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THE q–ITERATIVE METHODS IN NUMERICAL SOLVING OF SOME EQUATIONS WITH INFINITE PRODUCTS *

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(Dedicated to prof. dr Ljiljana Petković for her 60th birthday)

Abstract. We consider a few modifications of the well known methods for numerical solving of a equation or a system of equations. Especially, we included Newton's, the Newton-Kantorovich and gradient method. The purpose was to adapt them to cases when the functions are given in the form of infinite products. The examples comprehend the infinite q-power products and prove that the methods are pretty suitable for them.

1. Introduction

A lot of papers were written about the iterative methods for solving of a nonlinear equation

$$F(x) = 0,$$

where $F(x)$ is a continuous operator defined on a nonempty subset of a Banach space. In a few papers were considered some unusual functions such as continuous but non-differentiable functions [7] or noncontinuous functions.

Another perspective branch of mathematics is q–calculus. It appears as a connection between mathematics and physics (see [2], [5], [10]). It has a lot of applications in different mathematical areas, such as: number theory, combinatorics, orthogonal polynomials, basic hyper geometric functions and other sciences: quantum theory, mechanics and theory of relativity.

Let $q \neq 1$. A q–complex number $[a]_q$ is defined by

$$[a]_q = \frac{1-q^a}{1-q}, \quad a \in \mathbb{C}.$$

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The factorial of a number $[n]_q$ and q-binomial coefficient, we define by

$$[0]_q! = 1, \quad [n]_q! = [n]_q[n-1]_q \cdots [1]_q, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} .$$

The important role in q–calculus has q-Pochammer symbol defined by

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{i=0}^{n-1}(1-aq^i) \quad \Big(n \in \mathbb{N} \cup \{+\infty\}\Big).$$

The q-derivative of a function $f(x)$ is

$$D_q f(x) := \frac{f(x)-f(qx)}{x-qx} \ (x \neq 0), \quad D_q f(0) := \lim_{x\to 0}(D_q f)(x),$$

and high q-derivatives $D_q^0 f := f, \quad D_q^n f := D_q(D_q^{n-1} f), \quad n = 1,2,3,\ldots.$

From the above definition, it is obvious that a continuous function on an interval, which does not include 0, is continuous q-differentiable.

In q–analysis, q–integral is defined by

$$I_q(f) = \int_0^a f(t)d_q(t) := a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n.$$

Notice that according to [10], it holds

$$I(f) = \int_0^a f(t)\, dt = \lim_{q\uparrow 1} I_q(f).$$

Also,

$$\int_a^b f(t)d_q(t) := \int_0^b f(t)d_q(t) - \int_0^a f(t)d_q(t).$$

2. The partial q–derivatives and q–differential

Let $f(\vec{x})$, where $\vec{x} = (x_1, x_2, \ldots, x_n)$ be a multi variable real continuous function. We introduce an operator $\varepsilon_{q,i}$ which multiplies a coordinate of the argument by

$$(\varepsilon_{q,i} f)(\vec{x}) = f(x_1, \ldots, x_{i-1}, qx_i, x_{i+1}, \ldots, x_n).$$

Furthermore,

$$(\varepsilon_q f)(\vec{x}) = (\varepsilon_{q,1} \cdots \varepsilon_{q,n} f)(\vec{x}) = f(q\vec{x}).$$

We define partial q–derivative of a function $f(\vec{x})$ to a variable x_i by

$$D_{q,x_i} f(\vec{x}) = \frac{f(\vec{x}) - (\varepsilon_{q,i} f)(\vec{x})}{(1-q)x_i} \quad (x_i \neq 0), \quad D_{q,x_i} f(\vec{x})\Big|_{x_i=0} = \lim_{x_i\to 0} D_{q,x_i} f(\vec{x}).$$

At the similar way, high partial q–derivatives are

$$\begin{aligned} D_q^0 f(\vec{x}) &= f(\vec{x}), \qquad D^m_{q;x_1^{k_1}\dots x_i^{k_i}\dots x_n^{k_n}} f(\vec{x}) = \Big(D_{q,x_i}\Big(D^{m-1}_{q;x_1^{k_1}\dots x_i^{k_i-1}\dots x_n^{k_n}} f\Big)\Big)(\vec{x})\,, \\ m &= k_1+\cdots+k_n, \qquad m=1,2,\dots\,. \end{aligned}$$

Obviously,

$$D^{r+s}_{q,x_i^m,x_j^n} f(\vec{x}) = D^{r+s}_{q,x_j^n,x_i^r} f(\vec{x}) \quad (i,j=1,2\dots,n)\ (r,s=0,1,\dots).$$

