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IMPROVED HIGHER ORDER METHOD FOR THE INCLUSION OF MULTIPLE ZEROS OF POLYNOMIALS*

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Abstract. Starting from a suitable fixed point relation and employing Schröder's and Halley-like corrections, we derive some high order iterative methods for the simultaneous inclusion of polynomial multiple zeros in circular complex interval arithmetic. These methods are more efficient compared to the existing inclusion methods based on fixed point relations. Using the concept of the *R*-order of convergence of mutually dependent sequences, we present the convergence analysis of the obtained total-step and single-step methods. The proposed self-validated methods possess a great computational efficiency since the acceleration of the convergence rate from four to seven is achieved with only few additional calculations. Numerical examples illustrate the convergence properties of the presented methods.

1. Introduction

Self-validated methods for the simultaneous determination of complex zeros of a given polynomial, realized in complex interval arithmetic, are a very useful tool for the error estimates of a given set of approximate zeros. Based on the very important inclusion property, which provides the enclosure of the sought zeros in each iteration, this class of methods produces resulting disks each of which contains one and only one zero in every iteration. Due to this property, these methods are highly appreciated as the most powerful iterative methods for the inclusion of polynomial zeros.

In this paper we are concerned with the construction of advanced inclusion methods for the simultaneous determination of polynomial zeros with the special emphases to the fact that these zeros can be multiple. This work can be regarded as a continuation of the research on interval versions of Halley-like iterative method presented in [16] and [26], and discussed later in [17] and [22].

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The paper is organized as follows. Some basic definitions and operations of circular complex interval arithmetic, necessary for the construction and the convergence analysis of inclusion methods, are given at the end of Introduction. In Section 2 we develop the total-step methods without and with corrections, while the convergence analysis of these methods is presented in Section 3. The single-step versions of these methods are discussed in Section 4, and the numerical examples are given in Section 5.

The development and convergence analysis of the proposed inclusion methods need the basic properties of the so-called circular complex arithmetic, introduced by Gargantini and Henrici [5]. A circular closed region (disk) $Z := \{z : |z - c| \leq r\}$ with center $c :=$ mid Z and radius $r :=$ rad Z we will denote by parametric notation $Z := \{c; r\}$. In the proof of this lemma and forthcoming assertions, we will use the following operations and properties of circular complex arithmetic. Let $Z_k := \{c_k; r_k\}$ $(k = 1, 2)$, then

$$\begin{aligned} Z_1 \pm Z_2 &= \{c_1 \pm c_2; r_1 + r_2\}, \\ w \cdot \{c; r\} &= \{wc; |w|r\} \ \ (w \in C), \\ Z_1 \cdot Z_2 &= \{c_1c_2; |c_1|r_2 + |c_2|r_1 + r_1r_2\}, \end{aligned}$$

$$\{c; r\}^{-1} = \frac{\{\bar{c}; r\}}{|c|^2 - r^2} \quad (0 \notin \{c; r\}), \ \ \text{(exact inversion)}, \tag{1.1}$$

$$\{c; r\}^{I_c} = \left\{\frac{1}{c}; \frac{r}{|c|(|c| - r)}\right\} \quad (0 \notin \{c; r\}), \ \ \text{(centered inversion)}, \tag{1.2}$$

$$Z_1 : Z_2 = Z_1 \cdot Z_2^{-1} \ \text{ or } \ Z_1 : Z_2 = Z_1 \cdot Z_2^{I_c} \quad (0 \notin Z_2).$$

For the basic interval operations $+, -, \cdot, :$ the *inclusion property* is valid, that is,

$$Z_k \subseteq W_k \Rightarrow Z_1 * Z_2 \subseteq W_1 * W_2 \quad (k = 1, 2; * \in \{+, -, \cdot, :\}).$$

More details about circular arithmetic can be found in the books [1], [15], [18] and [25]. Throughout this paper disks in the complex plane will be denoted by capital letters.

