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ON CHARACTERISTICS OF NONLINEAR PARAMETERS FOR ANGULAR BODY ORIENTATION

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Abstract. *The objective of this work is to present comparative results a number of existing nonlinear angular body orientation parameters and a couple the new (recently given in literature) as to their efficiency in actual computer simulations. Comparations are made due to established unique criteria not existing up to now. Some sets of parameters that describe angular body orientation (examples of such parameter sets are Euler angles, Krilov angles, Euler and Rodrigues-Hamilton parameters, Direct cosines, quaternions, etc.), are quoted earlier for applications, but their application is not exemined for flight dynamics purposes. The sets of angular body parameters differ in various respects such as presence of singularities, trigonometric or purely algebric functions, number of additional constraint equations, feasibility for computer implementation, etc. Results have shown that kinematic differential equations should be used for numerical simulations of nonlinear angular body orientation parameters and not the suggested quaternion equations. The fact is that quaternions have no singular-pathological points which should be taken care about numerical simulations as it is case in some of the analyzed kinematic differential equations.*

INTRODUCTION

In rigid body dynamic applications a description of a position vector in space is needed. The position vector, the knowledge of which is required for Newton´s second law of motion and subsequent weighted residual methods, may be described using a vector giving the location of the origin of a dynamic reference frame, a position vector in that frame, and a transformation matrix T^{BA} between the inertial A and the dynamic B frame. In order to specify the angular orientation of a rigid body in a vector base $e^{(A)}$ it is sufficient to specify the angular orientation of a vector base $e^{(B)}$ which is rigidly attached to the body. This can be done, for instance, by means of the direction cosine matrix T^{BA}:

$$e^{(B)} = T^{BA} e^{(A)} \tag{1}$$

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The nine elements of matrix T^{BA} are generalized coordinates which describes the angular orientation of the body in base $e^{(A)}$. Between these coordinates there exist six constraint equations of the form

$$\sum_{j=1}^{j=3} a_{ij} = 1; \ (i,j = 1,2,3); \quad \sum_{j=1}^{j=3} a_{ij} a_{kj} = 0; \quad (i,k = 1,2,3, \ i \neq k) \tag{2}$$

It is often inconvenient to work with nine matrix coordinates and six constraint equations. There are several useful parameter systems of three coordinates without constraint equations and four coordinates with one constraint equation which can be used as alternatives of direction cosines.

Significant computing time can be saved (up to 40%) if transformation matrix T^{BA} components a_{ij} has simple form. This is significant where the main computational effort is spent in updating the attitude itself, by a runing integration for the angular orientation time history of a body subjected to rapid irregular angular rates ω Although this advantage is offset in other sittuations requiring frequent transformation of vector components (1), there are enough sets of four-parameter applications that warrant further development alternatives of direction cosines matrix.

Relationship between the angular velocity of a body ω and parameters describing the angular orientation of the body cannot be determined from $\omega_i(t)$, $(i = 1,2,3)$ by simple integration. Instead, differential equations must be solved in which $\omega_i(t)$ appear as variable coefficients. These kinematical differential equations can be formulated for every set of angular orientation parameters of a rigid body.

A number of sets of nonlinear parameters (whose numerical values can be obtained as solutions of apropriate differential equation) for transformation matrix have been previously developed and given in literature. Examples of such sets are Euler angles, Krilov angles, Euler parameters, Direct cosines, Rodrigues parameters (quaternions), Tait-Bryan angles (roll-pitch-yaw-rotations), parameters developed by Argyris, etc. Descriptions of these sets are found in [1,2,3,4] and also in [5,6,7,8] quoted earlier for applications. The sets of parameters differ in various respects such as presence of singularities (pathological cases), trigonometric or purely algebric functions, redundancy (a certain number of accompanying constraint equations), feasibility for computer implementation, etc. The objective of this work is to compare some sets of existing angular orientation parameters as to their efficiency in actual computer simulations.