Also, for an arbitrary $\vec{a}=(a_1,a_2,\dots,a_n)\in\mathbb{R}^n$, we can introduce q–differential

$$d_q f(\vec{x},\vec{a}) = (x_1-a_1)D_{q,x_1} f(\vec{a}) + (x_2-a_2)D_{q,x_2} f(\vec{a}) + \cdots + (x_n-a_n)D_{q,x_n} f(\vec{a}),$$

and high q–differentials:

$$\begin{aligned} d_q^k f(\vec{x},\vec{a}) &= \Big((x_1-a_1)D_{q,x_1}+\cdots+(x_n-a_n)D_{q,x_n}\Big)^k f(\vec{a}) \\ &= \sum_{\substack{i_1+\cdots+i_n=k \\ i_j\in\mathbb{N}_0}} \frac{[k]_q!}{[i_1]_q![i_2]_q!\cdots[i_n]_q!} D^k_{q,x_i^{i_1},\cdots,x_n^{i_n}} f(\vec{a}) \prod_{j=1}^{n} x_j^{i_j}\left(a_j/x_j;q\right)_{i_j}\,. \end{aligned}$$

Notice, that a continuous function $f(\vec{x})$ in a neighborhood, which does not include any point with a zero coordinate, has also continuous q-partial derivatives. More details can be found in [8] and [11].

3. Numerical computing of infinite products

The exact value of an infinite product can not be evaluated. That is why a few numerical methods are developed for its approximating. The most simple way is to omit factors from an index and to compute truncated product

$$P = \prod_{k=1}^{\infty} a_k \;\approx\; P_n = \prod_{k=1}^{n} a_k \tag{3.1}$$

But, it is not an effective method, neither to exactness nor to number of operations. L. Slater [13] mentioned some computing difficulties for $0.89 < q < 1$ that can be avoided by using the logarithmic form.

The *Euler identity*

$$P \equiv (aq;q)_\infty = \prod_{n=1}^{\infty}(1-aq^n) = \sum_{n=0}^{\infty} \frac{(-1)^n a^n q^{n(n+1)/2}}{(q;q)_n} \quad (|q|<1)\,, \tag{3.2}$$

enable us to compute an infinite sum instead of the infinite product. A.D. Sokal [14] proved that the truncated method (3.1) has linear convergence, which is weaker in comparison with the square convergence of the method (3.2). Computing by the Euler identity is more precise although it has troubles for q nearby 1 because of dividing by $(q;q)_n$.

Example 3.1. Numerical evaluating of infinite product $P \equiv (xq;q)_\infty$ for $x = 1/3$ and $q = 1/2$ are compared on the next table and illustrated by Figure 3.1.

Members	Error (3.1)	Error (3.2)
5	$4.909 \star 10^{-3}$	$1.483 \star 10^{-9}$
10	$1.523 \star 10^{-4}$	$1.765 \star 10^{-25}$
20	$1.488 \star 10^{-7}$	$6.395 \star 10^{-80}$

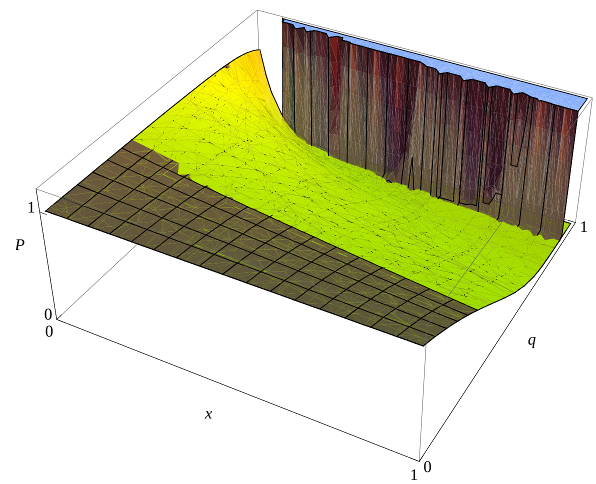

Fig. 3.1: The comparison of the truncated product (smooth yellow surface) and the Euler identity (mesh surface)

Remark 3.1. L. Gatteschi has introduced the iterative scheme (see [6] or [1])

$$x_{n+1} = x_n \frac{(q-1)x_n + (3-q)y_n}{2y_n}, \qquad y_{n+1} = x_{n+1} \frac{y_n}{qx_n + (1-q)y_n},$$

with initial values (x_0, y_0) which have to be related by $(1-q)x_0 = (1+2a-q)y_0$. Then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x_0(a;q)_\infty$.

