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SOLUTION OF THE DIRECT PROBLEM IN THEORY OF FLOW THROUGH STRAIGHT PLANE PROFILE CASCADE BY USING CONFORMAL MAPPING INTO BAND $-\pi/2 \leq \text{Im}\xi \leq \pi/2$

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***Abstract**. In this paper, the mapping nature of flow around the profile of a straight plane cascade into band flow $-\pi/2 \leq Im\xi \leq \pi/2$ with simetrically distributed singular points in $\xi = \pm k$, where k is a real number depending on geometric parameters of cascade, has been amalyzed. According to angles of flow at inlet and outlet of cascade as well as geometric parameters of cascade profiles, nine characteristic situations can occure, among them four belong to the group of basic mapping and five to the group of random mapping.*
According to the character of variation of the velocity potential along the band contour one can conclude that the whole contour is mapped into finite part of band, so that the infinite reach of band and the decaying conformity of mapping in infinity can't make troubles in the solution of problem. The Schwartz-integrals forming the mathematical model, can be reduced to the forms with finite boundaries.

1. INTRODUCTION

Direct problem in the theory of flow trough straight plane profile cascade is the task of the determination the flow through a given profile cascade for a given position of front stagnant point on profile (S_{01}) and for a given value of flow velocity far in front of cascade (V_1). The problem can be considered as solved if: the velocity distribution along the profile contour ($V(s)$), the value of velocity far behind the cascade (V_2) and the angles of flow direction far before (α_1) and far behind (α_2) the profile cascade, have been determined.

Method of conformal mapping enables the solution of the above problem according to the model of potential flow of incompressible fluid. In order to avoid the aglomeration of the profile contour mapped points it is recommended as a canonic region of mapping the usage of flow in the band $-\pi/2 \leq \text{Im}\xi \leq \pi/2$ with symetrically distributed singular points ($\xi = \pm k$) where the infinity far in front ($z = -\infty$) and far behind ($z = \infty$) of cascade is mapped.

Analysis of mapping nature; given in this paper, can show that the infinity of band and

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the decaying conformity of mapping in infinity do not make troubles in the solution of problem and that the closure of band by other contours, as it is recommended theoretically [1], is not necessary.

2. NATURE OF MAPPING OF PROFILE CONTOUR INTO BAND CONTOUR

By conformal mapping the flow between two neighboring congruentic streamlines in z-plane of profile cascade is mapped into flow in band $-\pi/2 \le \mathrm{Im}\xi \le \pi/2$, the profile contour ($s$) is mapped into upper $\xi_+(\xi = \xi + i\pi/2)$ and lower $\xi_-(\xi = \xi - i\pi/2)$ contour of band, while the infinite domains in front ($z = -\infty$) and behind ($z = \infty$) of profile cascade have been mapped into singular points $\xi = \pm k$ on real axis of ξ - plane. In singular point $\xi = \xi(z = -\infty) = -k$, the integrated source (Q) and vortex (Γ_1) are placed, while the integrated source ($-Q$) and vortex ($-\Gamma_2$) are placed in singular point $\xi = \xi(z = \infty) = k$, where:

$$Q = V_{1x}\, t = V_{2x}\, t, \qquad \Gamma_1 = V_{1y}\, t = Q\, \mathrm{tg}\alpha_1, \qquad \Gamma_2 = V_{2y}\, t = Q\, \mathrm{tg}\alpha_2 \tag{1}$$

t - spacing between profiles
α_1, V_{1x}, V_{2x} - angle of flow direction and velocity components in $z = -\infty$,
α_2, V_{1y}, V_{2y} - angle of flow direction and velocity components in $z = \infty$, for the case when y-axis of z-plane is in direction of profile cascade axis.

The value of real number k defining the position of singular points, as parameter of canonic domain, depends on geometry of mapped profile cascade. For boundary cases: $k = 0$ for a single profile ($t = \infty$) and $k \to \infty$ for infinite dense profile cascade ($t \to 0$).