2. Total-step methods

Let f be a monic polynomial of degree $n \geq 3$ with simple or multiple complex zeros $\zeta_1, \ldots, \zeta_\nu$ $(2 \leq \nu \leq n)$, with respective multiplicities $\mu_1, \ldots, \mu_\nu$ $(\mu_1 + \ldots + \mu_\nu = n)$ and let

$$\begin{aligned} \Delta_{0,i} &= 1, \\ \Delta_{k,i}(z) &= \sum_{\nu=1}^{k} (-1)^{k-\nu} \frac{1}{\mu_i}\Big(\frac{1}{\mu_i} + 1\Big) \ldots \Big(\frac{1}{\mu_i} + \nu - 1\Big) \sum \prod_{\lambda=1}^{k} \frac{1}{p_\lambda!}\Big(\frac{f^{(\lambda)}(z)}{\lambda!\, f(z)}\Big)^{p_\lambda}, \end{aligned}$$

where $k = 1, 2, \ldots$, and the second sum on the right-hand side is taken over all nonnegative integers $(p_1, \ldots, p_k)$ which satisfy $p_1+2p_2+\ldots+kp_k = k$, $p_1+p_2+\ldots+p_k = \nu$.

For example, we have

$$\Delta_{1,i}(z) = \frac{1}{\mu_i}\frac{f'(z)}{f(z)}, \quad \Delta_{2,i}(z) = \frac{1}{2\mu_i}\Big(\frac{1}{\mu_i}+1\Big)\Big(\frac{f'(z)}{f(z)}\Big)^2 - \frac{1}{2\mu_i}\frac{f''(z)}{f(z)}.$$

We observe that the function

$$N_i(z) = \frac{\Delta_{0,i}(z)}{\Delta_{1,i}(z)} = \mu_i\frac{f(z)}{f'(z)}$$

appears in the Schröder iterative method $\hat{z} = z - N_i(z)$ of the second order, and

$$H_i(z) = \frac{\Delta_{1,i}(z)}{\Delta_{2,i}(z)} = \left(\Big(\frac{1+1/\mu_i}{2}\Big)\frac{f'(z)}{f(z)} - \frac{f''(z)}{2f'(z)}\right)^{-1}$$

occurs in cubically convergent Halley's iterative formula $\hat{z} = z - H_i(z)$.

In our consideration we will use the abbreviations

$$\Sigma_{k,i} := \sum_{\substack{j=1\\ j\neq i}}^{\nu} \frac{\mu_j}{(z-\zeta_j)^k} \quad (k = 1, 2).$$

The following fixed point relation was derived in [26]:

$$\zeta_i = z - \cfrac{1}{H_i(z)^{-1} - \cfrac{f(z)}{2f'(z)}\Big(\cfrac{1}{\mu_i}\Sigma_{1,i}^2 + \Sigma_{2,i}\Big)}, \quad (i \in \boldsymbol{I}_\nu := \{1, \ldots, \nu\}). \tag{2.1}$$

Let us define the disk

$$S_{\lambda,i}(\boldsymbol{X}, \boldsymbol{W}) := \sum_{j=1}^{i-1} \mu_j\Big(\mathrm{INV}_1(z - X_j)\Big)^\lambda + \sum_{j=i+1}^{\nu} \mu_j\Big(\mathrm{INV}_1(z - W_j)\Big)^\lambda \quad (\lambda = 1, 2),$$

where $\boldsymbol{X} = (X_1, \ldots, X_\nu)$ and $\boldsymbol{W} = (W_1, \ldots, W_\nu)$ are vectors whose components are disks and $\mathrm{INV}_1 \in \{()^{-1}, ()^{I_c}\}$.

Remark 2.1 According to [14], we write $\Big(\mathrm{INV}_1(z - X_j)\Big)^k$ rather than $\mathrm{INV}_1(z - X_j)^k$ since $\mathrm{rad}\Big(\mathrm{INV}_1(z - X_j)\Big)^k \le \mathrm{rad}\,\mathrm{INV}_1(z - X_j)^k$ $(0 \notin X_j,\ k = 1, 2)$.