The expansions of $\prod_{k=1}^{\infty}(1-q^{ak})(1-q^{bk})$ were studied in [15].

4. On q-Newton method

If in the previous speculation we take $n = 1$, the system of equations reduce to one equation $f(x) = 0$ and the main objects of work are functions of one variable.

T. Ernst [4] introduced the following q-Taylor formula

$$f(z) = \sum_{k=0}^{n-1} \frac{D_q^k f(c)}{[k]_q!} z^k (c/z;q)_k + R_n(f,z,c,q),$$

where $R_n(f,z,c,q)$ is the remainder term determined by

$$R_n(f,z,c,q) = \int_{t=c}^{t=z} \frac{z^n (t/z;q)_n}{z-t} \frac{D_q^n f(t)}{[n-1]_q!} d_q(t).$$

Suppose that an equation $f(x) = 0$ has the unique isolated solution $x = \xi$. If x_n is an approximation to the exact solution ξ, by using Jackson's q-Taylor formula, we have

$$0 = f(\xi) \approx f(x_n) + D_q f(x_n)(\xi - x_n),$$

hence

$$\xi \approx x_n - \frac{f(x_n)}{D_q f(x_n)}.$$

So, we can construct *q-Newton method*

$$x_{n+1} = x_n - \frac{f(x_n)}{D_q f(x_n)}. \tag{4.1}$$

We can rearrange the above expression to the form

$$x_{n+1} = x_n \left\{ 1 - \frac{1-q}{1 - \frac{f(qx_n)}{f(x_n)}} \right\}. \tag{4.2}$$

This method written in the form

$$x_{n+1} = x_n - \frac{x_n - qx_n}{f(x_n) - f(qx_n)} f(x_n) \tag{4.3}$$

resembles the method of chords (secants).

Theorem 4.1. *Let the equation $f(x) = 0$ has a unique isolated root $x = \xi$ and $a > 0$, $1 \leq p \leq 2$. Let the function $f(x)$ satisfies*

$$\begin{aligned} &(1) \qquad |D_q f(x)| \geq M_1^{p-1} > 0, \\ &(2) \qquad |f(x) - f(y) - D_q f(y)(x-y)| < L^{p-1}|x-y|^p, \end{aligned}$$

where M_1 and L are positive constants. Then, for all initial values $x_0 \in (\xi - b, \xi + b)$, where $b = \min\{a, M_1/L\}$, the q-Newton method converges to the exact solution of the equation $f(x) = 0$ and it is valid

$$|\xi - x_n| \leq \left(\frac{L}{M_1}\right)^{p^n - 1} |\xi - x_0|^{p^n}.$$

Proof. We can write q-Newton method (4.1) in the form

$$D_q f(x_n)(x_{n+1} - x_n) = -f(x_n).$$

From the condition (2), we have

$$|f(\xi) - f(x_n) - D_q f(x_n)(\xi - x_n)| < L^{p-1}|\xi - x_n|^p.$$

Hence, using $f(\xi) = 0$, we yield

$$|D_q f(x_n)(\xi - x_{n+1})| < L^{p-1}|\xi - x_n|^p.$$

By the condition (1), we have

$$|\xi - x_{n+1}| < \frac{L^{p-1}}{|D_q f(x_n)|}|\xi - x_n|^p < \left(\frac{L}{M_1}\right)^{p-1}|\xi - x_n|^p.$$

Now, if $x_n \in (\xi - b, \xi + b)$, then

$$|\xi - x_{n+1}| < \left(\frac{L}{M_1}\right)^{p-1} b^p = \left(\frac{L}{M_1}\right)^{p-1} b^{p-1} b \leq b.$$

Denote by $c = L/M_1$. Now

$$|\xi - x_{n+1}| < c^{p-1}|\xi - x_n|^p \quad \Rightarrow \quad c\,|\xi - x_{n+1}| < c^p|\xi - x_n|^p,$$

wherefrom we get the final conclusion. □

Our purpose is to formulate and prove the theorem for scanning the convergence of an iterative process

$$x_{k+1} = \Phi(x_k) \quad (k = 0, 1, 2, \ldots),$$

by q-analysis.

