FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 28, No 4 (2013), 417–428

POINT ESTIMATION OF CUBICALLY CONVERGENT ROOT FINDING METHOD OF WEIERSTRASS' TYPE *

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Abstract. The aim of this paper is to state initial conditions for the safe and fast convergence of the simultaneous method of Weierstrass' type for finding simple zeros of algebraic polynomial. This conditions are computationally verifiable and they depend only on the available data - polynomial coefficients, its degree and initial approximations $z_1^{(0)}, \dots, z_n^{(0)}$ to the zeros. It is shown that under the stated conditions, the proposed iterative method is convergent.

1. Introduction

The problem of finding all zeros of a polynomial has always been very important issue in numerical analysis and applied scientific disciplines. The list of publications concerning this topic is very extensive (see McNamee's book [2]). Iterative methods for the simultaneous determination of all zeros of polynomial belong to the most efficient approaches. In connection with this, constructing computationally verifiable initial conditions which provide both the guaranteed and fast convergence of a numerical root-finding algorithm is of considerable practical importance.

The aim of this paper is to establish initial conditions which guarantee the convergence of an efficient third order method for the simultaneous approximation of all simple zeros of a polynomial $P(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n$, $(a_i \in \mathbb{C})$. In [6] the following method for finding simple roots of polynomial was proposed

$$\hat{z}_i = z_i - \frac{P(z_i)}{P'(z_i - \frac{1}{2}W_i)}\,,$$

where $z_1, \dots, z_n$ are some approximations to the zeros $\zeta_1, \dots, \zeta_n$ and W_i is the Weierstrass correction. defined in Section 2. The cubic convergence of this method was

Received December 20, 2013.
2000 *Mathematics Subject Classification.* Primary 65H05
*The authors were supported in part by the Serbian Ministry of Education Science and Technological Development under grant 174022

proved in [6] assuming that the initial approximations are good enough, but without detailed quantitative convergence analysis with regard to the initial conditions. Now, we study computationally verifiable initial conditions which depend only of available data: initial approximations $z_1^{(0)},\dots,z_n^{(0)}$ to the zeros $\zeta_1,\dots,\zeta_n$ of a polynomial P, its degree n and the polynomial coefficients $a_0,a_1,\dots,a_n$. Such an approach, known as *the theory of point estimation*, was introduced by Smale in 1981 [9]. Many authors have started their investigation in this field after another Smale's fundamental work [10]. More details on the point estimation theory concerning iterative methods for the simultaneous determination of polynomial zeros can be found in [4], [8] and [11], and, particularly, in the book [3] and the references cited there.

In Section 2 we present the convergence theorem which provides very simple verification of the convergence of a rather wide class of simultaneous iterative methods and we apply this theorem in Section 3 to prove the guaranteed convergence of the proposed method. In our convergence analysis we will estimate some complex quantities using an elegant and fruitful approach by circular complex arithmetic which deals with disks.

2. The third order method

For distinct complex numbers $z_1,\dots,z_n$ and polynomial $P(z)=z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n$ let us define

$$W_i(z)=\frac{P(z)}{\prod_{\substack{j=1\\ j\neq i}}^{n}(z-z_j)}, \quad W_i=W_i(z_i),$$

$$w=\max_{1\le i\le n}|W_i|, \qquad d=\min_{\substack{1\le i,j\le n\\ j\neq i}}|z_i-z_j|.$$

Let us consider the iterative method

$$z_i^{(m+1)}=z_i^{(m)}-\frac{P(z_i^{(m)})}{P'\Big(z_i^{(m)}-\frac{1}{2}W_i(z_i^{(m)})\Big)},$$

where m denotes the m-th iterative step. For simplicity, we will omit sometimes the iteration index m and denote quantities in the latter $(m+1)$-st iteration by $\widehat{\ }$ ("hat"), that is

$$\hat{z}_i=z_i-\frac{P(z_i)}{P'(z_i-\frac{1}{2}W_i)}. \tag{2.1}$$

In the sequel for $q=1,2$ we use the abbreviations

$$S_{1,i}=\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{z_i-\frac{1}{2}W_i-z_j}, \quad G_{q,i}=\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{W_j}{(z_i-\frac{1}{2}W_i-z_j)^q}. \tag{2.2}$$