The basic value of the velocity potential function on upper and lower contour of band is defined by relation (2):

$$\Phi o(\xi_\pm) = \frac{Q}{2\pi}\left[\ln\frac{ch(\xi_\pm + k)}{ch(\xi_\pm - k)} \pm tg\alpha_1 arcctgsh(\xi_\pm + k) \mp tg\alpha_2 arcctgsh(\xi_\pm - k)\right] \tag{2}$$

where the values of function *arcctg* are placed into interval ($0 \div \pi$) (in order to be continous functions $\Phi o(\xi_+)$ and $\Phi o(\xi_-)$ on ξ_+ and ξ_- contour of band).

For $\xi_\pm = \infty$ values of functions $\Phi o(\xi_+)$ and $\Phi o(\xi_-)$ are equal, while for $\xi_\pm = -\infty$ they differ between each other for $\Gamma_1 - \Gamma_2$.

Due to ambiguity of function *arcctg*, the other values of velocity potential function along band contour are:

$$\Phi n(\xi_\pm) = \Phi o(\xi_\pm) \mp \frac{\Gamma_1 - \Gamma_2}{2} n = \Phi o(\xi_\pm) \mp Q\frac{tg\alpha_1 - tg\alpha_2}{2} n, \qquad n = 0; \pm 1; \pm 2; ... \tag{3}$$

Positions of stagnant points on band contours, in which the front (S_{01}) and rear (S_{02}) stagnant points are mapped on profile contour, are defined by the following expressions:

$$\xi_{o+} = k + \ln\frac{sh2k(-1 \pm \sqrt{1+D})}{C}; \quad \xi_{o-} = k + \ln\frac{sh2k(1 \pm \sqrt{1+D})}{C}, \quad \text{for } C \ne 0 \tag{4}$$

where:

$$C = e^{2k} tg\alpha_2 - tg\alpha_1; \qquad D = \frac{2ch2k\, tg\alpha_1 tg\alpha_1 - (tg^2\alpha_1 + tg^2\alpha_2)}{sh^2 2k} \tag{4'}$$

For $C = 0$ ($tg\alpha_1 = e^{2k} tg\alpha_2$) the stagnant points are in:

$$\xi_{o+}(S_{01}) = k + \ln\frac{e^{2k} tg\alpha_1 - tg\alpha_2}{2e^{2k} sh2k} \quad \text{and} \quad \xi_{o\pm}(S_{02}) = \infty, \quad \text{for } (\alpha_1, \alpha_2) > 0$$

$$\xi_{o\pm}(S_{01}) = -\infty \quad \text{and} \quad \xi_{o-}(S_{02}) = k + \ln\frac{tg\alpha_2 - e^{2k} tg\alpha_1}{2e^{2k} sh2k}, \quad \text{for } (\alpha_1, \alpha_2) < 0 \qquad (5)$$

$$\xi_{o\pm}(S_{01}) = -\infty \quad \text{and} \quad \xi_{o\pm}(S_{02}) = \infty, \qquad \text{for } \alpha_1 = \alpha_2 = 0$$

According to the expressions (4) and (5) one can conclude thet, depending on the values of canonic region perameters (k), and on the flow direction angles in front (α_1) and behinde (α_2) the profile cascade, there are nine possible mapping forms shown in figure 1. a)÷i). Figures a), b), c) and d) show the four basic mapping forms, while figures e), f), g), h) and i) show the random mapping forms occuring when parameters k, α_1 and α_2 satisfy the functional relations of unambiguity (attached to the figures). By varying one of these parameters for a very small value, the random mapping forms are reduced to one of four basic mapping forms.

