FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 28, No 4 (2013), 403–416

ON AN INTERVAL METHOD FOR THE INCLUSION OF ONE POLYNOMIAL ZERO *

Dušan M. Milošević, Miodrag S. Petković and Mimica R. Milošević

Abstract. In this paper we construct a new interval method for the inclusion of one simple or multiple complex polynomial zero in circular complex arithmetic. We present the convergence analysis starting from the computationally verifiable initial condition that guarantees the convergence of this inclusion method. We also give two numerical examples in order to demonstrate convergence behavior of the proposed method.

1. Introduction

Starting from an appropriate zero-point relation we derive an interval method for the inclusion of one simple or multiple complex zero of a polynomial in circular complex arithmetic. Under computationally verifiable initial conditions we prove that the proposed method has the convergence order equals three. The considered method was realized in circular interval arithmetic, which means that the produced approximations have the form of disks containing the wanted zero. The main advantage of the inclusion methods is the feasibility to express the upper error bound of the approximation by the radii of the resulting disk. We note that an appropriate fourth-order circular arithmetic method for the simultaneous inclusion of all simple or multiple complex zero, obtained from the same zero-relation, was considered in the paper [11].

The presentation of the paper is organized as follows. Some basic definitions and operations of circular complex interval arithmetic, necessary for the convergence analysis and the construction of inclusion methods, are given at the end of the Introduction. The derivation of the method for the inclusion of one simple or multiple complex zero and its convergence analysis are presented in sections 2. and 3., while numerical examples are given in section 4.

First we give a review of the basic properties of the so-called circular complex arithmetic introduced by Gargantini and Henrici [4]. A circular closed region (disk)

Received December 20, 2013.
2000 *Mathematics Subject Classification.* Primary 65H05
*This work was supported by the Serbian Ministry of Science under grant 174022

$Z := \{z : |z - c| \le r\}$ with center $c :=$ mid Z and radius $r :=$ rad Z is denoted by parametric notation $Z := \{c; r\}$. Then

$$\begin{aligned} \alpha\{c; r\} &= \{\alpha c; |\alpha| r\} \quad (\alpha \in \mathbb{C}), \\ \{c_1; r_1\} \pm \{c_2; r_2\} &= \{c_1 \pm c_2; r_1 + r_2\}. \end{aligned}$$

The inversion of a non-zero disk Z is defined by the Möbius transformation,

$$Z^{-1} = \left\{z^{-1} \ : \ z \in Z\right\} = \left\{\frac{\bar{c}}{|c|^2 - r^2}; \frac{r}{|c|^2 - r^2}\right\}, \tag{1.1}$$

($|c| > r$, i.e., $0 \notin Z$). The addition, subtraction and inversion Z^{-1} are exact operations.

Let us define the disk $\left\{z \ : \ |z - a| \le R\right\}$, denoted by $\{a; R\}$, and the region $W = \left\{z \ : \ |z - a| > R\right\}$. If $z \notin W$ (that is, $|z - a| \le R$) the inversion of the region

$$z - W = \{w \ : \ w - (z - a) > R\}$$

is the closed interior of a circle given by (see [3])

$$\begin{aligned} V(z) &= (z - W)^{-1} = \{h(z); d(z)\} \\ &= \left\{w : \left|w + \frac{\bar{z} - \bar{a}}{R^2 - |z - a|^2}\right| \le \frac{R}{R^2 - |z - a|^2}\right\}, \end{aligned} \tag{1.2}$$

where

$$h(z) = \text{mid } V(z) = \frac{\bar{a} - \bar{z}}{R^2 - |z - a|^2}$$

and

$$d(z) = \text{rad } V(z) = \frac{R}{R^2 - |z - a|^2}.$$

The set $\{z_1 z_2 \ : \ z_1 \in Z_1, z_2 \in Z_2\}$, in general, is not a disk. In order to remain within the set of disks, Gargantini and Henrici [4] introduced the disk multiplication by

$$Z_1 \cdot Z_2 \quad := \quad \{c_1 c_2; |c_1| r_2 + |c_2| r_1 + r_1 r_2\} \supseteq \{z_1 z_2 \ : \ z_1 \in Z_1, z_2 \in Z_2\}. \tag{1.3}$$

Then the division is defined by

$$Z_1 : Z_2 = Z_1 \cdot Z_2^{-1}.$$

The square root of a disk $\{c; r\}$ that does not contains the origin, where $c = |c| e^{i\theta}$ and $|c| > r$, is defined as the union of two disjoint disks (see [2]):

$$\{c; r\}^{1/2} \quad := \quad \left\{\sqrt{|c|} e^{i\theta/2}; \frac{r}{\sqrt{|c|} + \sqrt{|c| - r}}\right\} \bigcup \left\{-\sqrt{|c|} e^{i\theta/2}; \frac{r}{\sqrt{|c|} + \sqrt{|c| - r}}\right\}.$$

In this paper we will use the following obvious properties:

$$\begin{aligned} z \in \{c; r\} &\iff |z - c| \le r, \\ \{c_1; r_1\} \cap \{c_2; r_2\} = \emptyset &\iff |c_1 - c_2| > r_1 + r_2. \end{aligned}$$

More details about circular arithmetic can be found in the books [1], [5], [6] and [13].