Also, for the sums and products we will write

$$\sum_{j\neq i} \quad \text{and} \quad \prod_{j\neq i} \quad \text{instead of} \quad \sum_{\substack{j=1\\ j\neq i}}^{n} \quad \text{and} \quad \prod_{\substack{j=1\\ j\neq i}}^{n}.$$

The Lagrange representation of the polynomial P is valid:

$$P(z) = W_i \prod_{j\neq i}(z-z_j) + \prod_{j=1}^{n}(z-z_j)\Big(\sum_{j\neq 1}\frac{W_j}{z-z_j}+1\Big). \tag{2.3}$$

Applying logarithmic derivative to (2.3), we obtain

$$\begin{aligned} P'(z) \;=\;& W_i\Big(\sum_{j\neq i}\frac{1}{z-z_j}\Big)\prod_{j\neq i}(z-z_j) + \sum_{j=1}^{n}\frac{1}{z-z_j}\prod_{j=i}^{n}(z-z_j)\Big(\sum_{j\neq 1}\frac{W_j}{z-z_j}+1\Big) \\ &+ \prod_{j=1}^{n}(z-z_j)\sum_{j\neq i}\frac{-W_j}{(z-z_j)^2}\,. \end{aligned}$$

Putting $z = z_i - \frac{1}{2}W_i$ in the last relation and using the introduced abbreviations (2.2) we find

$$\begin{aligned} P'\Big(z_i - \tfrac{1}{2}W_i\Big) &= \Big(1 + G_{1,i} + \tfrac{1}{2}W_i(S_{1,i}+G_{2,i}) - \tfrac{1}{2}W_iS_{1,i}G_{1,i}\Big)\textstyle\prod_{j\neq i}(z_i - \frac{1}{2}W_i - z_j) \\ &= (1+H_i)\prod_{j\neq i}(z_i - \tfrac{1}{2}W_i - z_j), \end{aligned}$$

where

$$H_i = G_{1,i} + \frac{1}{2}W_i(S_{1,i}+G_{2,i}) - \frac{1}{2}W_iS_{1,i}G_{1,i}. \tag{2.4}$$

Using the last equality and the introduced abbreviation (2.4), we can represent the iterative method (2.1) in the form

$$\hat{z}_i \;=\; z_i - \frac{W_i}{1+H_i}\cdot\prod_{j\neq i}\Big(1+\frac{\frac{1}{2}W_i}{z_i-\frac{1}{2}W_i-z_j}\Big). \tag{2.5}$$

This form of the considered method is more suitable for the convergence analysis that we are about to perform in the sequel.

3. Convergence analysis

In this section we present the convergence analysis of the method (2.5) using the approach based on Smale's point estimation theory [10]. This approach, regarded as a significant advance in the theory of iterative processes, states computationally verifiable initial convergence conditions that guarantee the convergence of the considered methods. As mentioned in the introduction, in the case of algebraic polynomials $P(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n$ it is assumed that initial conditions depend only on the polynomial coefficients $a_1, \ldots, a_n$, the polynomial degree n and the initial approximations $z_1^{(0)}, \ldots, z_n^{(0)}$.

Before stating the main result concerned with the guaranteed convergence of the simultaneous method (2.5) , we give a general theorem which can be applied to a general class of simultaneous methods of the form

$$z_i^{(m+1)} = z_i^{(m)} - C_i\Big(z_1^{(m)}, \ldots, z_n^{(m)}\Big) \quad (m = 0, 1, \ldots), \tag{3.1}$$

where $i \in I_n = \{1, \ldots, n\}$ is the index set and $z_1^{(m)}, \ldots, z_n^{(m)}$ are some distinct approximations to the simple zeros $\zeta_i, \ldots, \zeta_n$, respectively, obtained in the m-th iterative step by (3.1). In what follows the term

$$C_i^{(m)} = C_i\Big(z_1^{(m)}, \ldots, z_n^{(m)}\Big) \quad (i \in I_n)$$

will be called the *correction*.

Let $\Lambda(\zeta_i)$ be a reasonably close neighborhood of a zero ζ_i $(i \in I_n)$ and let the corrections C_i, occurring in (3.1), can be represented as

$$C_i(z_1, \ldots, z_n) = \frac{P(z_i)}{F_i(z_1, \ldots, z_n)} \quad (i \in I_n), \tag{3.2}$$

where the function $(z_1, \ldots, z_n) \mapsto F_i(z_1, \ldots, z_n)$ satisfies the following conditions for each $i \in I_n$:

1° $F_i(\zeta_1, \ldots, \zeta_n) \neq 0$,
2° $F_i(z_1, \ldots, z_n) \neq 0$ for $(z_1, \ldots, z_n) \in \Lambda(\zeta_1) \times \cdots \times \Lambda(\zeta_n)$,
3° $F_i(z_1, \ldots, z_n)$ is continuous in $\mathbb{C}^n$.