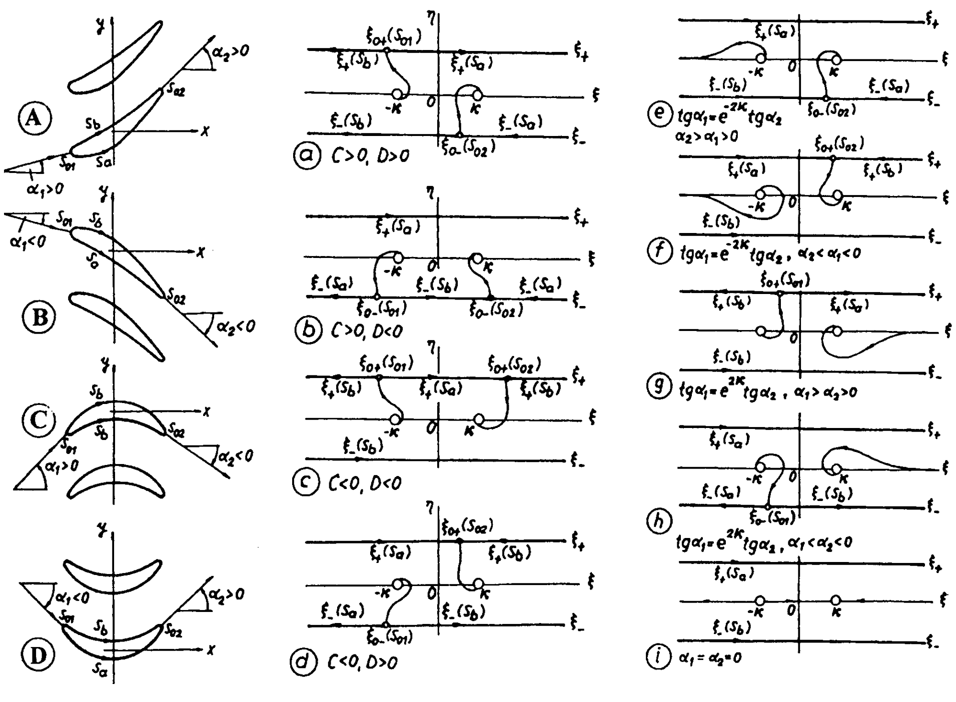

Fig. 1

The nature of *lnchx* and *arcctgshx* functions describing the velocity potential along the band contours, is such that with distancing from x = 0 the function *arcctgshx* tends quickly to zero (for $x > 0$) or to π (for $x < 0$), while *lnchx* function tends more quickly to $|x| - ln2$ function. By assuming that for $x = Xo(Xo > 0)$, with negligible error one can take that *arcctgsh*(–*X*o) = π, *arcctgsh X*o = 0, and *lnch*(±*Xo*) = *Xo* – *ln*2, the velocity potential functions along the band contours obtain constant values

$$\Phi o(\xi_\pm) = -\frac{Q}{\pi}k \pm \frac{Q}{2}(tg\alpha_1 - tg\alpha_2) = const., \qquad \text{for} \quad \xi_\pm \le -(k + Xo)$$

and

$$\Phi o(\xi_\pm) = \frac{Q}{\pi}k = const., \qquad \text{for} \quad \xi_\pm \ge k + Xo$$

it means, according to the nature of conformaly mapped flows that two points of profile contour are mapped into contours $\xi_\pm \le -(k + Xo)$ and $\xi_\pm \ge k + Xo$ with infinite stretching. Precisely said, the very small part of profile contour is mapped into contour $\xi_\pm \le -(k + Xo)$ that it can be treated as point on profile contour. The same is valid for mapping into contour $\xi_\pm \ge k + Xo$.

According to the above mentioned, the whole (entire) profile contour is mapping into contours of limited part of band $-(k + Xo) \le \xi_\pm \le k + X$o, while two points on profile contour, with infinite stretching, are mapped one into $\xi_\pm \le -(k + Xo)$ and the second one into $\xi_\pm \ge k + Xo$. On which sides of profile these two points will be placed it depends on the form of mapping (position of stagnant points on band contours).

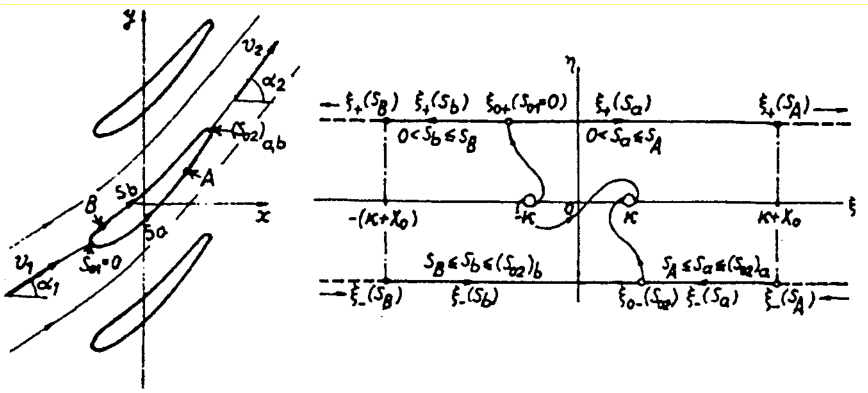

Fig. 2.

