equations. This procedure of determining the relative stress factor is quite complex and is justified at highly responsible gear teeth pairs, and is required when researching the operating and the critical tooth root condition and optimizing the gear teeth pairs. For the purpose of decreasing the calculation complexity level when determining the relative stress factor, the exact model at less responsible gear teeth pairs may be simplified by introducing the following approximations:
– the relative stress factor equality for the small and large gear teeth ($Y_{\varepsilon x1}=Y_{\varepsilon x2}=Y_{\varepsilon x}$) and
– the relative stress factor linear change during the contact period.

On the basis of these approximations, the approximate expressions for the relative stress factor within the characteristic contact points may be created. For the contact in B point, when $1<\varepsilon_\alpha<2$, the approximate expression follows:

$$Y_{\varepsilon B} \approx \frac{\varepsilon_\alpha - 1}{\varepsilon_\alpha}\left(1 - Y_{\varepsilon E}\right) \tag{12}$$

and when $2<\varepsilon_\alpha<3$

$$Y_{\varepsilon B} \approx 1 - \frac{\varepsilon_\alpha - 2}{\varepsilon_\alpha}\left(1 - Y_{\varepsilon E}\right) \tag{13}$$

If the assumption is that $Y_{\varepsilon E} \approx 0{,}25$, Eq.(12) and Eq.(13) may be written in the following form:

$$Y_{\varepsilon B} \approx 0{,}25 + \frac{0{,}75}{\varepsilon_\alpha} \tag{14}$$

$$Y_{\varepsilon B} \approx 0{,}25 + \frac{1{,}5}{\varepsilon_\alpha} \tag{14a}$$

In case that $Y_{\varepsilon E} \approx 0$, Eq.(12) and Eq.(13) get the form:

$$Y_{\varepsilon B} \approx \frac{2}{\varepsilon_\alpha} \tag{15}$$

$$Y_{\varepsilon B} \approx \frac{1}{\varepsilon_\alpha} \tag{15a}$$

In the expert and scientific literature we can find the approximate expressions (14) and (15) and terms "contact ratio factor ", "force arm factor", but not the exact expressions and terms based on the fundamental meaning of these factors. The first term is the consequence of the explicit function of the teeth profile meshing level approximate expressions, and the other one is the consequence of the fact that the $Y_{\varepsilon B}$ factor depends on the teeth bending moment arm value.

## 3. THE STRESS AUTHORITATIVE FACTOR

When the total load of the gear teeth pair is transmitted by engaging the larger number of the teeth pair numbers, then during the contact period the maximum stress at the tooth root may appear in A point or B point [5]. On the basis of the general expression (1) we may determine the stress at the tooth root for the contact in these points:

$$\sigma_{FAi} = \sigma_0 K_{\alpha A} Y_{FAi} Y_{SAi} \tag{16}$$

$$\sigma_{FBi} = \sigma_0 K_{\alpha B} Y_{FBi} Y_{SBi} \tag{17}$$

where:

$Y_{FAi}$, $Y_{SAi}$ – tooth form factor and stress correction factor for contact in A point;
$Y_{FBi}$, $Y_{SBi}$ - tooth form factor and stress correction factor for contact in B point;
$K_{\alpha A}$, $K_{\alpha B}$ – load distribution factor for contact in A point and B point.

To make the criterion for finding the contact points on the meshed teeth profiles (A or B point), for determining the stress relevant for checking the teeth volume strenght, the stress ratio for the contact in these points is observed:

$$K_{F\alpha i} = \frac{\sigma_{FAi}}{\sigma_{FBi}} = \frac{Y_{FAi} Y_{SAi}}{Y_{FBi} Y_{SBi}} \cdot \frac{K_{\alpha A}}{K_{\alpha B}} \tag{18}$$

By these factor ratio the relevant stress factor is defined.

On the basis of the relevant stress factor (9) and Eq.(18) factor definition the following expressions for the meshed gear teeth relevant stress factor appear:

$$K_{F\alpha i} = \frac{K_{\alpha A}}{K_{\alpha B}} \cdot \frac{1}{Y_{\varepsilon Bi}} \tag{19}$$

By the expression (18) analysis the exact criteria for identifying the contact points on the meshed teeth profiles may be created for determining the stress relevant for checking the tooth root strenght.

On the basis of the boundary conditions (5) and the expression (18) the maximum – physically actual $K_{F\alpha}$ factor value follows. If on the basis of (19) we get that it is $K_{F\alpha i} > \frac{1}{Y_{\varepsilon Bi}}$ we must take

$$K_{F\alpha i} = \frac{1}{Y_{\varepsilon Bi}} \tag{20}$$

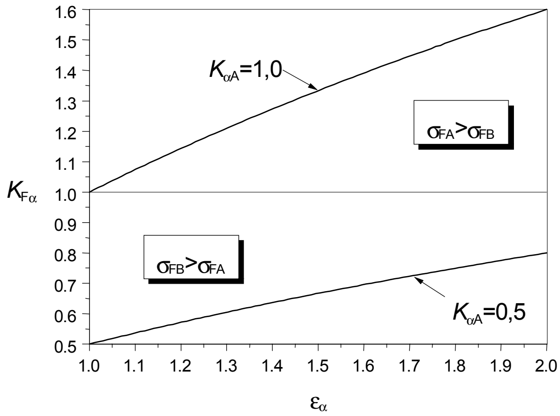
In the literature, the term "transverse load distribution factor" is regularly used for the $K_{F\alpha}$ factor. On the basis of the conducted analysis, it follows that this term is not based on the fundamental meaning of this factor, and is the consequence of its dependence on the load distribution at the simultaneously meshed teeth pairs.