An interval function F is called *complex circular extension* of a complex function f if

$$F(z) = f(z), \ (z \in Z), \quad F(Z) \supseteq \{f(z) : \ z \in Z\}.$$

If f is a rational function and F is its complex circular extension, then

$$Z_k \subseteq W_k \ (k = 1, \dots, q) \Rightarrow F(Z_1, \dots, Z_q) \subseteq F(W_1, \dots, W_q).$$

In particular, we have

$$w_k \in W_k \ (k = 1, \dots, q;\ w_k \in C) \Rightarrow f(w_1, \dots, w_q) \in F(W_1, \dots, W_q). \tag{2.2}$$

Taking disks $Z_1, \dots, Z_\nu$ containing the zeros $\zeta_1, \dots, \zeta_\nu$ instead of these zeros, and taking $z = z_i := \operatorname{mid} Z_i$ in (2.1), using the inclusion property (2.2) we obtain the following inclusion,

$$\zeta_i \in z_i - \mathrm{INV}_2\left(H_i(z_i)^{-1} - \frac{f(z_i)}{2f'(z_i)}\left[\frac{1}{\mu_i} S_{1,i}^2(\boldsymbol{Z}, \boldsymbol{Z}) + S_{2,i}(\boldsymbol{Z}, \boldsymbol{Z})\right]\right), \tag{2.3}$$

where $\boldsymbol{Z} = (Z_1, \dots, Z_\nu)$ and $\mathrm{INV}_2 \in \{()^{-1}, ()^{I_c}\}$.

Let $(Z_1, \dots, Z_\nu) := (Z_1^{(0)}, \dots, Z_\nu^{(0)})$ be initial disjoint disks containing the zeros $\zeta_1, \dots, \zeta_\nu$, that is, $\zeta_i \in Z_i^{(0)}$ for all i, and let $z_i = \operatorname{mid} Z_i$. The relation (2.3) suggests the following *total-step* method for the simultaneous inclusion of all zeros of f :

$$\widehat{Z}_i = z_i - \mathrm{INV}_2\left(H_i(z_i)^{-1} - \frac{f(z_i)}{2f'(z_i)}\left[\frac{1}{\mu_i} S_{1,i}^2(\boldsymbol{Z}, \boldsymbol{Z}) + S_{2,i}(\boldsymbol{Z}, \boldsymbol{Z})\right]\right), \tag{2.4}$$

for every $i \in \boldsymbol{I}_\nu$, where $\widehat{Z}_i$ denotes a new outer disk approximation to the zero ζ_i. The iterative method (2.4) has the order of convergence equal to *four* (see [15]).

Remark 2.2 Evidently, the main part in the iteration formulas (2.4) is Halley's correction $H(z)$. For this reason, these methods as well as their modifications, which will be considered in this paper, are referred to as *Halley-like* methods.

Remark 2.3 A discussion concerning the choice between the exact and centered inversion in the implementation of the method (2.4) is necessary here. Namely, since the exact inversion gives smaller disks, it seems natural to use the exact inversion. However, the application of the centered inversion produces the sequence of centers of resulting disks $\hat{Z}_i$, which coincides with the very fast iterative methods (in ordinary complex arithmetic). These fast methods significantly force the convergence of the radii, that is, the contraction of the disks which leads to the accelerated convergence of interval methods. On the other side, the exact inversion gives the "shifted" centers of inverted disks. For this reason, the use of the exact inversion can accelerate the convergence to a certain extent when the Schröder corrections are used and cannot increase the convergence rate applying corrections that appear in iterative methods of the order higher than two (see [17] for a detailed analysis). Due to this, we will apply only the centered inversion (1.2), i.e. $\mathrm{INV}_1 = \mathrm{INV}_2 = ()^{I_c}$ in the sequel.

Remark 2.4 The presented methods require initial disks containing the desired zeros and the knowledge of their multiplicities. Both tasks are very important in the theory of iterative interval processes. The problem of obtaining initial disks containing the desired zeros was studied, for instance, in [2], [7] and [20], while efficient procedures for the determination of the order of multiplicity can be found in [8], [9], [10], [12] and [13].

In order to express the proposed methods with Schröder's correction $N_i(z_i)$ and Halley's correction $H_i(z_i)$ in a unique form, we proceed similarly as in [24] (see, also, [3], [19], [21], [22], [23]). Both methods can be expressed uniquely by introducing the additional index λ: $\lambda = 1$ (for Schröder's correction) and $\lambda = 2$ (for Halley's correction)

$$\widehat{Z}_i = z_i - \mathrm{INV}_2\left(H(z_i)^{-1} - \frac{f(z_i)}{2f'(z_i)}\left[\frac{1}{\mu_i}S_{1,i}^2(\boldsymbol{Z}^{(\lambda)}, \boldsymbol{Z}^{(\lambda)}) + S_{2,i}(\boldsymbol{Z}^{(\lambda)}, \boldsymbol{Z}^{(\lambda)})\right]\right) \tag{2.5}$$

for $i \in \boldsymbol{I}_\nu$. In [24] it was proved that the order of convergence of the obtained improved methods (2.5) is equal $\lambda + 4$. Both corrections $N(z_i)$ and $H(z_i)$ will be denoted by $C^{(\lambda)}(z_i)$ ($\lambda = 1, 2$).