For the mapping form as in Fig. 1. a) one of these points is on S_a - side, and the other one on S_b- side of profile, as it is shown in Fig. 2. According to Fig. 2 the point A on S_a - side of profile is mapped into $\xi_\pm \ge k + Xo$, while the point B on S_b-side is mapped into $\xi_\pm \le -(k + Xo)$. For mapping form as in Fig.1.d) the points that are mapped with infinite stretching have their positions on different sides of profile, while for mapping forms as in Figures 1 b) and c) these points are on one side of profile. The special cases are: when the front stagnant point on profile (S_{01}) with infinite stretching, is mapped into $\xi_\pm \le -(k + Xo)$ (Fig. 1 e) and f)), when the rear stagnant point on profile (S_{02}), with infinite stretching, is mapped into $\xi_\pm \ge k + Xo$ (Fig. 1 g) and h)), and when both stagnant points are mapped, with infinite stretching.

As far as the value of parameter *X*o is concerned, which is defining the part of the band contour $-(k + Xo) \le \xi_\pm \le k + Xo$ where the whole contour of profile is practically mapped; its value can be found from the condition that arcctgsh($-Xo$) differs for a little from π. For $Xo = 5$ this difference is smaller than 0,5%, and can be treated as negligible. For $Xo = 5$, the value of *lnch*(Xo) function differs from $Xo-ln2$ function for about 0,001%.

3. System of equations for solving of problem

According to Zukovski-Chapligin the rear stagnant point on profile (S_{02}) is placed on outlet profile edge. For a fixed position of rear stagnant point, that is enough to determine by conformal mapping only one (all the same which one) flow through the given profile cascade. As a result of this solution one can obtain: parameter of canonic region (k) and law for correspodence between points on profile contour and band contour ($\xi_\pm(s)$) ; then the determination of other flows is very simple [3].

Since the velocity distribution along the profile contour has the best shape in the case of nonshocking incident flow at profile cascade, as basic problem this flow situation (the front stagnant point is on inlet profile edge) has been determined by using conformal mapping method. For given velocity values (V_1 or V_{1x}) in front of cascade it is necessary to find the flow direction angles in front (α_1) and behind (α_2) the cascade, the velocity value (V_2) behind the cascade and the velocity distribution along the profile contour ($V(s)$), under the condition that the front stagnant point is on inlet profile edge.

The problem is solved as a contour task for analytic function $F(\xi) = i \ln \overline{V} = \alpha\, i \ln V$; where: α - angle of flow direction, V - flow velocity in the plane of profile cascade. The angle of flow direction (α) is known along the profile contour. For known function of correspodence between contour points on profile and band ($\xi_\pm(s)$) it is known also the real part of function $F(\xi)$, and so the quantities: $\alpha_1 = \mathrm{Re}[F(\xi = -k)]$, $V_1 = \mathrm{Im}[F(\xi = -k)]$, $\alpha_2 = \mathrm{Re}[F(\xi = k)]$, $V_2 = \mathrm{Im}[F(\xi = k)]$, and $V(\xi_\pm) = \mathrm{Im}[F(\xi = \xi \pm i\pi/2)]$, can be determined by using Schwartz-integrals [3] :

$$\alpha_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\alpha_+ + \alpha_-}{ch(t+k)}dt \tag{6}$$

$$\ln V_1 = \frac{shk}{2\pi}\int_{-\infty}^{\infty}\frac{\alpha_+ - \alpha_-}{cht\, ch(t+k)}dt + \ln V(0) \tag{7}$$

$$\alpha_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\alpha_+ + \alpha_-}{ch(t-k)}dt \tag{8}$$