Let us define a real function $t \mapsto g(t)$ on the open interval $(0, 1)$ by

$$g(t) = \begin{cases} 1 + 2t, & 0 < t \leq \dfrac{1}{2}, \\ \dfrac{1}{1-t}, & \dfrac{1}{2} < t < 1. \end{cases}$$

The following theorem (see [4] and [5]), involving corrections C_i and the function g, plays the key role in our convergence analysis of the simultaneous method (2.5).

Theorem 3.1. *Let the iterative method (3.1) have the correction term of the form (3.2) for which the conditions $1^\circ - 3^\circ$ hold, and let $z_1^{(0)}, \ldots, z_n^{(0)}$ be distinct initial approximations to the zeros of P. If there exists a real number $\beta \in (0, 1)$ such that the following two inequalities*

(i) $\left|C_i^{(m+1)}\right| \le \beta \left|C_i^{(m)}\right| \quad (m = 0, 1, \ldots)$,

(ii) $\left|z_i^{(0)} - z_j^{(0)}\right| > g(\beta)\left(\left|C_i^{(0)}\right| + \left|C_j^{(0)}\right|\right)$,

($i, j \in I_n,\ i \ne j$) are valid, then the iterative method (3.1) is convergent.

In our convergence analysis we will estimate some complex quantities using an approach by circular complex arithmetic which deals with disks. A disk Z with center c and radius r, that is $Z := \{z : |z - c| \le r\}$, will be denoted briefly by the parametric notation $Z = \{c; r\}$. In the sequel for $Z_k = \{c_k; r_k\}$ $(k = 1, 2)$, we will use the relations:

$$z \in \{c; r\} \Longleftrightarrow |c| - r \le |z| \le |c| + r, \tag{3.3}$$

$$|c_1 - c_2| > r_1 + r_2 \Longleftrightarrow Z_1 \cap Z_2 = \emptyset, \tag{3.4}$$

$$\alpha Z = \{\alpha c; |\alpha| r\},\ \alpha + Z = \{\alpha + c; r\},\ \alpha \in \mathbb{C} \tag{3.5}$$

and

$$\prod_{k=1}^{n} z_k \in \prod_{k=1}^{n} Z_k = \left\{\prod_{k=1}^{n} c_k; \prod_{k=1}^{n}(|c_k| + r_k) - \prod_{k=1}^{n}|c_k|\right\}, \tag{3.6}$$

where $z_k \in Z_k = \{c_k; r_k\}$ $(k = 1, \ldots, n)$. For more details about properties of circular complex interval arithmetic see the books [1] and [7].

The following assertion was proved in [**?**]:

Theorem 3.2. *Let $z_1, \ldots, z_n$ be distinct numbers satisfying the inequality $w < c_n d$, $c_n < 1/(2n)$. Then the disks*

$$D_1 := \left\{z_1; \frac{|W_1|}{1 - nc_n}\right\}, \ldots, D_n := \left\{z_n; \frac{|W_n|}{1 - nc_n}\right\}$$

are mutually disjoint and each of them contains one and only one zero of the polynomial P, that is

$$\zeta_i \in \left\{z_i; \frac{1}{1 - nc_n}|W_i|\right\} \quad (i \in I_n). \tag{3.7}$$

In the sequel, we will assume that the following conditions

$$w < c_n d, \quad c_n = \frac{1}{6n}, \tag{3.8}$$

are fulfilled. The inequality in (3.8) is stronger than $w < d/(2n)$ so that the assertions of Theorem 3.2 hold.