Theorem 4.2. *Suppose that* $\Phi(x)$ *is a continuous function on* $[a, b]$ $(0 \notin [a, b])$, *which satisfies the following conditions:*

$$\begin{aligned} &(1) \qquad \Phi : [a, b] \mapsto [a, b], \\ &(2) \qquad \Big(\forall q \in (\min\{a, b\}/\max\{a, b\}, 1)\Big)\Big(\forall x \in (a, b)\Big) : \ \Big|D_q f(x)\Big| \leq \lambda < 1. \end{aligned}$$

Then the iterative process $x_{k+1} = \Phi(x_k)$, $k = 0, 1, 2, \ldots$, *with initial value* $x_0 \in [a, b]$, *is converging to the fixed point of* $\Phi(x)$, *i.e.,*

$$\lim_{k\to\infty} x_k = \xi, \quad \Phi(\xi) = \xi.$$

Proof. Notice that for a continuous function $\Phi(x)$ on $[a,b]$ $(0 \notin [a,b])$, for all x and y such that $a < x < y < b$, it is valid

$$\Phi(y) - \Phi(x) = D_{x/y}\Phi(y)(y-x), \quad \Phi(y) - \Phi(x) = D_{y/x}\Phi(x)(y-x).$$

Consider

$$\xi = x_0 + \sum_{k=0}^{\infty}(x_{k+1} - x_k), \quad S_n = x_0 + \sum_{k=0}^{n}(x_{k+1} - x_k). \tag{4.4}$$

Let $x_k^{(M)} = \max\{x_k, x_{k-1}\}$, $x_k^{(m)} = \min\{x_k, x_{k-1}\}$ and $q = x_k^{(m)}/x_k^{(M)}$. Now, we have

$$\Phi(x_k) - \Phi(x_{k-1}) = D_q\Phi(x_k^{(M)})(x_k - x_{k-1}).$$

So, it is valid

$$\left|x_{k+1} - x_k\right| = \left|D_q\Phi(x_k^{(M)})\right| |x_k - x_{k-1}| \le \lambda \, |x_k - x_{k-1}|.$$

Since

$$\left|x_{k+1} - x_k\right| \le \lambda^k |x_1 - x_0|,$$

we get

$$\sum_{k=0}^{\infty} |x_{k+1} - x_k| \le |x_1 - x_0| \sum_{k=0}^{\infty} \lambda^k = \frac{|x_1 - x_0|}{1-\lambda}.$$

Hence, the series (4.4) converges and

$$\xi = \lim_{n\to\infty} S_n = \lim_{n\to\infty} x_{n+1}.$$

Since $\Phi(x)$ is a continuous function, we have

$$\xi = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \Phi(x_n) = \Phi(\lim_{n\to\infty} x_n) = \Phi(\xi). \ \square$$

Definition 4.1. An iterative method $x_{n+1} = \Phi(x_n)$ $(n = 0,1,2,\ldots)$ with the fixed point ξ, has $(r;q)$-order of convergence, if there exists $C_r \in \mathbb{R}^+$ such that, for large enough n, it is valid

$$|\xi - x_{n+1}| < C_r |\xi^r (x_n/\xi; q)_r|.$$

Theorem 4.3. ([11]) *Let $f(x)$ be a continuous function on $[a,b]$ and $R_n(f,z,c,q)$, $z,c \in (a,b)$ be the remainder term in q–Taylor formula. Then it exists $\hat{q} \in (0,1)$ such that for all $q \in (\hat{q},1)$, can be found $\xi \in (a,b)$ between c and z, which satisfies*

$$R_n(f,z,c,q) = \frac{D_q^n f(\xi)}{[n-1]_q!} \int_{t=c}^{t=z} \frac{z^n (t/z;q)_n}{z-t} d_q(t) = \frac{D_q^n f(\xi)}{[n]_q!} z^n (c/z;q)_n.$$

Now, we are ready to prove the main theorem of this section.

Theorem 4.4. *Suppose that a function $f(x)$ is continuous on a segment $[a,b]$ and that the equation $f(x)=0$ has a unique isolated solution $\xi \in (a,b)$. If the conditions*

$$|D_q f(x)| \geq M_1, \qquad |D_q^2 f(x)| \leq M_2,$$

are satisfied for some positive constants M_1 and M_2 and all $x \in (a,b)$, then there exists $\hat{q} \in (0,1)$, such that for all $q \in (\hat{q},1)$, the iterations obtained by q–Newton method satisfy

$$|\xi - x_{k+1}| \leq \frac{M_2}{(1+q)M_1} |\xi^2 (x_k/\xi; q)_2|,$$

i.e., q-Newton method has $(2;q)$–order of convergence.