$$\ln V_2 = -\frac{sh2k}{2\pi}\int_{-\infty}^{\infty}\frac{\alpha_+ - \alpha_-}{cht\, ch(t-k)}dt + \ln V(0) \tag{9}$$

$$\ln V(\xi_\pm) = \frac{1}{2\pi}\left[\pm\int_{-\infty}^{+\infty}\alpha_\pm cth\frac{t-\xi_\pm}{2}dt \mp \int_{-\infty}^{\infty}\alpha_\mp th\frac{t-\xi_\pm}{2}dt - \int_{-\infty}^{+\infty}(\alpha_+ - \alpha_-)tht\, dt\right] + \ln V(0) \tag{10}$$

where: α_+ - value of $\alpha(t)$ on upper band contour,
α_- - value of $\alpha(t)$ on lower band contour

For given V_1, the constant $\ln V(0) = \ln V(\xi{=}0)$ can be found by using equation (7).

The complex relation (7) can be replaced by simpler one:

$$V_2 = V_1 \frac{\cos\alpha_1}{\cos\alpha_2} \tag{11}$$

which is obtained from the condition of equal flow rates reduced to one profile cascade step.

As one point of profile contour is mapped into $\xi_\pm \le -(k + Xo)$ and the other one into $\xi \ge k + Xo$, by separation of integration boundaries into $(-\infty,-(k + Xo)),(-(k + Xo), k + Xo)$ and $(k + Xo, \infty)$ the relations (6)÷(11) can be reduced to the forms where the integrals in boundaries from $-(k + Xo)$ to $k + Xo$ are occuring.

For the mapping having the form as in Fig. 2 it is existing $\alpha_\pm = \alpha_A$ for $t \ge k + Xo$ and $\alpha_\pm = \alpha_B$ for $t \le -(k + Xo)$, and equations (6), (8) and (10) can be reduced to the forms:

$$\alpha_1 = \frac{1}{2\pi}\int_{-(k+Xo)}^{k+Xo} \frac{\alpha_+ + \alpha_-}{ch(t+k)}dt + \frac{2\alpha_B}{\pi} arctge^{-Xo} + \frac{2\alpha_A}{\pi}\left(\frac{\pi}{2} - arctge^{(2k+Xo)}\right) \tag{12}$$

$$\alpha_2 = \frac{1}{2\pi}\int_{-(k+Xo)}^{k+Xo} \frac{\alpha_+ + \alpha_-}{ch(t-k)}dt + \frac{2\alpha_B}{\pi} arctge^{-(2k+Xo)} + \frac{2\alpha_A}{\pi}\left(\frac{\pi}{2} - arctge^{Xo}\right) \tag{13}$$

$$\begin{aligned}\ln V(\xi_\pm) = \frac{1}{2\pi}\left[\pm \int_{-(k+Xo)}^{k+Xo} \alpha_\pm cth\frac{t-\xi_\pm}{2}dt \mp \int_{-(k+Xo)}^{k+Xo} \alpha_\mp th\frac{t-\xi_\pm}{2}dt - \int_{-(k+Xo)}^{k+Xo}(\alpha_+ - \alpha_-)thtdt\right] \pm \\ \pm \frac{\alpha_B}{\pi}\ln th\frac{k+Xo+\xi_\pm}{2} \mu \frac{\alpha_A}{\pi}\ln th\frac{k+Xo-\xi_\pm}{2} + \ln V(0)\end{aligned} \tag{14}$$

where the constant $\ln V(0)$ can be calculated by using relation (7):

$$\ln V(0) = \ln V_1 - \frac{shk}{2\pi}\int_{-(k+Xo)}^{k+Xo} \frac{\alpha_+ - \alpha_-}{cht\, ch(t+k)}dt \tag{14'}$$

In order to close system of equations (11), (12), (13) and (14) two equations more are needed-equation for determinantion the canonic region parameter (k) and correspodence between contour points on profile and band ($\xi_\pm(s)$).