2. Interval method for the inclusion of one zero

Let $P(z) = z^N + a_{N-1}z^{N-1} + \ldots + a_1 z + a_0$ be a monic polynomial with simple or multiple complex zeros $\zeta_1, \ldots, \zeta_n$ $(2 \le n \le N)$, with multiplicities $\mu_1, \ldots, \mu_n$ $(\mu_1 + \ldots + \mu_n = N)$, respectively, and let

$$s_{k,i} = \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{\mu_j}{(z_i - \zeta_j)^k} \ (k = 1, 2), \ u(z) = \frac{P(z)}{P'(z)}. \tag{2.1}$$

The following zero-relation has been derived in [8]

$$\begin{aligned} \zeta_i \quad = \quad & z_i - \mu_i u(z_i) - \frac{1}{2\Big(1 - u(z_i)\, s_{1,i}\Big)^2}\Bigg[\mu_i u(z_i)\Bigg(1 - \mu_i \\ & + \mu_i u(z_i)\frac{P''(z_i)}{P'(z_i)} - u^2(z_i)(s_{1,i}^2 - \mu_i s_{2,i})\Bigg)\Bigg] \end{aligned} \tag{2.2}$$

for $i \in \mathcal{I}_n := \{1, 2, \ldots, n\}$.

Assume that we have found the inclusion disk $\{z : |z - a| \le R\}$ with center a and radius R containing only one zero ζ_i of P. All other zeros are supposed to lie in the region $W = \{z : |z - a| > R\}$. Using the inclusion isotonicity property we obtain for $z \in \{a; R\}$

$$(z - \zeta_j)^{-1} \in \left(\frac{1}{z - W}\right) \quad (j = 1, \ldots, i-1, i+1, \ldots, n). \tag{2.3}$$

If $z \notin W$ (that is, $|z - a| < R$), using (1.2) we obtain the inversion

$$V(z) = (z - W)^{-1} = \{h(z); d(z)\}.$$

According to the inclusion isotonicity property, and by (2.1) and (2.3), we find

$$s_{k,i} \in \sum_{\substack{j=1\\ j\neq i}}^{n} (z - W)^{-k} = (N - \mu_i)\, V(z)^k =: S_{k,i},$$

$(k = 1, 2, \ i \in \mathcal{I}_n)$. In view of this, from the zero-relation (2.2) we get for $z = z_i$

$$\begin{aligned} \zeta_i \ \in \ & z_i - \mu_i u(z_i) - \frac{1}{2\Big(1 - u(z_i)\, S_{1,i}\Big)^2}\Bigg[\mu_i u(z_i)\Bigg(1 - \mu_i \\ & + \mu_i u(z_i)\frac{P''(z_i)}{P'(z_i)} - u^2(z_i)(S_{1,i}^2 - \mu_i S_{2,i})\Bigg)\Bigg], \quad (i \in \mathcal{I}_n). \end{aligned} \tag{2.4}$$

In our consideration only one zero is requested so that, without loss of generality we may adopt that this zero is denoted with ζ_1 and suppose that all other zeros

$\zeta_2, \ldots, \zeta_n$ lie in the exterior of $\{a; R\}$. Moreover, for brevity, we will write ζ instead of ζ_1. Also, we will write s_1, s_2, S_1 and S_2 instead of $s_{1,1}, s_{2,1}, S_{1,1}$ and $S_{2,1}$, respectively.