Proof. From the formulation of q–Newton method, we have

$$x_{k+1} - \xi = x_k - \xi - \frac{f(x_k)}{D_q f(x_k)},$$

hence

$$f(x_k) + D_q f(x_k)(\xi - x_k) = D_q f(x_k)(\xi - x_{k+1}).$$

By using q–Taylor formula of order $n=2$ at the point x_k for $f(\xi)$, we have

$$f(\xi) = f(x_k) + D_q f(x_k)(\xi - x_k) + R_2(f, \xi, x_k, q).$$

Since $f(\xi) = 0$, we get

$$D_q f(x_k)(\xi - x_{k+1}) = -R_2(f, \xi, x_k, q),$$

i.e.

$$|\xi - x_{k+1}| = \frac{|R_2(f, \xi, x_k, q)|}{|D_q f(x_k)|}.$$

According to Theorem 4.3, there exists $\hat{q} \in (0,1)$ such that for all $q \in (\hat{q},1)$ it can be found $\xi \in (a,b)$ such that

$$R_2(f, \xi, x_k, q) = \frac{D_q^2 f(\xi)}{[2]_q} \xi^2 (x_k/\xi; q)_2.$$

Now,

$$|\xi - x_{k+1}| = \frac{|D_q^2 f(\xi)|}{|D_q f(x_k)|} \, \frac{|\xi^2 (x_k/\xi; q)_2|}{1+q}.$$

Using the conditions which function $f(x)$ and its q–derivatives satisfy, we yield the statement of the theorem. □

Example 4.1. For fixed real numbers b and $q \in (-1,1)$, we consider the equation

$$f(x) \equiv (x;q)_\infty - b = 0,$$

with unknown value x. Fine thing in q-Newton method (4.3) is appearance of $f(qx) = (qx;q)_\infty - b$. It is the function of the same type as $f(x)$ what makes easier the required computations.

For example, taking $b = \sqrt[24]{64/e^\pi}$ and $q = 1/e^{2\pi}$, starting with initial value $x_0 = 1/2$, after two iterations by q-Newton method we get the approximation with 6 exact digits of the exact solution $x = -1/e^\pi \approx -0.0432139$. It is quite in harmony with the known product

$$\prod_{n=0}^{\infty}\left(1 + e^{-(2k+1)\pi)}\right) = \sqrt[24]{\frac{64}{e^\pi}}\,.$$

Example 4.2. For given b and $q \in (-1,1)$, we look for unknown x from

$$(q;x)_\infty \equiv \prod_{n=0}^{\infty}(1 - qx^n) = b.$$

So, for the particular $b = 0.0895642804083$ and $q = 3/4$, starting from $x_0 = 0.5$, after 3 iterations by q-Newton method for $q = 3/4$, we find the approximation for the exact solution $x = 0.538463\ldots \approx 7/13$.

Example 4.3. For a given $q \in (-1,1)$, for the equation

$$(-q;x)_\infty^8 - (q;x)_\infty^8 - 16q(-q^2;x)_\infty^8 = 0$$

we get very close approximation of the exact solution $x = q^2$. It is the cutting curve (parabola) of two surfaces drawn on the Figure 4.1. It comes from known identity

$$(-q;q^2)_\infty^8 = (q;q^2)_\infty^8 + 16q(-q^2;q^2)_\infty^8.$$

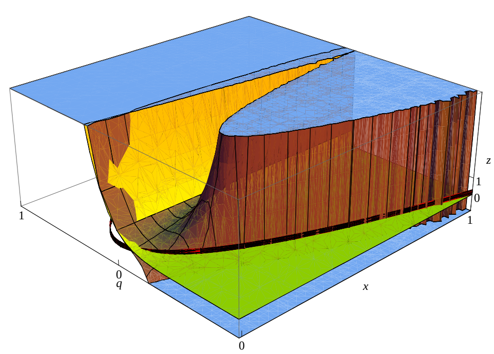

FIG. 4.1: Graphics of the functions $z = f_1(x,q) \equiv (-q;x)_\infty^8$ (yellow-green surface), $z = f_2(x,q) \equiv (q;x)_\infty^8 + 16q(-q^2;x)_\infty^8$ (brown surface) and $x = q^2$ (black line)

5. On q–Newton-Kantorovich method

For this section, we find motivation in the paper [12] where a modification of the Newton–Kantorovich method [9] was exposed.