RODRIGE-HAMITON PARAMETERS - QUATERNIONS

Rodrige-Hamiton parameters - quaternions $\Lambda(\lambda_0,\lambda_1,\lambda_2,\lambda_3)^T$ is four-parameter quantity having two parts $\Lambda(\lambda_0,\lambda_1 e_1,\lambda_2 e_2,\lambda_3 e_3) = \Lambda(\lambda_0,\lambda)$ where vector part is $\lambda = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$ and scalar is λ_0 (when $\lambda_0 = 0$, quaternion is ordinary vector). A special algebra is developed for quaternions, for example, inverse quaternion is $\underline{\Lambda}(\lambda_0,-\lambda)$. Main qaternion differential equation is:

$$\dot{\Lambda} = \Lambda + \frac{1}{2}\Lambda \circ \Omega \tag{3}$$

where Ω - is quaternion expressin of angular velocity ω, $\Omega = (0,\omega)$ and o sign for quaternion product.

ROTATION OPERATOR EQUATIONS

General procedure for obtaining various sets of differential equations whose solutions gives angular orientation parameters is given in [9,10]. The description with a rotation vector x hence uses three independent parameters and describes a rotation by it´s natural axis and angle of rotation. These vector equations are:

$$x = \Re\omega;\ \ \omega = \Re^{-1}x;\ \ \Re = \mathbf{x}'E - \frac{1}{2}X + \delta X^2;\ \ \Re^{-1} = \frac{1}{\mathbf{x}'}E + \beta X + \gamma X^2 \tag{4}$$

Where in (4) are: $\Re$ -linear orthogonal rotation operator ($\Re^{-1}$ inverse); X - operator given as $X = \mathbf{x}U(e)$ where U - skew symmetric vector product operator dimension 3 × 3; E - unit matrix dimension 3 × 3; and expressions for β,γ,δ and module of vector $x, \mathbf{x}, \mathbf{x}', |\dots|$ are given below. Vector equations (4) for x after inserting $\Re$ in it are:

$$\overset{\bullet(*)}{x} = \mathbf{x}'\omega \pm \frac{1}{2}x \times \omega + \delta x \times (x \times \omega) \tag{5}$$

$$\overset{\bullet(*)}{x} = \frac{\mathbf{x}}{2}ctg\frac{\chi}{2}\omega \pm \frac{1}{2}x \times \omega + \delta(x\omega)x \tag{6}$$

$$\overset{\bullet(*)}{x} = \frac{\mathbf{x}}{2}ctg\frac{\chi}{2}\omega \pm \frac{1}{2}x \times \omega + \delta\{xx\} \tag{7}$$

$$\omega = \frac{1}{\mathbf{x}'}\overset{\bullet(*)}{x} \mp \frac{1}{2}\beta x \times \overset{\bullet(*)}{x} + \gamma x \times \left(x \times \overset{\bullet(*)}{x} \right) \tag{8}$$

$$\omega = \alpha \overset{\bullet(*)}{x} \mp \frac{1}{2}\beta x \times \overset{\bullet(*)}{x} + \gamma x \left(x \overset{\bullet(*)}{x} \right) x \tag{9}$$

$$\omega = \alpha \overset{\bullet(*)}{x} \mp \frac{1}{2}\beta x \times \overset{\bullet(*)}{x} + \gamma\{xx\} \overset{\bullet(*)}{x} \tag{10}$$

$$\overset{\bullet}{\mathbf{x}} = \frac{\mathbf{x}'(x\omega)}{\mathbf{x}};\ \ \mathbf{x}' = \frac{d\mathbf{x}}{d\chi};\ \ \chi = \int_{t_0}^{t} \omega dt \tag{11}$$

$$\alpha = \frac{1}{\mathbf{x}}\sin\chi;\ \ \beta = \frac{1}{\mathbf{x}^2}(1 - \cos\chi) \tag{12}$$