To determine the tooth root stress relevant for checking the tooth volume strenght it is required:

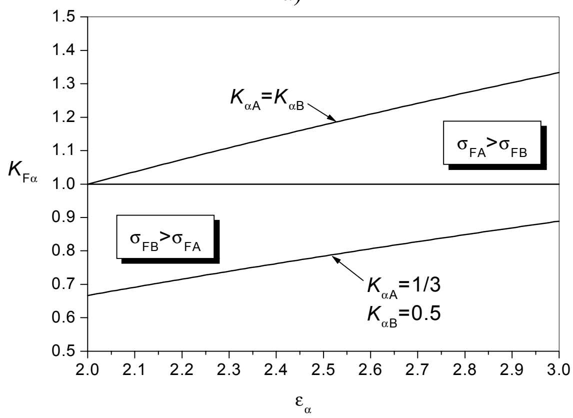
a) according to (19), to conclude which of the contact points (A or B) is relevant for determining the tooth root stress and

b) according to (16) and (17), to determine the stress relevant for the meshed gear teeth.

For this analysis, it is required to determine the $K_{\alpha A}$ factor according to (4),or (4a) and (4b) depending on the number of the simultaneously meshed teeth pairs and the required calculation accuracy, and the $K_{\alpha B}$ factor according to (4c) (when $2<\varepsilon_\alpha<3$), the stress constant part $\sigma_0$ according to (2) and the relative stress factor $Y_{\varepsilon Bi}$ by the procedure described under item 2.3.




a)




b)

Fig.4. a) The ranges of the tooth root stress at the single and duble mesh;
b) The ranges of the tooth root stress at the duble and triple mesh.

On the basis of the boundary load distribution and the approximate expressions (14) and (14a) the areas of allowable solutions for the tooth root stress are shown, diagram on Fig.4. When during the contact period the single and double mesh follow each other (Fig.4a), the area of allowable solutions is larger than the area corresponding to double and triple mesh follow (Fig.4b). According to diagram on Fig.4, we may carry out the analysis of the load distribution effect in the relatively simple way the profile mesh level transverse contact ratio value of the tooth root stress condition.

## 4. CONCLUSION

The great number of stress effective factors, at the tooth root makes difficult the model making for the relatively simple determination of the stress, relevant for checking the teeth operating capability from the volume strenght aspect. On the basis of the carried analysis, it is shown that in up to now development of these models some of the relevant factors have lost their fundamental meaning. The terms "meshing level factor", "load distribution face factor" does not correspond to their original meaning. For the purpose of forming the exact mathematical model, the relative and the relevant stress factors are defined. Their boundary values are determined and the exact expressions for the stress condition precise analysis at the tooth root are given. Also, the appropriate approximate expressions and assumptions under which they are valid are given. To eliminate the assumption on the tangent force action on the meshed gear reference circles, the contact point angle factor is introduced.

The created model represents the basis for further development of the calculation of tooth strenght from the aspect of geometry, load distribution and the teeth manufacturing accuracy and for optimization of gear trains.

## REFERENCES


1. Buckingham, E.: Analytical mechanics of gears, 1963
2. eriga, S.: Machine part III, Gear transmision, Belgrade, 1993
3. Nieman, G., Winter, H.: Maschinenelemente, Band II, Berlin, 1983
4. Airapetov, E. A.: Improving of methods for calculation of gear transmission capacity, Tehnika mashinostroenia, №2, pp. 8-33, 2000, (in Russian)
5. Ristivojevic, M.: Analysis of load distribution on simultaneously conjugated pair of teeth with application on root strenght of tooth of the gear, M.Sc.Thesis, Faculty of Mechanical Engineering, Belgrade, 1985
6. Ristivojevic, M.: Physical base for definition of the "transverse load distribution factor" of cylindrical spur gears, International Conference on Engineering Design ICED, pp. 1881-1888, Dubrovnik, 1990
7. Costopoulos, T.: Gear modelling and tooth stress, International Congress – Gear Transmissions, Sofia, 1995
8. Albert, M.: Berechung der Zahnfuβtragfähigkeit – ein schwieriges Normungsproblem, Konstruktion, №39, pp. 447-455, 1987
9. Winter, H., Hohn, B.: New ISO standards for the calculation of gear load capacity, International Gearing Conference, Newcastle, 1994
10. ISO/TC60/SC2, 2000.


# RAZVOJ TEORIJSKOG MODELA ZA ODREĐIVANJE NAPONA MERODAVNOG ZA PROVERU ZAPREMINSKE ČVRSTOĆE ZUPCA ZUPČANIKA


**Mileta Ristivojević**



*U radu je analiziran uticaj geometrije zubaca i raspodele opterećenja kod istovremeno spregnutih pari zubaca na napon u podnožju zupca, kada se u toku dodirnog perioda smenjuju jednostruka i dvostruka i dvostruka i trostruka sprega. Na osnovu analitičkih istraživanja formirani su egzaktni i približni matematički modeli za određivanje radnog i kritičnog napona u podnožju zupca merodavnih za proveru zapreminske čvrstoće zupca zupčanika.*