Let

$$\vec{f}(\vec{x}) = \mathbf{0}$$

be a system of nonlinear equations, where

$$\vec{f}(\vec{x}) = \Big(f_1(\vec{x}), f_2(\vec{x}), \dots f_n(\vec{x})\Big), \qquad \vec{x} = (x_1, x_2, \dots x_n) \qquad (n \in \mathbb{N}).$$

The numerous methods have been developed for their solving (see, for example, [3]).

We will suppose that this system has an isolated real solution $\vec{\xi}$. Using q-Taylor series of the function $\vec{f}(\vec{x})$ around some value $\vec{x}^{(m)} \approx \vec{\xi}$, we have

$$f_i(\vec{\xi}) \approx f_i\Big(\vec{x}^{(m)}\Big) + \sum_{j=1}^{n} D_{q,x_j} f_i\Big(\vec{x}^{(m)}\Big)\Big(\xi_j - x_j^{(m)}\Big) \quad (i = 1, 2, \dots, n).$$

We can rewrite this in the following matrix form

$$\vec{f}(\vec{\xi}) \approx \vec{f}\Big(\vec{x}^{(m)}\Big) + W_q\Big(\vec{x}^{(m)}\Big)\Big(\vec{\xi} - \vec{x}^{(m)}\Big),$$

where

$$W_q(\vec{x}) = D_q \vec{f}(\vec{x}) = \Big[D_{q,x_j} f_i(\vec{x})\Big]_{n\times n}$$

is the Jacobi matrix of partial q–derivatives. If the matrix W_q is regular, there exists the inverse matrix W_q^{-1}, so that we can formulate q–Newton–Kantorovich method in the form

$$\vec{x}^{(m+1)} = \vec{x}^{(m)} - W_q\Big(\vec{x}^{(m)}\Big)^{-1} \vec{f}\Big(\vec{x}^{(m)}\Big).$$

Theorem 5.1. *Let the function $\vec{f}(\vec{x})$ has q-partial derivatives to all variables x_j $(i, j = 1, \dots, n)$ in a ball $K[\vec{x}^{(0)}, R] = \{\vec{x} : \|\vec{x} - \vec{x}^{(0)}\| \le R\}$. Suppose that the matrix $W_q(\vec{x})$ is regular in this ball and the conditions*

$$\|W_q(\vec{x}) - W_q(\vec{y})\| \le L\|\vec{x} - \vec{y}\|,$$

$$\|\vec{f}(\vec{x}) - \vec{f}(\vec{y}) - W_q(\vec{y})(\vec{x} - \vec{y})\| \le \frac{L}{2}\|\vec{x} - \vec{y}\|^2,$$

are satisfied for all $\vec{x}, \vec{y} \in K[x^{(0)}, R]$ and a constant $L > 0$. If there are fulfilled the inequalities

$$\|W_q\Big(\vec{x}^{(0)}\Big)^{-1}\| \le b, \quad \|W_q\Big(\vec{x}^{(0)}\Big)^{-1} \vec{f}\Big(\vec{x}^{(0)}\Big)\| \le a, \quad h = abL \le 1/2$$

and

$$R > r = \frac{1 - \sqrt{1 - 2h}}{h}\, a,$$

then the sequence $\{\vec{x}^{(m)}\}_{m\in\mathbb{N}_0}$ *converges to the solution* $\vec{\xi} \in K[\vec{x}^{(0)}, r]$ *and it is valid*

$$\|\vec{\xi} - \vec{x}^{(m)}\| \leq \frac{a}{2^{m-1}}\Big(2h\Big)^{2^m-1} \quad (m \in \mathbb{N}).$$

Example 5.1. For known $q \in (-1, 1)$, consider a system of equations

$$f_1(x, y) \equiv 2\,(xq; q)_\infty - (y^3q; q)_\infty = 0\,, \qquad f_2(x, y) \equiv (x^2q; q)_\infty + e^{(yq;q)_\infty} - 2 = 0\,,$$

with unknown values x and y.

Especially, for $q = 3/4$, starting with initial values $x_0 = y_0 = 3/4$, after $n = 8$ iterations provided by q–Newton–Kantorovich method, we find solution with six exact decimal digits $x = 0.303135835\ldots$, $y = 0.45229252\ldots$, what is shown on the Figure 5.1.