The value of the canonic region parameter (k) can be found from the condition that the variation of velocity potential along one side of profile contour is equal to the variation of potential along the corresponding band contour. For mapping nature as in Fig. 2 this is described by equation:

$$\int_{s_{01}=0}^{(s_{02})_a} V(S_a)dS_a = \varphi_o^*(\xi_{o-}(S_{02})) - \varphi_o^*(\xi_{o-}(S_{01})) \tag{15}$$

where: $(S_{02})_a$ - length coordinate of rear stagnant point measured along S_a - side of profile.

The corresponding points on contours of profile and band ($\xi_\pm(s)$) can be found from the condition of equal variations of velocity potentials along the corresponding contours. For mapping nature as in Fig. 2, taking into account that the velocity potential functions are continous along the band contours, this can be described by equations:

$$\int_0^{S_a} V(S_a)dS_a = \varphi_o^*(\xi_+) - \varphi_o^*(\xi_{o+}), \quad \text{for} \quad \xi_+(S_a), \quad 0 \le S_a \le S_A$$
$$\int_0^{S_a} V(S_a)dS_a = \varphi_o^*(\xi_-) - \varphi_o^*(\xi_{o+}), \quad \text{for} \quad \xi_-(S_a), \quad S_A \le S_a \le (S_{02})_a$$
$$\int_0^{S_b} V(S_b)dS_b = \varphi_o^*(\xi_+) - \varphi_o^*(\xi_{o+}), \quad \text{for} \quad \xi_+(S_b), \quad 0 \le S_b \le S_B \tag{16}$$
$$\int_0^{S_b} V(S_b)dS_b = \varphi_o^*(\xi_-) + Q(tg\alpha_1 - tg\alpha_2) - \varphi_o^*(\xi_{o+}), \quad \text{for} \quad \xi_-(S_b), \quad S_B \le S_b \le (S_{02})_b$$

where: S_A - point on S_a - profile side which is mapping into $\xi_\pm \ge k + Xo$
S_B - point on S_b - profile side which is mapping into $\xi_\pm \le -(k + Xo)$

For mapping form as in Fig. 2, equations (11), (12), (13),(14),(15) and (16) are making a closed system for solution of a direct task, for the case of nonshocking incident flow at profile cascade. According to the given equations, the infinite reach of band contours does not impose troubles during solution of problem.

For other mapping forms shown in Fig. 1 a) ÷i) the systems of equations can be also constructed for the solution of problems in which Schwartz-integrals can be reduced into forms with finite integration boundaries. The systems of equations for all possible mapping forms are given in [3]. In the interest of brevity these systems will be not mentioned here.

For mapping form as in Fig. 2, the computer programme [4] has been done, which can easily be enlarged for all other possible mapping forms presented in the paper. According to the mapping form as in Fig. 2 the flow situations in almost all reaction profile cascades can be solved.

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REŠAVANJE DIREKTNOG ZADATKA TEORIJE STRUJANJA KROZ PRAVE RAVANSKE REŠETKE PROFILA KONFORMNIM PRESLIKAVANJEM STRUJANJA NA POJAS $-\pi/2 \le \text{Im}\xi \le \pi/2$

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U radu je analiziran karakter preslikavanja strujanja oko profila prave ravanske rešetke na strujanje u pojasu $-\pi/2 \le Im\xi \le \pi/2$ *sa simetrično raspoređenim singularnim tačkama u* $\xi = \pm k$,

gde je k - realan broj, koji zavisi od geometrijskih parametara preslikavanja rešetke. Zavisno od uglova pravaca strujanja ispred i iza rešetke i geometrijskih parametara rešetke profila mogu se javiti devet karakterističnih slučajeva preslikavanja, od kojih se četiri mogu svrstati u grupu osnovnih preslikavanja, a pet u grupu slučajnih preslikavanja.

Prema karakteru promene potencijala brzine po konturi pojasa zaključuje se da se cela kontura profila praktično preslikava na ograničeni deo pojasa, pa beskrajno prostiranje pojasa i narušena konformnost preslikavanja u beskonačnosti ne stvara teškoće pri rešavanju zadatka. Schwartz-ovi integrali, koji ulaze u sistem jednačina za rešavanje zadatka, svode se na oblike sa konačnim granicama integraljenja.