Let $\varepsilon_i = z_i - \zeta_i$ denote the error of the approximation z_i to the zero ζ_i. Then, starting from (3.7) on the basis of properties of circular complex arithmetic, we obtain

$$|\varepsilon_i| = |z_i - \zeta_i| < \frac{1}{1 - nc_n}|W_i| < \frac{c_n}{1 - nc_n} d = \frac{1}{5n} d.$$

Besides, we have the estimation

$$|z_i - \zeta_j| \geq |z_i - z_j| - |z_j - \zeta_j| > d - \frac{1}{5n}d = \frac{5n-1}{5n}d.$$

According to (3.8) we find

$$\left|z_i - \frac{1}{2}W_i - z_j\right| \geq |z_i - z_j| - \frac{1}{2}|W_i| \geq \frac{12n-1}{12n}d. \tag{3.9}$$

Starting from (2.2) and taking into account (3.8) and (3.9), we estimate

$$|S_{1,i}| \leq \sum_{j\neq i} \frac{1}{\left|z_i - \frac{1}{2}W_i - z_j\right|} \leq \frac{12n(n-1)}{(12n-1)d}, \tag{3.10}$$

$$|G_{1,i}| \leq \sum_{j\neq i} \frac{|W_j|}{\left|z_i - \frac{1}{2}W_i - z_j\right|} \leq \frac{2(n-1)}{12n-1}, \tag{3.11}$$

$$|G_{2,i}| \leq \sum_{j\neq i} \frac{|W_j|}{\left|z_i - \frac{1}{2}W_i - z_j\right|^2} \leq \frac{24n(n-1)}{(12n-1)^2 d}. \tag{3.12}$$

According to (2.4) and (3.10)–(3.12) we find

$$|H_i| \leq |G_{1,i}| + \frac{1}{2}|W_i|\Big(|S_{1,i}| + |G_{2,i}|\Big) + \frac{1}{2}|W_i||S_{1,i}||G_{1,i}| \leq \frac{38n^2 - 41n + 3}{(12n-1)^2} =: h_n$$

and

$$|1 + H_i| \leq 1 + h_n = \frac{182n^2 - 65n + 4}{(12n-1)^2}, \tag{3.13}$$

$$|1 + H_i| \geq 1 - h_n = \frac{106n^2 + 17n - 2}{(12n-1)^2}. \tag{3.14}$$

Hence, by (3.8) and (3.9) we estimate

$$\prod_{j\neq i}\left|1 + \frac{\frac{1}{2}W_i}{z_i - \frac{1}{2}W_i - z_j}\right| \leq \left(1 + \frac{1}{12n-1}\right)^{n-1} =: p_n. \tag{3.15}$$

Starting from the iterative formula (2.5) we find

$$\hat{z}_i - z_i = -C_i = -\frac{W_i}{1 + H_i} \cdot \prod_{j\neq i}\left(1 + \frac{\frac{1}{2}W_i}{z_i - \frac{1}{2}W_i - z_j}\right). \tag{3.16}$$

By (3.14)–(3.16) we estimate

$$\begin{aligned}
|\hat{z}_i - z_i| &= |C_i| = \frac{|W_i|}{|1 + H_i|} \prod_{j\neq i}\left|1 + \frac{\frac{1}{2}W_i}{z_i - \frac{1}{2}W_i - z_j}\right| \\
&< \frac{p_n}{1 - h_n}|W_i| < 1.48 c_n d < \frac{1}{4n}d, \qquad (3.17)
\end{aligned}$$

The sequence $\{p_n/(1-h_n)\}_{n=3,4,\ldots}$ has a complicated form and we used symbolic computation in the programming package *Mathematica* to find its upper bound. It was found

$$\frac{p_n}{1-h_n} < 1.48.$$

Therefore, we have proved

$$|C_i| < 1.48|W_i|. \tag{3.18}$$

According to (3.17) we obtain

$$|\hat{z}_i - z_j| > \left(1 - \frac{1}{4n}\right)d = \frac{4n-1}{4n}d \tag{3.19}$$

and

$$|\hat{z}_i - \hat{z}_j| > \left(1 - 2\frac{1}{4n}\right)d = \frac{2n-1}{2n}d. \tag{3.20}$$

The inequality (3.20) gives

$$\hat{d} > \frac{2n-1}{2n}d, \quad \text{that is} \quad \frac{d}{\hat{d}} < \frac{2n}{2n-1}. \tag{3.21}$$

The following lemma is concerned with some necessary bounds and estimates.

Lemma 3.1. *Let the inequality* (3.8) *hold. Then*

(i) $|\widehat{W}_i| < 0.4|W_i|$;

(ii) $\widehat{w} < c_n\hat{d}, \ \ c_n = 1/(6n)$.