Let $Z^{(m)} = \left\{z^{(m)}; r^{(m)}\right\}$ be a disk with center $z^{(m)} = \operatorname{mid} Z^{(m)}$ and radius $r^{(m)} = \operatorname{rad} Z^{(m)}$ for $m = 0, 1, \ldots$. For the initial inclusion disk $Z^{(0)}$ we have $Z^{(0)} = \{a; R\}$, that is, $z^{(0)} = a$, $r^{(0)} = R$. We introduce the notation

$$V^{(m)} = \left(z^{(m)} - W\right)^{-1} = \left\{h^{(m)}; d^{(m)}\right\}$$

and

$$S_k^{(m)} = (N - \mu)\left(V^{(m)}\right)^k \quad (k = 1, 2).$$

The relation (2.4) suggests the following iterative method for finding a multiple complex zero of a polynomial P, starting from the isolated inclusion disk $Z^{(0)} = \{a; R\}$:

$$\begin{aligned} Z^{(m+1)} &= z^{(m)} - \mu u\left(z^{(m)}\right) - \frac{1}{2\left(1 - u\left(z^{(m)}\right) S_1^{(m)}\right)^2} \\ &\quad \times\Bigg[\mu u\left(z^{(m)}\right)\Bigg(1 - \mu + \mu u\left(z^{(m)}\right)\frac{P''\left(z^{(m)}\right)}{P'\left(z^{(m)}\right)} \\ &\quad - u^2\left(z^{(m)}\right)\left(\left(S_1^{(m)}\right)^2 - \mu S_2^{(m)}\right)\Bigg)\Bigg]. \end{aligned} \tag{2.5}$$

3. Convergence analysis

In this section we will analyze the iterative method (2.5) which can be rewritten in the form

$$Z^{(m+1)} = z^{(m)} - \mu u\left(z^{(m)}\right) - \frac{\{b(z^{(m)}); \eta(z^{(m)})\}}{2\{c(z^{(m)}); \gamma(z^{(m)})\}^2}, \tag{3.1}$$

$(m = 1, 2, \ldots)$, where

$$\begin{aligned} b(z^{(m)}) &= \mu u\left(z^{(m)}\right)\Bigg(1 - \mu + \mu u\left(z^{(m)}\right)\frac{P''(z^{(m)})}{P'(z^{(m)})} - u^2\left(z^{(m)}\right)(N - \mu)(N - 2\mu) \\ &\quad \times\left(\frac{\bar{a} - \bar{z}^{(m)}}{R^2 - \left|z^{(m)} - a\right|^2}\right)^2\Bigg), \\ \eta(z^{(m)}) &= \mu\left|u\left(z^{(m)}\right)\right|^3 (N - \mu)(N - 2\mu)\frac{2\left|a - z^{(m)}\right|R + R^2}{\left(R^2 - \left|z^{(m)} - a\right|\right)^2}, \\ \gamma(z^{(m)}) &= \left|u\left(z^{(m)}\right)\right|(N - \mu)\frac{R}{R^2 - \left|z^{(m)} - a\right|^2}, \\ c\left(z^{(m)}\right) &= 1 - u\left(z^{(m)}\right)(N - \mu)\frac{\bar{a} - \bar{z}^{(m)}}{R^2 - \left|z^{(m)} - a\right|^2}. \end{aligned}$$

Assume that we have found the initial disk $Z^{(0)} = \{a; R\}$ so that the conditions

$$|u(a)| = \left|\frac{P(a)}{P'(a)}\right| < \frac{R}{8(N-\mu)\mu^2}, \qquad \left|\frac{P''(a)}{P'(a)}\right| < \frac{8(N-\mu)}{R} \tag{3.2}$$

are satisfied. Also, for $m = 1, 2, \ldots$ let us introduce

$$\delta^{(m)} = R - \left|z^{(m)} - a\right|.$$

At the beginning, let us consider the first iteration ($m = 0$). According to (2.1), (3.2) and the inequality $(N - 2\mu)/(N - \mu) < 1$, we obtain

$$\eta(a) = \mu|u(a)|^3(N-\mu)(N-2\mu)\frac{1}{R^2} < \frac{R}{8^3(N-\mu)\mu^5}$$

and

$$\begin{aligned} |b(a)| &\leq \mu|u(a)|\left(1 + \mu + \mu|u(a)|\left|\frac{P''(a)}{P'(a)}\right|\right) \\ &< \frac{\mu^2+\mu+1}{\mu^2}\,\frac{R}{8(N-\mu)} \leq \frac{3R}{8(N-\mu)}. \end{aligned}$$

Similarly, $c(a) = 1$ and

$$\gamma(a) < |u(a)|(N-\mu)\frac{1}{R} < \frac{1}{8\mu^2} \leq \frac{1}{8}.$$

Let us examine the disk in the denominator in the first iteration. Using formula (1.3) and the bound for $\gamma(a)$, we estimate

$$\{1; \gamma(a)\}^2 = \{1; 2\gamma(a) + \gamma^2(a)\} \subset \left\{1; \frac{17}{64}\right\}.$$