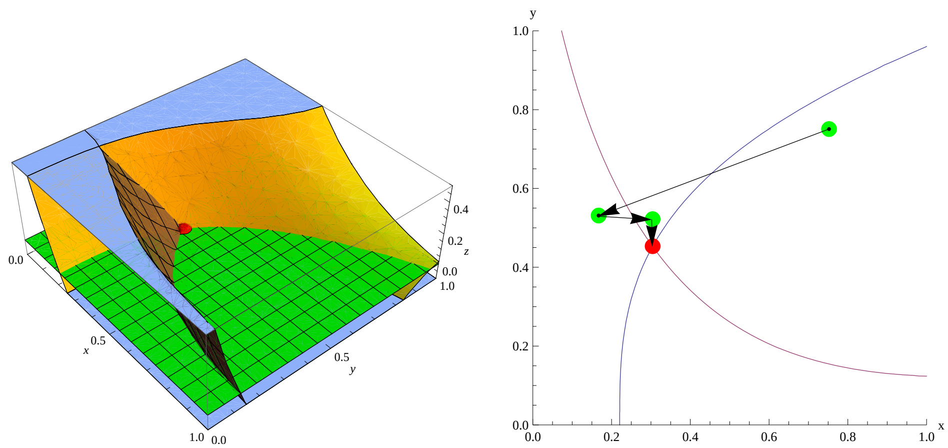

Fig. 5.1: a) Graphics of the functions $z = f_1(x, y)$, $z = f_2(x, y)$ and $z = 0$ (green); b) approximations of the red solution provided by q-gradient method

6. On q–gradient method

We will consider again a system of nonlinear equations

$$\vec{f}(\vec{x}) = 0,$$

where $\vec{f}(\vec{x}) = \Big(f_1(\vec{x}), f_2(\vec{x}), \ldots f_n(\vec{x})\Big)$ with $\vec{x} = (x_1, x_2, \ldots x_n) \in \mathbb{R}^n$, $n \in \mathbb{N}$, and assume that thus system has an isolated real solution $\vec{\xi}$.

Here, we will associate to the function $\vec{f}(\vec{x})$ a new function $\vec{U}(\vec{x})$ by

$$U(\vec{x}) = \Big(\vec{f}(\vec{x}), \vec{f}(\vec{x})\Big) = \sum_{i=1}^{n}\Big(f_i(x_1, \ldots, x_n)\Big)^2.$$

The function $U(\vec{x})$ is nonnegative for all $\vec{x} \in \mathbb{R}^n$ and vanishes when $\vec{f}(\vec{x})$ vanishes.

For solving the mentioned system of equations, we want to construct an iterative method of the form

$$\vec{x}^{(k+1)} = \vec{x}^{(k)} - \lambda_k \nabla_q U(\vec{x}^{(k)}), \quad (k = 0, 1, \ldots),$$

where

$$\nabla_q U(\vec{x}) = \mathrm{grad}_q U(\vec{x}) = [D_{q,x_1} U(\vec{x}) \quad \ldots \quad D_{q,x_n} U(\vec{x})]^T$$

and λ_k is the solution of minimization problem

$$\min_{\lambda} U\Big(\vec{x}^{(k)} - \lambda \nabla_q U(\vec{x}^{(k)})\Big).$$

Denote by $\Phi(\lambda)$ the function

$$\Phi(\lambda) = U\Big(\vec{x}^{(k)} - \lambda \nabla_q U(\vec{x}^{(k)})\Big) = \sum_{i=1}^{n} \Big(f_i\Big(\vec{x}^{(k)} - \lambda \nabla_q U(\vec{x}^{(k)})\Big)\Big)^2.$$

By q-Taylor expansion of the function f_i around point $\vec{x}^{(k)}$, the function $\Phi(\lambda)$ can be approximated by

$$\tilde{\Phi}(\lambda) = \sum_{i=1}^{n} \Big(f_i(\vec{x}^{(k)}) - \lambda\Big(\nabla_q f_i(\vec{x}^{(k)}), \nabla_q U(\vec{x}^{(k)})\Big)\Big)^2.$$

Since

$$\frac{d\tilde{\Phi}(\lambda)}{d\lambda} = -2 \sum_{i=1}^{n} \Big(f_i(\vec{x}^{(k)}) - \lambda\Big(\nabla_q f_i(\vec{x}^{(k)}), \nabla_q U(\vec{x}^{(k)})\Big)\Big) \Big(\nabla_q f_i(\vec{x}^{(k)}), \nabla_q U(\vec{x}^{(k)})\Big),$$

we will choose λ_k as the solution of equation

$$\frac{d}{d\lambda} \tilde{\Phi}(\lambda) = 0,$$

i.e.