Proof. Putting $z = \hat{z}_i$ in (2.3), we obtain

$$P(\hat{z}_i) = \left(\frac{W_i}{\hat{z}_i - z_i} + 1 + \sum_{j\neq i}\frac{W_j}{\hat{z}_i - z_j}\right)\prod_{j=1}^{n}(\hat{z}_i - z_j).$$

Hence, after dividing by $\prod_{j\neq i}(\hat{z}_i - \hat{z}_j)$, we find

$$\widehat{W}_i = \frac{P(\hat{z}_i)}{\prod_{j\neq i}(\hat{z}_i - \hat{z}_j)} = (\hat{z}_i - z_i)\left(\frac{W_i}{\hat{z}_i - z_i} + 1 + \sum_{j\neq i}\frac{W_j}{\hat{z}_i - z_j}\right)\prod_{j\neq i}\left(1 + \frac{\hat{z}_j - z_j}{\hat{z}_i - \hat{z}_j}\right). \tag{3.22}$$

Starting from the iterative formula (2.5) we obtain

$$\frac{W_i}{\hat{z}_i - z_i} = -(1 + H_i)\prod_{j\neq i}\left(1 - \frac{\frac{1}{2}W_i}{z_i - z_j}\right). \tag{3.23}$$

Using (3.8) and the definition of the minimal distance d we obtain

$$\frac{\frac{1}{2}|W_i|}{|z_i - z_j|} \leq \frac{1}{2}c_n.$$

According to (3.3), (3.5) and the last inequality, we estimate

$$-\frac{\frac{1}{2}W_i}{z_i - z_j} \in \left\{0; \frac{1}{2}c_n\right\} \Longrightarrow 1 - \frac{\frac{1}{2}W_i}{z_i - z_j} \in \left\{1; \frac{1}{2}c_n\right\}$$

wherefrom, due to (3.6), we find

$$\prod_{j\neq i}\left(1 - \frac{\frac{1}{2}W_i}{z_i - z_j}\right) \in \prod_{j\neq i}\left\{1; \frac{1}{2}c_n\right\} = \left\{1; \left(1 + \frac{1}{2}c_n\right)^{n-1} - 1\right\}. \tag{3.24}$$

Now, using (3.5), (3.23) and (3.24) we obtain

$$\begin{aligned} \frac{W_i}{\hat{z}_i - z_i} &= -(1 + H_i)\prod_{j\neq i}\left(1 - \frac{\frac{1}{2}W_i}{z_i - z_j}\right) \\ &\in -(1 + H_i)\left\{1; \left(1 + \frac{1}{2}c_n\right)^{n-1} - 1\right\} \\ &\subset \left\{-1 - H_i; (1 + h_n)\left(\left(1 + \frac{1}{2}c_n\right)^{n-1} - 1\right)\right\} \\ &\subset \{-1 - H_i; r_n\}. \end{aligned} \tag{3.25}$$

By (3.8) and (3.13) we find the upper bound

$$r_n = (1 + h_n)\left(\left(1 + \frac{1}{2}c_n\right)^{n-1} - 1\right) < 0.11. \tag{3.26}$$

To prove (i) we use (2.4), (3.5) and (3.25) and find:

$$\begin{aligned} \frac{W_i}{\hat{z}_i - z_i} + 1 + \sum_{j\neq i}\frac{W_j}{\hat{z}_i - z_j} &\in \{-1 - H_i; r_n\} + 1 + \sum_{j\neq i}\frac{W_j}{\hat{z}_i - z_j} \\ &= \Bigg\{-1 - G_{1,i} - \frac{1}{2}W_i(S_{1,i} + G_{2,i}) + \frac{1}{2}W_iS_{1,i}G_{1,i} \\ &\qquad +1 + \sum_{j\neq i}\frac{W_j}{\hat{z}_i - z_j}; r_n\Bigg\} \\ &= \Bigg\{-(\hat{z}_i + \frac{1}{2}W_i - z_i)\sum_{j\neq i}\frac{W_j}{(\hat{z}_i - z_j)(z_i - \frac{1}{2}W_i - z_j)} \\ &\qquad -\frac{1}{2}W_i(S_{1,i} + G_{2,i}) + \frac{1}{2}W_iS_{1,i}G_{1,i}; r_n\Bigg\} \\ &= \{\theta_i; r_n\}. \end{aligned}$$