$$\gamma = \frac{1}{\mathbf{x}'\mathbf{x}^2}(1 - \mathbf{x}'\alpha);\ \ \delta = \frac{1}{\mathbf{x}^2}\left(\mathbf{x}' - \frac{\mathbf{x}}{2}ctg\frac{\chi}{2} \right) \tag{13}$$

where: $(x\omega)$, $(x \times \omega)$, $\{xx\}$, are scalar, vector and dyad product (out of date terminology, adequate is dual which is for two vectors a and b scalar $c = a_1b_1 + a_2b_2 + a_3b_3$).

Equations (5-13) are valid for both reference frames, i.e. inertial and dynamic. Local derivatives in dynamic reference frame are signed with point and absolute derivatives in inertial reference frame with -*-. For example, in (5) for absolute derivative sign is - and for local +. From general equations (5-13), particular can be obtained after inserting various expressions for modules of vector x – $\mathbf{x}$, |...|, for example:

$$\begin{aligned}&\mathbf{x}=\mathbf{e}=1;\ \mathbf{x}=|\chi|;\ \mathbf{x}=|\vartheta|=k_\vartheta tg\frac{\chi}{2};\ \mathbf{x}=|\upsilon|=k_\upsilon tg\frac{\chi}{2}\\ &\mathbf{x}=|\tau|=k_\tau tg\frac{\chi}{4};\mathbf{x}=|\rho|=k_\rho tg\frac{\chi}{4}\end{aligned}\tag{14}$$

where $k_\mu(\mu=\chi,\vartheta,\upsilon,\tau,\rho)$ are real positiv arbitrary constants. Inserting (14) in (5-13), we obtain set of nonlinear differential equations for angular body orientation:

1. $\mathbf{x}=\mathbf{e}=1;\mathbf{e}'=0$

$$\overset{\bullet(*)}{e}=\frac{1}{2}ctg\frac{|\chi|}{2}\omega\pm\frac{1}{2}e\times\omega-\frac{1}{2}ctg\frac{|\chi|}{2}(e\omega)e\quad -POJ\tag{15}$$

2. $\mathbf{x}=\chi=|\chi|e;\ \ |\chi|'=1$

$$\overset{\bullet(*)}{\chi}=\frac{|\chi|}{2}ctg\frac{|\chi|}{2}\omega\pm\frac{1}{2}\chi\times\omega+\frac{1}{|\chi|^2}\left(1-\frac{|\chi|}{2}ctg\frac{|\chi|}{2}\right)(\chi\omega)\chi\quad -VOR\tag{16}$$

$$\overset{*}{\chi}=\frac{1}{|\chi|}(\chi\omega)=e\omega$$

3. $x=\vartheta=|\vartheta|e;\ \ |\vartheta|'=\frac{1}{2}\left(k_\vartheta+\frac{1}{k_\vartheta}|\vartheta|^2\right);\ \ ctg\frac{|\chi|}{2}=\frac{k_\vartheta}{|\vartheta|}-VKR$

$$\overset{\bullet(*)}{\vartheta}=\frac{k_\vartheta}{2}\omega\pm\frac{1}{2}\vartheta\times\omega+\frac{1}{2k_\vartheta}(\vartheta\omega)\vartheta\tag{17}$$

$$\overset{*}{\chi}=\frac{1}{2}\left(\frac{k_\vartheta}{|\vartheta|}+\frac{|\vartheta|}{k_\vartheta}\right)(\vartheta\omega)=\frac{1}{2}\left(k_\vartheta+\frac{1}{k_\vartheta}|\vartheta|^2\right)(e\omega)$$