The obtained disk does not contain the origin (because of $1 > 17/64$) and we can find its inverse disk using (1.1). We get

$$\left\{1; \frac{17}{64}\right\}^{-1} = \frac{\left\{1; \frac{17}{64}\right\}}{1 - \left(\frac{17}{64}\right)^2} < \frac{6}{5}\left\{1; \frac{17}{64}\right\}.$$

Starting from (3.1), using (1.3) and the obtained bounds for $|b(a)|$ and $\eta(a)$, we find the upper bound for $r^{(1)}$,

$$r^{(1)} = \operatorname{rad} Z^{(1)} = \frac{3}{5}\left(|b(a)|\frac{17}{64} + \eta(a) + \eta(a)\frac{17}{64}\right),$$

wherefrom

$$r^{(1)} < \frac{2}{25} \cdot \frac{R}{N-\mu}. \tag{3.3}$$

Under the conditions (3.2) and the obtained bound for $|b(a)|$ we estimate

$$\left|z^{(1)} - a\right| \le \mu|u(a)| + \frac{3}{5}|b(a)| < \frac{R}{8(N-\mu)\mu} + \frac{9R}{40(N-\mu)}$$

and find

$$\left|z^{(1)} - a\right| < \frac{7}{20} \cdot \frac{R}{N-\mu}. \tag{3.4}$$

Now we shall prove that the conditions (3.2) imply the inequality

$$\delta^{(1)} > 8(N-\mu)r^{(1)}. \tag{3.5}$$

Using the inequality (3.4) we find

$$\delta^{(1)} = R - \left|z^{(1)} - a\right| > R - \frac{7R}{20(N-\mu)} = R\left[1 - \frac{7}{20(N-\mu)}\right],$$

so that, according to (3.3) and (3.5), it is sufficient to show that

$$R\left[1 - \frac{7}{20(N-\mu)}\right] > 8(N-\mu)\frac{2R}{25(N-\mu)}.$$

The last inequality follows directly from the inequalities

$$1 - \frac{7}{20(N-\mu)} \ge \frac{13}{20} = \frac{65}{100} > \frac{64}{100} = \frac{16}{25}.$$

The analysis of the first iterative step shows that

(i) a new disk approximation $Z^{(1)}$ includes the zero ζ;

(ii) this disk is contracted because of

$$r^{(1)} < \frac{2R}{25}. \tag{3.6}$$

Besides, the initial conditions (3.2) induces the condition (3.5).

Now, we can analyze the iterative process (3.1) beginning with $m \ge 1$ and starting from the inclusion disk $Z^{(1)}$ with the assumption that the condition (3.5) holds. For simplicity, in our analysis we omit the iteration index always when the possibility of any confusion does not exist. Also, let us include the abbreviations

$$\{c_2(z); \gamma_2(z)\} = \{c_1(z); \gamma_1(z)\}^{-1} = \left(\{c(z); \gamma(z)\}^2\right)^{-1},$$

and $\varepsilon = z - \zeta$.

Lemma 3.1. *If the inequality*

$$\delta > 8(N-\mu)r \tag{3.7}$$

holds, then

(i) $|u(z)| = \left|\frac{\varepsilon}{\mu+\varepsilon s_1}\right| < \frac{8r}{7\mu}$;

(ii) $\frac{8}{7} > |c(z)| > \frac{6}{7}$;

(iii) $|\gamma(z)| < \frac{8(N-\mu)r}{7\mu\delta} < \frac{1}{7\mu}$;

(iv) $|c_2(z)| < \frac{9}{5}$;

(v) $\gamma_2(z) < \frac{7(N-\mu)r}{\mu\delta} < \frac{7}{8\mu}$;

(vi) $|b(z)| < \frac{18(N-\mu)r^2}{5\delta}$;

(vii) $\eta(z) < \frac{3(N-\mu)(N-2\mu)r^3}{2\mu^2\delta^2}$.

Proof. Of (i): Since

$$|s_k| < \frac{N-\mu}{\delta^k} \quad (k = 1, 2),$$

under the condition (3.7) we estimate

$$\begin{aligned} |u(z)| &= \left|\frac{P(z)}{P'(z)}\right| = \left|\sum_{j=1}^{n} \frac{\mu_j}{z-\zeta_j}\right|^{-1} = \left|\frac{\mu}{\varepsilon} + s_1\right|^{-1} = \left|\frac{\varepsilon}{\mu+\varepsilon s_1}\right| \\ &\leq \frac{|\varepsilon|}{\mu\left(1-\frac{|\varepsilon s_1|}{\mu}\right)} < \frac{r}{\mu\left(1-\frac{1}{8\mu}\right)} < \frac{8r}{7\mu}. \end{aligned}$$