Using the bounds (3.8) and (3.17) we find

$$\left|\hat{z}_i + \frac{1}{2}W_i - z_i\right| \le |\hat{z}_i - z_i| + \frac{1}{2}|W_i| \le \frac{1}{3n}d.$$

According to (3.8), (3.9), (3.19) and the last inequality we estimate

$$\begin{aligned} &\left|(\hat{z}_i + \frac{1}{2}W_i - z_i)\sum_{j\ne i}\frac{W_j}{(\hat{z}_i - z_j)(z_i - \frac{1}{2}W_i - z_j)}\right| \\ &\le \frac{8(n-1)}{3(4n-1)(12n-1)}\,. \end{aligned} \tag{3.27}$$

Now, by the inequalities (3.10)–(3.12) and (3.27) we can find the upper bound to the center θ_i,

$$\begin{aligned} |\theta_i| \quad &\le \quad \left|(\hat{z}_i + \frac{1}{2}W_i - z_i)\sum_{j\ne i}\frac{W_j}{(\hat{z}_i - z_j)(z_i - \frac{1}{2}W_i - z_j)}\right| \\ &\quad + \frac{1}{2}|W_i|(|S_{1,i}| + |G_{2,i}|) + \frac{1}{2}|W_i||S_{1,i}||G_{1,i}| \\ &\le \quad \frac{168n^3 - 126n^2 - 47n + 5}{3(4n-1)(12n-1)^2} < 0.1. \end{aligned}$$

Taking into account (3.3) and (3.26) we find

$$\left|\frac{W_i}{\hat{z}_i - z_i} + 1 + \sum_{j\ne i}\frac{W_j}{\hat{z}_i - z_j}\right| \le |\theta_i| + r_n < 0.21. \tag{3.28}$$

With regard to the bounds (3.17) and (3.20) we estimate

$$\left|\prod_{j\ne i}\left(1 + \frac{\hat{z}_j - z_j}{\hat{z}_i - \hat{z}_j}\right)\right| < \left(1 + \frac{1}{2(2n-1)}\right)^{n-1}. \tag{3.29}$$

Taking absolute value of (3.22), we use the inequalities (3.17), (3.28) and (3.29), and obtain

$$|\widehat{W}_i| < 1.48|W_i| \cdot 0.21\left(1 + \frac{1}{2(2n-1)}\right)^{n-1} < 0.4|W_i|, \tag{3.30}$$

which proves (i) of Lemma 3.1. Hence, for $n \ge 3$ it follows $0.8n/(2n-1) < 1$. Now we have by (3.21)

$$\widehat{w} < 0.4w < 0.4c_n d < \frac{0.8n}{2n-1}\cdot c_n\hat{d} < c_n\hat{d},$$

which completes the proof of the assertion (ii) of Lemma 3.1. Note that the assertion (ii) plays one of the key roles in convergence analysis since the stated initial condition (3.8) keeps its form in the next iteration. □

Now, we state the main result concerning the initial conditions which guarantee the convergence of the iterative method (2.5).

Theorem 3.3. *Under the initial condition*

$$w^{(0)} < \frac{d^{(0)}}{6n}, \tag{3.31}$$

the iterative method (2.5) *is convergent.*

Proof. In Lemma 3.1 (assertion (ii)) we have proved the implication

$$w < c_n d \Rightarrow \hat{w} < c_n \hat{d}, \quad c_n = \frac{1}{6n}.$$

Similarly, we prove by induction that the condition (3.31) implies the inequality $w^{(m)} < c_n d^{(m)}$ for each $m = 1, 2, \ldots$. Therefore, all assertions of Lemmas 3.1 hold for each $m = 1, 2, \ldots$ if the initial condition (3.31) is valid. In particular, the following inequalities

$$|W_i^{(m+1)}| < 0.4|W_i^{(m)}| \tag{3.32}$$

and

$$|C_i^{(m)}| = |z_i^{(m+1)} - z_i^{(m)}| < 1.48|W_i^{(m)}| \tag{3.33}$$

hold for $i \in I_n$ and $m = 0, 1, \ldots$.