4. $x=\upsilon=|\upsilon|e;\ \ |\upsilon|'=-\frac{1}{2}\left(k_\upsilon+\frac{1}{k_\upsilon}|\upsilon|^2\right);\ \ ctg\frac{|\chi|}{2}=\frac{|\upsilon|}{k_\upsilon}-KAP$

$$\overset{\bullet(*)}{\upsilon}=\frac{1}{2k_\upsilon}|\upsilon|^2\ \omega\pm\frac{1}{2}\upsilon\times\omega-\left(\frac{k_\upsilon}{2|\upsilon|^2}+\frac{1}{k_\upsilon}\right)(\upsilon\omega)\upsilon\tag{18}$$

$$\overset{*}{\upsilon}=\frac{1}{2}\left(\frac{k_\upsilon}{|\upsilon|}+\frac{|\upsilon|}{k_\upsilon}\right)(\upsilon\omega)=-\frac{1}{2}\left(k_\upsilon+\frac{1}{k_\upsilon}|\upsilon|^2\right)(e\omega)$$

5. $x = \tau = |\tau| e;\ \ |\tau|' = \frac{1}{4}\left(k_\tau + \frac{1}{k_\tau}|\tau|^2\right);\ \ ctg\frac{|\chi|}{2} = \frac{1}{2}\left(\frac{k_\tau}{|\tau|} - \frac{|\tau|}{k_\tau}\right) - NEW(1)$

$$\overset{\bullet(*)}{\tau} = \frac{1}{4}\left(k_\tau - \frac{1}{k_\tau}|\tau|^2\right)\omega \pm \frac{1}{2}\tau\times\omega + \frac{1}{2k_\tau}(\tau\omega)\tau \tag{19}$$

$$\overset{*}{\tau} = \frac{1}{4}\left(\frac{k_\tau}{|\tau|} + \frac{|\tau|}{k_\tau}\right)(\tau\omega) = \frac{1}{4}\left(k_\tau + \frac{1}{k_\tau}|\tau|^2\right)(e\omega)$$

6. $x = \rho = |\rho| e;\ \ |\rho|' = -\frac{1}{4}\left(k_\rho + \frac{1}{k_\rho}|\rho|^2\right);\ \ ctg\frac{|\chi|}{2} = -\frac{1}{2}\left(\frac{k_\rho}{|\rho|} - \frac{|\rho|}{k_\rho}\right) - NEW(2)$

$$\overset{\bullet(*)}{\rho} = -\frac{1}{4}\left(k_\rho - \frac{1}{k_\rho}|\rho|^2\right)\omega \pm \frac{1}{2}\rho\times\omega - \frac{1}{2k_\rho}(\rho\omega)\rho \tag{20}$$

$$\overset{*}{\rho} = -\frac{1}{4}\left(\frac{k_\rho}{|\rho|} + \frac{|\rho|}{k_\rho}\right)(\rho\omega) = -\frac{1}{4}\left(k_\rho + \frac{1}{k_\rho}|\rho|^2\right)(e\omega)$$

SIMULATION RESULTS

Results of performed numerical integration of differential equations (3,15-20) for arbitrary choosen ω are given in following table where function $f(t)$ has next form:

$$f(t) = [\sin(1000t + 1),\ \ 2\sin(t + 2),\ \ 3\sin(0.001t + 3)]^{\mathrm{T}} \tag{21}$$

In above table are: k - number of integration steps, f - number of function evaluations per step, cpu - central procesor unit time, Δt_p - initial integration step size (minimal Δt_m during integration steps was greater or equal $\Delta t_m = \Delta t_p/100$).

Next subroutines are coded [7] for differential equations (3,15-19) signed as: ROH - (3), POJ -(15), VOR-(16), VKR -(17), KAP -(18), NEW -(19) (TOL□0.1E-6 -relative precision tolerance used as stopping criteria during numerical integration in all subroutines). Obtained results for the number of existing nonlinear angular body orientation parameters and the new (NEW recently given in literature) can be compared as to their efficiency in actual computer simulations due to established unique criteria not existing up to now. This results clearly shows the effect of preferred axes for various angular orientation parametres, and also the effect of the increase of the magnitude of ω on the quaternions. From given table we can see that execution time (cpu) for VKR angular orientation parametres for all angular velocities ω is less for 67% from time necessary for execution quaternions -ROH. Results have shown that kinematic differential equations should be used for numerical simulations of nonlinear angular body orientation parameters and not the suggested quaternion equations. The fact is that quaternions have no singular-pathological points which should be taken care about numerical simulations as it is case in some of the analyzed kinematic differential equations.