Of (ii): Using the bound

$$\frac{|a-z|}{R^2-|z-a|^2} = \frac{R-\delta}{R^2-(R-\delta)^2} < \frac{1}{\delta} \tag{3.8}$$

and the assertion (i), from the relation

$$c(z) = 1 - u(z)(N-\mu)\frac{\bar{a}-\bar{z}}{R^2-\left|z-a\right|^2}$$

we obtain

$$|c(z)| > 1 - \frac{8r}{7\mu}\frac{N-\mu}{\delta} > \frac{6}{7}$$

and

$$|c(z)| < 1 + \frac{8r}{7\mu}\frac{N-\mu}{\delta} < \frac{8}{7}.$$

Of (iii): Starting from the relation

$$\gamma(z) = u(z)(N-\mu)\frac{R}{R^2 - \left|z-a\right|^2},$$

using the assertion (i) and the bound

$$\frac{R}{R^2 - \left|z-a\right|^2} < \frac{1}{\delta}$$

we obtain

$$\gamma(z) < \frac{8(N-\mu)r}{7\mu\delta} < \frac{1}{7\mu}.$$

Of (iv) and (v): Using (1.3) we find

$$\{c_1(z); \gamma_1(z)\} = \{c(z); \gamma(z)\}^2 = \{c(z)^2; 2|c(z)|\gamma(z) + \gamma(z)^2\}.$$

According to the assertion (ii), from the last relation we estimate

$$|c_1(z)| > \frac{36}{49}. \tag{3.9}$$

Similarly, in regard to the assertions (ii) and (iii) we find

$$\gamma_1(z) < 2\,\frac{8}{7}\,\frac{8r}{7\mu}\,\frac{N-\mu}{\delta} + \left(\frac{8r}{7\mu}\,\frac{N-\mu}{\delta}\right)^2 < \frac{14r(N-\mu)}{5\mu\delta} < \frac{7}{20}. \tag{3.10}$$

Since $|c_1(z)| > \frac{36}{49} > \frac{7}{20} > \gamma_1(z)$, we conclude that $0 \notin \{c_1; \gamma_1\}$ and we can find the inverse of the disk $\{c_1(z); \gamma_1(z)\}$,

$$\{c_2(z); \gamma_2(z)\} = \{c_1(z); \gamma_1(z)\}^{-1} = \frac{\{\bar{c}_1(z); \gamma_1(z)\}}{|c_1(z)|^2 - \gamma_1^2(z)}.$$

From the last relation and (3.9) and (3.10), we estimate

$$|c_2(z)| = \frac{1}{|c_1(z)| - \gamma_1^2(z)/|c_1(z)|} < \frac{1}{\frac{36}{49} - \frac{49/400}{36/49}} < \frac{9}{5}$$

and

$$\gamma_2(z) = \frac{\gamma_1(z)}{|c_1(z)|^2 - \gamma_1(z)^2} < \frac{\frac{14r(N-\mu)}{5\mu\delta}}{\left(\frac{36}{49}\right)^2 - \left(\frac{7}{20}\right)^2} < \frac{7r(N-\mu)}{\mu\delta} < \frac{7}{8\mu}.$$

Of (vi) and (vii): Now we consider the disk in the nominator $\{b(z); \eta(z)\}$. Using the abbreviations

$$\Delta_1(z) = \frac{P'(z)}{P(z)} \quad \text{and} \quad \Delta_2(z) = \frac{P'(z)^2 - P(z)P''(z)}{P(z)^2}$$

we obtain

$$\frac{P''(z)}{P'(z)} = u(z)(\Delta_1(z)^2 - \Delta_2(z)). \tag{3.11}$$

Since

$$\Delta_1(z) = \frac{d}{dz}\Big(\log P(z)\Big) = \sum_{j=1}^{n} \frac{\mu_j}{z-\zeta_j} = \frac{\mu}{\varepsilon} + s_1$$

and

$$\Delta_2(z) = -\frac{d^2}{dz^2}\Big(\log P(z)\Big) = \sum_{j=1}^{n} \frac{\mu_j}{(z-\zeta_j)^2} = \frac{\mu}{\varepsilon^2} + s_2,$$

using (3.11) we find

$$1 - \mu + \mu u(z)\frac{P''(z)}{P'(z)} = \frac{2\mu\varepsilon s_1 + \varepsilon^2 s_1^2 - \mu\varepsilon^2 s_2}{(\mu + \varepsilon s_1)^2}. \tag{3.12}$$