Further acceleration of the convergence speed can be obtained by using corrections of higher order for finding a multiple zero. In this paper we use the correction of the fourth order obtained from the following two-point method for solving nonlinear equations proposed in [11]

$$\hat{z} = z - u(z) \cdot \frac{\beta + \gamma t(z)}{1 + \delta t(z)}, \quad t(z) = \frac{f'(z - \theta u(z))}{f'(z)}, \tag{2.6}$$

where

$$\theta = \frac{2m}{m+2}, \ \ \beta = -\frac{m^2}{2}, \ \ \gamma = \frac{m(m-2)}{2}\left(\frac{m}{m+2}\right)^{-m}, \ \ \delta = -\left(\frac{m}{m+2}\right)^{-m},$$

and m is the multiplicity of the sought zero ζ of a function f (not necessarily algebraic polynomial in general). The order of convergence of the iterative method (2.6) is four, that is,

$$\hat{z} - \zeta = O_M\left((z - \zeta)^4\right) \tag{2.7}$$

holds (for the proof, see [11]). Here O_M is a symbol which points to the fact that two complex numbers w_1 and w_2 have moduli of the same order (that is, $|w_1| = O(|w_2|)$, O is the Landau symbol), written as $w_1 = O_M(w_2)$.

In the sequel, we substitute z by the disk approximation Z_j of ζ_j, and m by the corresponding multiplicity μ_j of ζ_j. The approximation Z_j is replaced by Z_j^*, calculated by (2.4), that is,

$$Z_j^* = Z_j - u_j \cdot \frac{\beta_j + \gamma_j t_j}{1 + \delta_j t_j} = Z_j - C_j^{(3)},$$

where we put $u_j = u(z_j)$, $t_j = f'(z_j - \theta_j u_j)/f'(z_j)$ and

$$\theta_j = \frac{2\mu_j}{\mu_j + 2}, \ \ \beta_j = -\frac{\mu_j^2}{2}, \ \ \gamma_j = \frac{\mu_j(\mu_j - 2)}{2}\left(\frac{\mu_j}{\mu_j + 2}\right)^{-\mu_j}, \ \ \delta_j = -\left(\frac{\mu_j}{\mu_j + 2}\right)^{-\mu_j}.$$

In that manner we have obtained a new method for the simultaneous inclusion of all simple or multiple zeros of a given polynomial in the form (2.5) for $\lambda = 3$.

Remark 2.5 To decrease the total computational cost, before executing any iteration step it is necessary to calculate first all corrections $C_j^{(3)}$.

3. Convergence analysis

Let us introduce the abbreviations

$$r = \max_{1\le i\le \nu} r_i, \quad \varepsilon_i = z_i - \zeta_i, \quad \varepsilon = \max_{1\le i\le \nu} |\varepsilon_i|,$$

$$h_{ij} = \mathrm{mid}\left(z_i - Z_j + C^{(3)}(z_j)\right), \quad d_{ij} = \frac{r_j}{|h_{ij}|\left(|h_{ij}| - r_j\right)}, \quad g_{ij} = \frac{1}{h_{ij}}, \tag{3.1}$$

$$s_{k,i} = \sum_{j\ne i} \mu_j g_{ij}^k \ (k = 1, 2), \ \rho_{1,i} = \sum_{j\ne i} \mu_j d_{ij}, \ \rho_{2,i} = \sum_{j\ne i} \mu_j (2|g_{ij}|d_{ij} + d_{ij}^2). \tag{3.2}$$

First we will prove the following assertion.

Lemma 3.1 *For the inclusion method* (2.5), *for* $\lambda = 3$, *the following relations can be stated:*

(i) $\hat{r} = O_M\left(\varepsilon^3 r\right)$;

(ii) $\hat{\varepsilon} = O_M\left(\varepsilon^7\right)$.