From the iterative formula (3.1) we see that the corrections $C_i^{(m)}$ are expressed by

$$C_i^{(m)} = \frac{P(z_i^{(m)})}{(1 + H_i^{(m)}) \prod_{j\neq i}\Big(z_i^{(m)} - \frac{1}{2} W_i(z_i^{(m)}) - z_j^{(m)}\Big)}, \tag{3.34}$$

where the abbreviations $C_i^{(m)}$ and $H_i^{(m)}$ are related to the m-th iterative step. Omitting the iteration index for simplicity, we find by (3.2),

$$C_i = \frac{P(z_i)}{(1 + H_i) \prod_{j\neq i}\Big(z_i - \frac{1}{2} W_i - z_j\Big)} = \frac{P(z_i)}{F_i(z_1, \ldots, z_n)},$$

where

$$\begin{aligned} |F_i(z_1, \ldots, z_n)| &= |1 + H_i| \prod_{j\neq i} \Big| z_i - \frac{1}{2} W_i - z_j \Big| \\ &> (1 - h_n)\Big(1 - \frac{1}{2} c_n\Big)^{n-1} d^{n-1} > 0.6 d^{n-1} > 0 \end{aligned}$$

for $i \in I_n$. It proves that the iterative process (2.5) is well defined in each iteration.We show that the function $F_i(z_1, \ldots, z_n) = P(z_i)/C_i$ appearing in (3.2), cannot be 0.

Now we will prove that the sequences $\{|C_i^{(m)}|\}$ $(i \in I_n)$ are monotonically decreasing.

Omitting the iteration index for simplicity, we find by (3.13), (3.17), (3.18), (3.30) and (3.32)–(3.34)

$$\begin{aligned} |\widehat{C}_i| &< 1.48|\widehat{W}_i| < 1.48 \cdot 0.4 \frac{|W_i|}{|1+H_i|} \prod_{j \neq i} \Big| 1 + \frac{\frac{1}{2} W_i}{z_i - \frac{1}{2} W_i - z_j} \Big| \\ & \quad \times |1+H_i| \prod_{j \neq i} \Big| 1 - \frac{\frac{1}{2} W_i}{z_i - z_j} \Big| \\ &\leq 0.592|C_i|(1+h_n)\Big(1 + \frac{1}{2} c_n\Big)^{n-1} < 0.82|C_i|, \end{aligned}$$

that is,

$$|\widehat{C}_i| < 0.82|C_i|.$$

Therefore, the constant β which appears in Theorem 3.1 is equal to $\beta = 0.82$. In this way we have proved the inequality $|C_i^{(m+1)}| < 0.82|C_i^{(m)}|$, which holds for each $i = 1, \dots, n$, $m = 0, 1, \dots$.

The quantity $g(\beta)$ appearing in (ii) of Theorem 3.1 is equal to $g(0.82) = 1/(1 - 0.82) \leq 5.56$. It remains to prove the disjunctivity of the inclusion disks

$$S_1 = \{z_1^{(0)}; g(0.82)|C_1^{(0)}|\}, \dots, S_n = \{z_n^{(0)}; g(0.82)|C_n^{(0)}|\}$$

(assertion (ii) of Theorem 3.1). Due to the inequality (3.18), there holds the estimate

$$|C_i^{(0)}| < 1.48 w^{(0)}$$

for every correction $|C_i^{(0)}|$ for all $i = 1, \dots, n$. Let p be the index $p \in I_n$ such that

$$|C_p^{(0)}| = \max_{1 \leq i \leq n} |C_i^{(0)}|.$$

Then

$$\begin{aligned} d^{(0)} &> 6nw^{(0)} > \frac{1}{1.48} 6n|C_p^{(0)}| \geq \frac{6n}{2 \cdot 1.48}\Big(|C_i^{(0)}| + |C_j^{(0)}|\Big) \\ &> g(0.82)\Big(|C_i^{(0)}| + |C_j^{(0)}|\Big) \end{aligned}$$

since

$$\frac{6n}{2 \cdot 1.48} \geq 6.08 > 5.56 \geq g(0.82)$$

for all $n \geq 3$. This means that

$$|z_i^{(0)} - z_j^{(0)}| \geq d^{(0)} > g(0.82)\Big(|C_i^{(0)}| + |C_j^{(0)}|\Big) = \text{rad } S_i + \text{rad } S_j.$$

Hence, according to a simple geometric construction, it follows that the inclusion disks $S_1, \ldots, S_n$ are disjoint (see (3.4)), which completes the proof of Theorem 3.3. □

Theorem 3.3 gives sufficient initial conditions that guarantee the convergence of the iterative method (2.5). In practice, these conditions can be relaxed and a greater value for the constant c_n (instead of $1/(6c_n)$) can be taken.

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