Taking into account that

$$|\varepsilon| \le r \quad \text{and} \quad s_k \le \frac{N-\mu}{\delta^k} \quad (k = 1, 2),$$

we find the bounds

$$|2\mu\varepsilon s_1 + \varepsilon^2 s_1^2 - \mu\varepsilon^2 s_2| < \frac{9r\mu(N-\mu)}{4\delta} \tag{3.13}$$

and

$$|\mu + \varepsilon s_1| \ge \mu - \frac{(N-\mu)r}{\delta} = \mu\left(1 - \frac{(N-\mu)r}{\mu\delta}\right) \ge \frac{7\mu}{8}. \tag{3.14}$$

Using the inequalities (3.13) and (3.14) we estimate from (3.12)

$$\left|1 - \mu + \mu u(z)\frac{P''(z)}{P'(z)}\right| < \frac{144(N-\mu)r}{49\mu\delta}. \tag{3.15}$$

By applying the assertion (i) and the inequalities (3.8) and (3.15), we find the upper bound of $|b(z)|$,

$$|b(z)| \le \frac{8r}{7}\left(\frac{144r(N-\mu)}{49\mu\delta} + \frac{64r^2(N-\mu)(N-2\mu)}{49\mu^2\delta^2}\right) < \frac{18(N-\mu)r^2}{5\delta}.$$

Similarly, by virtue of the inequality (3.8) and the assertion (i), from the relation

$$\eta(z) = \mu u^3(z)(N-\mu)(N-2\mu)\frac{2\big|a-z\big|R + R^2}{\Big(R^2 - \big|z-a\big|\Big)^2}$$

we obtain

$$\eta(z) < \frac{3(N-\mu)(N-2\mu)r^3}{2\mu^2\delta^2}. \qquad \square$$

Now we are able to prove that the order of convergence of the method (2.5) is three.

Theorem 3.1. *Let the sequence of circular intervals* $\left\{Z^{(m)}\right\}_{m=1,2,\ldots}$ *be defined by the iterative formula* (2.5), *assuming that the initial disk* $Z^{(0)} = \{a; R\}$ *is chosen so that it contains the zero* ζ *and the conditions* (3.2) *are satisfied. Then, in each iterative step, the following is true:*

(i) $\zeta \in Z^{(m)}$;

(ii) $r^{(m+1)} < \frac{68(N-\mu)}{R^2}\left(r^{(m)}\right)^3$.

Proof. The assertion (i) follows from the zero-relation (2.2) on the basis of the inclusion isotonicity property and the fact that $z^{(m)} \in \{a; R\}$ for each $m = 0, 1, \ldots,$ which is obvious because of

$$R - \left|z^{(m)} - a\right| = \delta^{(m)} > 8(N-\mu)r^{(m)} > 0.$$

We now prove that the convergence rate of the iterative method (2.5) is cubic (the assertion (ii)). Using the inequality (3.5) and the bounds obtained in the Lemma 3.1, we find for $r^{(2)} = \operatorname{rad} Z^{(2)}$

$$r^{(2)} = \frac{1}{2}\Big(|b(z)|\gamma_2(z) + |c_2(z)|\eta(z) + \eta(z)\gamma_2(z)\Big) < \frac{15(N-\mu)^2\left(r^{(1)}\right)^3}{\left(\delta^{(1)}\right)^2}$$

and

$$r^{(2)} < \frac{1}{4}r^{(1)}. \tag{3.16}$$

By the inequality (3.4) we estimate

$$\delta^{(1)} = R - \left|z^{(1)} - a\right| > R - \frac{7}{20}R = \frac{13}{20}R. \tag{3.17}$$

Besides, starting from the inequality

$$\begin{aligned}
\delta^{(2)} &= R - \left|z^{(2)} - a\right| \\
&= R - \left|z^{(1)} - a - \mu u(z) - \frac{1}{2}c_2(z)b(z)\right| \\
&> R - \left|z^{(1)} - a\right| - \left|\mu u(z) + \frac{1}{2}c_2(z)b(z)\right|
\end{aligned}$$

we obtain

$$\left|\mu u(z) + \frac{1}{2}\, c_2(z) b(z)\right| \quad < \quad \frac{8}{5}\, r^{(1)}$$

and conclude that

$$\delta^{(2)} > \delta^{(1)} - \frac{8}{5}\, r^{(1)}.$$

Applying the inequalities (3.5) and (3.16), we get

$$\begin{aligned} \delta^{(2)} \quad &> \quad \delta^{(1)} - \frac{8}{5}\, r^{(1)} > 8(N-\mu) r^{(1)} - \frac{8}{5}\, r^{(1)} \\ &= \quad \left[8(N-\mu) - \frac{8}{5}\right] r^{(1)} \\ &> \quad 4\left[8(N-\mu) - \frac{8}{5}\right] r^{(2)} > 8(N-\mu) r^{(2)}. \end{aligned}$$