Proof. Let $Z_j = \{z_j; r_j\}$. Then $z_i - Z_j + C_j^{(3)} =: \{h_{ij}; r_j\}$. According to this we obtain

$$S_{1,i} = \sum_{j\ne i} \frac{\mu_j}{\{h_{ij}; r_j\}} = \sum_{j\ne i} \mu_j \{g_{ij}; d_{ij}\} = \{s_{1,i}; \rho_{1,i}\} \tag{3.3}$$

and

$$\begin{aligned} S_{2,i} &= \sum_{j\ne i} \mu_j \left(\frac{1}{\{h_{ij}; r_j\}}\right)^2 = \sum_{j\ne i} \mu_j \{g_{ij}; d_{ij}\}^2 \\ &= \sum_{j\ne i} \mu_j \{g_{ij}^2; 2|g_{ij}|d_{ij} + d_{ij}^2\} = \{s_{2,i}; \rho_{2,i}\}. \end{aligned} \tag{3.4}$$

Starting from (3.3) we find

$$S_{1,i}^2 = \{s_{1,i}; \rho_{1,i}\}^2 = \{s_{1,i}^2; 2|s_{1,i}|\rho_{1,i} + \rho_{1,i}^2\}. \tag{3.5}$$

Since

$$H(z_i) = \frac{f(z_i)}{\left(\frac{1 + 1/\mu_i}{2}\right) f'(z_i) - \frac{f(z_i) f''(z_i)}{2 f'(z_i)}},$$

using the notations

$$\delta_{1,i} = \frac{f'(z_i)}{f(z_i)} \quad \text{and,} \quad \delta_{2,i} = \frac{f'(z_i)^2 - f(z_i)f''(z_i)}{f(z_i)^2},$$

$$Y_i := \{y_i; \eta_i\} = \frac{1}{\mu_i}\Big(\delta_{1,i}^2 - S_{1,i}^2(\boldsymbol{Z}^{(3)}, \boldsymbol{Z}^{(3)})\Big) + (\delta_{2,i} - S_{2,i}(\boldsymbol{Z}^{(3)}, \boldsymbol{Z}^{(3)})),$$

we find

$$\begin{aligned} \frac{f(z_i)}{2f'(z_i)}\, Y(z_i) &= \frac{f(z_i)}{2f'(z_i)}\{y_i; \eta_i\} \\ &= H(z_i)^{-1} - \frac{f(z_i)}{2f'(z_i)}\Big[\frac{1}{\mu_i}S_{1,i}^2(\boldsymbol{Z}^{(3)}, \boldsymbol{Z}^{(3)}) + S_{2,i}(\boldsymbol{Z}^{(3)}, \boldsymbol{Z}^{(3)})\Big] \\ &= \frac{f(z_i)}{2f'(z_i)}\Big[\frac{1}{\mu_i}\Big(\delta_{1,i}^2 - S_{1,i}^2(\boldsymbol{Z}^{(3)}, \boldsymbol{Z}^{(3)})\Big) + (\delta_{2,i} - S_{2,i}(\boldsymbol{Z}^{(3)}, \boldsymbol{Z}^{(3)}))\Big]. \end{aligned} \tag{3.6}$$

Finally, the method (2.5) can be written in the form

$$\hat{Z}_i = z_i - \frac{2f'(z_i)/f(z_i)}{\{y_i; \eta_i\}} = z_i - \frac{2\delta_{1,i}}{\{y_i; \eta_i\}} = z_i - 2\delta_{1,i}\Big\{\frac{1}{y_i}; \frac{\eta_i}{|y_i|(|y_i| - \eta_i)}\Big\}. \tag{3.7}$$

From the introduced abbreviations (3.1) and (3.2) we obtain the estimates

$$\begin{aligned} h_{ij} &= O_M(1), \quad g_{ij} = O_M(1), \quad d_{ij} = O_M(r), \\ \rho_{1,i} &= O_M(r), \quad \rho_{2,i} = O_M(r), \quad s_{k,i} = O_M(1)\ (k = 1, 2). \end{aligned} \tag{3.8}$$