$$\lambda_k = \frac{\sum_{i=1}^{n} f_i(\vec{x}^{(k)}) \Big(\nabla_q f_i(\vec{x}^{(k)}), \nabla_q U(\vec{x}^{(k)})\Big)}{\sum_{i=1}^{n} \Big(\nabla_q f_i(\vec{x}^{(k)}), \nabla_q U(\vec{x}^{(k)})\Big)^2}.$$

Again, let

$$W_q(\vec{x}) = D_q \vec{f}(\vec{x}) = \Big[D_{q,x_j} f_i(\vec{x})\Big]_{n\times n}$$

be the Jacobi matrix of q-partial derivatives. Hence

$$\lambda_k = \frac{\Big(f(\vec{x}^{(k)}), W_q(\vec{x}^{(k)}) \nabla_q U(\vec{x}^{(k)})\Big)}{\Big(W_q(\vec{x}^{(k)}) \nabla_q U(\vec{x}^{(k)}), W_q(\vec{x}^{(k)}) \nabla_q U(\vec{x}^{(k)})\Big)}.$$

Example 6.1. We want to solve the system of nonlinear equations

$$f_1(x,y) \equiv 2(xq;q)_\infty - (yq;q)_\infty = 0, \qquad f_2(x,y) \equiv (x^2yq;q)_\infty - (q;q)_\infty = 0.$$

For the concrete value $q = 3/4$, taking the initial values $x_0 = y_0 = 1/2$, we get the tenth iteration $x = 1.022328910$, $y = 0.955965476$, which is the approximation with 3 exact digits. On the Figure 6.1, the functions $z = f_1(x,y)$ and $z = f_2(x,y)$ are drawn. Here, again we notice the troubles with computing $f_2(x,y)$ for some values $x > 1$ and $y > 1$.

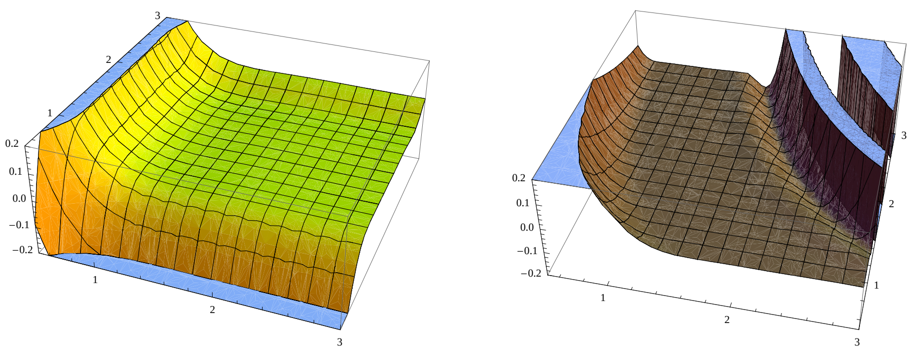

FIG. 6.1: The functions $z = f_1(x,y)$ (yellow surface) and $z = f_2(x,y)$ (brown surface)

On the first figure of Figure 6.2, they are both shown with the horizontal plane and on the second one, the implicit functions $f_1(x,y) = 0$ and $f_2(x,y) = 0$ and approximations are drawn.

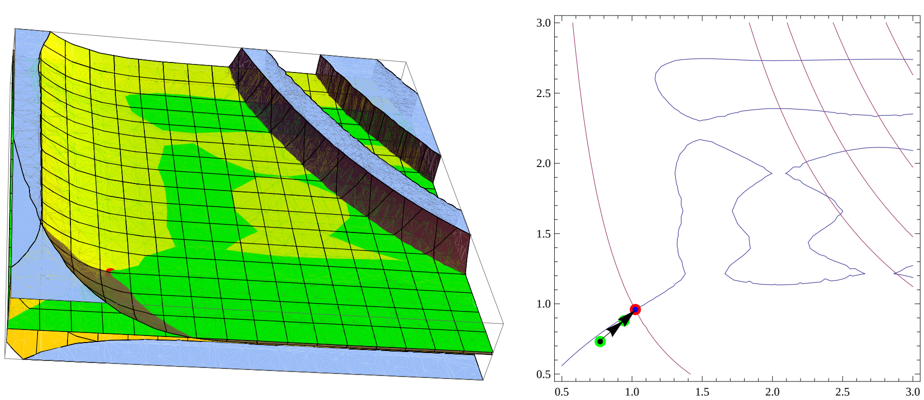

FIG. 6.2: a) Graphics of the functions $z = f_1(x,y)$, $z = f_2(x,y)$ and $z = 0$ (green); b) approximations of the red solution provided by q-gradient method

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