Using the same consideration as for $m = 2$, by induction we prove that for $m \geq 2$ the following relations (already proved for $m = 2$) are true:

$$r^{(m+1)} \quad < \quad \frac{15(N-\mu)^2}{\left(\delta^{(m)}\right)^2} \left(r^{(m)}\right)^3, \tag{3.18}$$

$$r^{(m+1)} \quad < \quad \frac{r^{(m)}}{4}, \tag{3.19}$$

$$\delta^{(m)} \quad > \quad 8(N-\mu) r^{(m)} \tag{3.20}$$

and

$$\delta^{(m+1)} > \delta^{(m)} - \frac{8}{5}\, r^{(m)}. \tag{3.21}$$

By successive application of the inequalities (3.19) and (3.21), using the bounds (3.6) and (3.17) we obtain

$$\begin{aligned} \delta^{(m)} \quad &> \quad \delta^{(m-1)} - \frac{8}{5}\, r^{(m-1)} > \dots \\ &> \quad \delta^{(1)} - \frac{8}{5}\, r^{(1)} \left(1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \dots\right) \\ &> \quad \delta^{(1)} - \frac{32}{15}\, r^{(1)} > \frac{13}{20}\, R - \frac{32}{15} \cdot \frac{2}{25}\, R > \frac{47}{100}\, R. \end{aligned}$$

According to this, from the inequality (3.18) it follows

$$r^{(m+1)} < \frac{68(N-\mu)^2}{R^2} \left(r^{(m)}\right)^3.$$

We complete the proof of the theorem providing that the iterative method (2.5) is defined in each iterative step under the initial conditions (3.2), that is, $0 \notin \left\{c^{(m)}; r^{(m)}\right\}$ for each $m = 1, 2, \dots$. Indeed, from the conditions (3.2) the inequality (3.20) follows for each $m = 1, 2, \dots$ so that Lemma 3.1 is applicable. $\square$

4. Numerical examples

The proposed algorithm (2.5) has been tested in solving many polynomial equations. To provide the enclosure of the zeros in the second and third iteration that produce very small disks, we used the programming package *Mathematica* 7 with multi-precision arithmetic. We also tested the third order Halley-like method for the inclusion of one polynomial zero [7]

$$Z^{(m+1)} = z^{(m)} - \frac{1}{A\left(z^{(m)}\right)}, \tag{4.1}$$

where

$$\begin{aligned} A(z) &= \left(1+\frac{1}{\mu}\right)\frac{P'(z)}{2P(z)} - \frac{P''(z)}{2P'(z)} \\ &\quad -\frac{P(z)}{2P'(z)}\,\frac{N(N-\mu)}{\mu}\Big\{h(z);d(z)\Big\}^2, \end{aligned}$$

and the third order Euler-like method [10] (see also [9], [12] and [14])

$$\begin{aligned} Z^{(m+1)} &= z^{(m)} - \frac{2\mu}{\delta_1(z) + \left[2\mu\delta_2(z) - \delta_1^2(z) + 2N(N-\mu)\Big\{h(z);d(z)\Big\}^2\right]_*^{1/2}}. \\ &\qquad (i=1,\ldots,n;\ m=0,1,\ldots). \end{aligned} \tag{4.2}$$

Example 4.1. To find the circular inclusion approximations to a simple zero of the polynomial

$$\begin{aligned} P(z) &= z^{17} - z^{16} + 28z^{15} - 390z^{14} + 6\,002z^{13} - 10\,762z^{12} - 29\,484z^{11} \\ &\quad +846\,040z^{10} - 76\,809\,707z^9 + 130\,583\,427z^8 - 2\,113\,327\,216z^7 \\ &\quad +24\,795\,890\,990z^6 - 339\,342\,802\,696z^5 + 178\,957\,763\,336z^4 \\ &\quad +7\,226\,702\,364\,672z^3 - 88\,957\,569\,392\,640z^2 \\ &\quad +1\,984\,671\,888\,998\,400z - 1\,902\,803\,374\,080\,000 \end{aligned}$$

we implemented the interval methods (2.5), (4.1) and (4.2). The isolated zero of P is $\zeta_1 = 1$. The initial disk was selected to be $Z_1^{(0)} = \{0.9 + 0.1i; 6\}$. The radii of the inclusion disks produced in the first three iterative steps are given in Table 4.1, where the denotation $A(-q)$ means $A \times 10^{-q}$.