On the other hand, from the difference

$$\Sigma_{1,i} - s_{1,i} = \sum_{j \neq i} \mu_j\Big(\frac{1}{z_i - \zeta_j} - g_{ij}\Big) = -\sum_{j \neq i} \mu_j \frac{z_j - \zeta_j + C^{(3)}(z_j)}{(z_i - \zeta_j)h_{ij}},$$

using relation (2.7) we obtain

$$\Sigma_{1,i} - s_{1,i} = O_M\big(\varepsilon^4\big).$$

According to this we get

$$\Sigma_{1,i}^2 - s_{1,i}^2 = \Big(\Sigma_{1,i} - s_{1,i}\Big)\Big(\Sigma_{1,i} + s_{1,i}\Big) = O_M(\varepsilon^4) \tag{3.9}$$

and

$$\begin{aligned} \Sigma_{2,i} - s_{2,i} &= \sum_{j \neq i} \mu_j\Big(\frac{1}{(z_i - \zeta_j)^2} - g_{ij}^2\Big) \\ &= \sum_{j \neq i} \mu_j\Big(\frac{1}{z_i - \zeta_j} - g_{ij}\Big)\Big(\frac{1}{z_i - \zeta_j} + g_{ij}\Big) = O_M(\varepsilon^4). \end{aligned} \tag{3.10}$$

By virtue of the obtained estimates (3.8), (3.9) and (3.10), and the equalities

$$\delta_{1,i} = \frac{\mu_i}{\varepsilon_i} + \Sigma_{1,i} \quad \text{and} \quad \delta_{2,i} = \frac{\mu_i}{\varepsilon_i^2} + \Sigma_{2,i}$$

we obtain from (3.6)

$$y_i = \frac{1}{\mu_i}(\delta_{1,i}^2 - s_{1,i}^2) + (\delta_{2,i} - s_{2,i}) = \frac{2\mu_i + 2\varepsilon_i\Sigma_{1,i} + O_M(\varepsilon^6)}{\varepsilon_i^2} = O_M(\varepsilon^{-2}) \tag{3.11}$$

and

$$\eta_i = \frac{1}{\mu_i}(2|s_{1,i}|\rho_{1,i} + \rho_{1,i}^2) + \rho_{2,i} = O_M(r). \tag{3.12}$$

Using (3.11) and (3.12) we obtain from (3.7)

$$\hat{\varepsilon}_i = \varepsilon_i - \frac{2(\mu_i/\varepsilon_i + \Sigma_{1,i})}{\dfrac{2\mu_i + 2\varepsilon_i\Sigma_{1,i} + O_M(\varepsilon^6)}{\varepsilon_i^2}} = O_M(\varepsilon^7)$$

and

$$\hat{r}_i = \frac{2(\mu_i/\varepsilon_i + \Sigma_{1,i})\eta_i}{|y_i|(|y_i| - \eta_i)} = O_M(\varepsilon^3 r). \quad \square$$

The convergence analysis of inclusion methods (2.5) with corrections requires the following assertion which is a special case of Theorem 3 given in [4] (see also [6]):

Theorem 3.1 *Given the error-recursion*

$$w_i^{(m+1)} \le \alpha_i \prod_{j=1}^{k} \left(w_j^{(m)}\right)^{t_{ij}}, \quad (i = 1, \ldots, k;\ m \ge 0), \tag{3.13}$$

where $t_{ij} \ge 0$, $\alpha_i > 0$, $1 \le i, j \le k$, and $w_i^{(m)} = \varepsilon_i^{(m)}$ or $w_i^{(m)} = r_i^{(m)}$. Denote the matrix of exponents appearing in (3.13) with T_k, that is $T_k = [t_{ij}]_{k\times k}$. If the non-negative matrix T_k has the spectral radius $\rho(T_k) > 1$ and a corresponding eigenvector $\boldsymbol{x}_\rho > 0$, then all sequences $\{w_i^{(m)}\}$ $(i = 1, \ldots, k)$ have the R-order at least $\rho(T_k)$.

Let $O_R(IM)$ denote the R-order of convergence of an iterative method IM. The matrix $T_k = [t_{ij}]$ will be called the *R-matrix* since it is concerned with the R-order of convergence. Finally, for the inclusion methods (2.5) we can state the following theorem.