Numerical integration results

ω	subr.	k	f	cpu	Δt_p
$2\pi\cdot\begin{bmatrix}1\\0\\0\end{bmatrix}$	ROH	24	199	0,078	0,1E-3
	NOV	26	222	0,077	0,1E-3
	KAP	24	206	0,071	0,1E-3
	VKR	25	207	0,074	0,1E-3
	VOR	25	207	0,076	0,1E-3
	POJ	25	252	0,111	0,1E-3
$2\pi\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}$	ROH	39	319	0,135	0,1E-4
	NOV	40	334	0,124	0,1E-4
	KAP	43	379	0,142	0,1E-4
	VKR	40	334	0,125	0,1E-4
	VOR	40	334	0,128	0,1E-4
	POJ	46	438	0,196	0,1E-4
$2\pi\cdot\begin{bmatrix}10\\10\\10\end{bmatrix}$	ROH	281	2381	1,019	0,1E-6
	NOV	173	1998	0,526	0,1E-6
	KAP	187	1503	0,558	0,1E-6
	VKR	151	1222	0,470	0,1E-6
	VOR	151	1222	0,477	0,1E-6
	POJ	327	3036	1,390	0,1E-6
$2\pi\cdot\begin{bmatrix}10\\100\\1000\end{bmatrix}$	ROH	17063	152534	65,160	0,1E-7
	NOV	12536	112027	42,738	0,1E-7
	KAP	13630	122109	46,576	0,1E-7
	VKR	10651	95512	36,909	0,1E-7
	VOR	10651	95512	37,880	0,1E-7
	POJ	18139	169892	77,926	0,1E-7
$2\pi\cdot f(t)$	ROH	646	6519	5,041	0,1E-6
	NOV	728	7245	5,292	0,1E-6
	KAP	773	7766	5,665	0,1E-6
	VKR	762	7629	5,631	0,1E-6
	VOR	762	7629	5,706	0,1E-6
	POJ	796	8734	7,029	0,1E-6

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KARAKTERISTIKE NELINEARNIH PARAMETARA ZA DEFINISANJE UGLOVNOG POLOŽAJA TELA

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Istraživanje osobina nelinearnih parametara za definisanje uglovnog položaja tela u prostoru nije sveobuhvatno vršeno zbog nepostojanja kriterijuma za ocenu njihove efikasnosti (primeri grupa takvih parametara su Euler-ovi uglovi, Krilov-ljevi uglovi, Euler-ovi i Rodrigues-Hamilton-ovi parametri, Kosinusi pravaca, kvaternioni, itd.). Definisan je kriterijum za ocenu efikasnosti kada se parametri koriste za numeričke proračune dinamike tela čime su stvoreni uslovi da se kvalitativno analiziraju različite grupe parametara i njihovih kinematičkih diferencijalnih jednačina i poredi njihova efikasnost u pogledu vremena potrebnog za kompjutersku simulaciju. Izloženi su rezultati analize koji ukazuju na to da za proračun uglovnih parametara treba koristiti jednu grupu kinematičkih diferencijalnih jednačina, a odbaciti, radi uštede u vremenu potrebnom za kompjutersku simulaciju, preporučivane kvaternione jednačine. Ostaje činjenica da kvaternioni nemaju singularnih-patoloških tačaka o kojima se mora voditi računa prilikom numeričke integracije kinematičkih diferencijalnih jednačina kao što je slučaj kod nekih od analiziranih jednačina za dobijanje uglovnih parametara.