Example 4.2. To find the circular inclusion approximations to a multiple zero of the polynomial

$$\begin{aligned} P(z) &= z^{14} - 9z^{13} + 57z^{12} - 343z^{11} - 1\,830z^{10} + 22\,644z^9 - 147\,528z^8 \\ &\quad +889\,056z^7 - 295\,488z^6 - 13\,343\,616z^5 + 95\,178\,240z^4 \\ &\quad -576\,108\,288z^3 + 1\,279\,867\,392z^2 - 1\,148\,857\,344z + 362\,797\,056, \end{aligned}$$

Table 4.1: The radii of inclusion disks

	$r^{(1)}$	$r^{(2)}$	$r^{(3)}$
(4.1)	1.08(−2)	2.07(−9)	8.75(−36)
(4.2)	4.85(−2)	3.66(−9)	1.12(−34)
(2.5)	9.01(−2)	1.01(−7)	3.58(−30)

we implemented the same interval methods. The isolated multiple zero of P is $\zeta_1 = 1$ of multiplicity $\mu_1 = 3$. The initial disk was selected to be $Z_1^{(0)} = \{0.9 + 0.1i; 2\}$. The radii of the inclusion disks produced in the first three iterative steps, are given in Table 4.2.

Table 4.2: The radii of inclusion disks

	$r^{(1)}$	$r^{(2)}$	$r^{(3)}$
(4.1)	6.03(−3)	4.05(−11)	1.50(−38)
(4.2)	2.60(−2)	5.70(−11)	7.21(−39)
(2.5)	9.04(−3)	2.01(−10)	4.29(−37)

From Tables 4.1 and 4.2 we observe that theoretical results, concerning the convergence order of the considered method (2.5), mainly well coincide with the convergence behavior in practice. The disks obtained by the iterative method (2.5) and the methods (4.1) and (4.2) are comparable in size.

Also we remark that the third iterations are displayed only to demonstrate remarkable accuracy of the produced approximations, which are rarely requested in practice.

REFERENCES

1. G. Alefeld and J. Herzberger: *Introduction to Interval Computations*. Academic Press, New York, 1983.
2. I. Gargantini: *Parallel Laguerre iterations: The complex case"*. Numer. Math. **26** (1976), 317–323.
3. I. Gargantini: *Further application of circular arithmetic: Schröder-like algorithms with error bound for finding zeros of polynomials*. SIAM J. Numer. Anal. **15** (1978), 497–510.
4. I. Gargantini and P. Henrici: *Circular arithmetic and the determination of polynomial zeros*. Numer. Math. **18** (1972), 305-320.
5. M. S. Petković: *Iterative Methods for Simultaneous Inclusion of Polynomial Zeros*. Springer-Verlag, Berlin-Heidelberg-New York, 1989.
6. M. S. Petković: *Point Estimation of Root Finding Methods*. Springer, Berlin-Heidelberg-New York, 2008.

7. M. Petković, Đ. Herceg and S. Ilić: *Point Estimation Theory and its Applications*. Institute of Mathematics, Novi Sad, 1997.
8. M. S. Petković, J. Džunić and M. Milošević: *Traub's accelerating generator of iterative root finding methods*. Appl. Math. Lett. **24** (2011), 1443–1448.
9. M. S. Petković and D. M. Milošević: *A new higher-order family of inclusion zero-finding methods*. J. Comput. Appl. Math. **182** (2005), 416–432.
10. M. S. Petković and D. M. Milošević: *A higher order family for the simultaneous inclusion of multiple zeros of polynomials*. Numer. Algorithms **39** (2005), 415–435.
11. M. S. Petković, M. R. Milošević and D. M. Milošević: *Efficient method for the inclusion of polynomial zeros*. Appl. Math. Comput. **217** (2011), 7636–7652.
12. M. S. Petković, D. M. Milošević and L. D. Petković: *High order Euler-like method for the inclusion of polynomial zeros*. Appl. Math. Comput. **196** (2008), 762–773.
13. M. S. Petković and L. D. Petković: *Complex Interval Arithmetic and its Applications*. Wiley-VCH: Berlin-Weinheim-New York, 1998.
14. M. S. Petković and D. V. Vranić: *The convergence of Euler-like method for the simultaneous inclusion of polynomial zeros*. Comput. Math. Appl. **39** (2000), 95–105.

Dušan M. Milošević
Faculty of Electronic Engineering
Department of Mathematics
P.O. Box 73
18000 Niš, Serbia
dusan.milosevic@elfak.ni.ac.rs

Miodrag S. Petković
Faculty of Electronic Engineering
Department of Mathematics
P.O. Box 73
18000 Niš, Serbia
msp@junis.ni.ac.rs

Mimica R. Milošević
Faculty of Management in Civil Engineering
University UNION Nikola Tesla
11000 Belgrade, Serbia
mimica.milosevic@gmail.com