Theorem 3.2 *If $Z_1^{(0)}, \ldots, Z_\nu^{(0)}$ are sufficiently close initial disjoint disks containing the distinct zeros $\zeta_1, \ldots, \zeta_\nu$, then the lower bound of the R-order of convergence of the interval method (2.5) $(\lambda = 3)$ is seven.*

Proof. For simplicity, as usual in this type of analysis, we adopt the relation $1 > |\varepsilon^{(0)}| = r^{(0)} > 0$, which means that we deal with the "worst case" model. This assumption is of no relevance to the final result of the limit process applied in order to obtain the lower bound for the R-order of convergence. By virtue of Lemma 3.1 we notice that these sequences behave as follows

$$\varepsilon^{(m+1)} \sim \left(\varepsilon^{(m)}\right)^7, \quad r^{(m+1)} \sim \left(\varepsilon^{(m)}\right)^3 r^{(m)}.$$

From these relations and Theorem 3.1 we form the R-matrix

$$T_2 = \begin{bmatrix} 7 & 0 \\ 3 & 1 \end{bmatrix}$$

whose spectral radius is $\rho(T_2) = 7$, and the corresponding eigenvector is $\boldsymbol{x}_\rho = (2, 1\big) > 0$. Hence, according to Theorem 3.1, we obtain

$$O_R\Big((2.5)_{\lambda=3}\Big) \geq \rho\,(T_2) = 7. \quad \square$$

4. Single-step methods

The convergence of methods (2.4) and (2.5) can be accelerated by applying the Gauss-Seidel approach. In this manner, applying the Gauss-Seidel approach, we obtain from (2.4) the *single-step* method

$$\widehat{Z}_i = z_i - \mathrm{INV}_2\bigg(H_i(z_i)^{-1} - \frac{f(z_i)}{2f'(z_i)}\Big[\frac{1}{\mu_i}S_{1,i}^2(\widehat{\boldsymbol{Z}}, \boldsymbol{Z}) + S_{2,i}(\widehat{\boldsymbol{Z}}, \boldsymbol{Z})\Big]\bigg) \ (i \in \boldsymbol{I}_\nu) \tag{4.1}$$

and from (2.5) the *single-step* methods with corrections

$$\widehat{Z}_i = z_i - \mathrm{INV}_2\bigg(H_i(z_i)^{-1} - \frac{f(z_i)}{2f'(z_i)}\Big[\frac{1}{\mu_i}S_{1,i}^2(\widehat{\boldsymbol{Z}}, \boldsymbol{Z}^{(\lambda)}) + S_{2,i}(\widehat{\boldsymbol{Z}}, \boldsymbol{Z}^{(\lambda)})\Big]\bigg) \tag{4.2}$$

for $i \in \boldsymbol{I}_\nu$ and $\lambda = 1, 2, 3$.

The methods (4.1), and (4.2) for $\lambda = 1, 2$ were examined in [24], where it was proved that the R-order of convergence of the single-step method (4.1) is at least $3 + x_\nu$, where $x_\nu > 1$ is the unique positive root of the equation $x^\nu - x - 3 = 0$ and that the ranges of the lower bounds of the R-order of convergence of the methods (4.2) are

$$\Omega^{(1)} = (5, 6.646), \quad \Omega^{(2)} = (6, 7.855),$$

for the methods with Schröder's ($\lambda = 1$) and Halley's corrections ($\lambda = 2$), respectively.

Let us examine now the single-step method (4.2) for $\lambda = 3$. It is very difficult to find the R-order of convergence of this method since 2ν sequences of centers and radii, and the number of zeros ν are involved in the convergence analysis.

However, we can estimate easily the limit bounds of the R-order taking the limit cases $\nu = 2$ and very large ν.

First, since the convergence rate of a single-step method becomes almost the same as the one of the corresponding total-step method when the polynomial degree is very large, according to Theorem 3.2 we have $O_R((4.2)_{\lambda=3}, \nu) \geq 7$ for very large ν.

Consider now the single-step method (4.2) ($\lambda = 3$) for $\nu = 2$ and assume that $|\varepsilon_1^{(0)}| = |\varepsilon_2^{(0)}| = r_1^{(0)} = r_2^{(0)}$ (the "worst case" model). After an extensive calculation we derive the following estimates

$$|\hat{\varepsilon}_1| \sim |\varepsilon_1|^3|\varepsilon_2|^4, \quad |\hat{\varepsilon}_2| \sim |\varepsilon_1|^3|\varepsilon_2|^7, \quad \hat{r}_1 \sim |\varepsilon_1|^3 r_2, \quad \hat{r}_2 \sim |\varepsilon_1|^3|\varepsilon_2|^3 r_2.$$

The corresponding R-matrix and their spectral radii and eigenvectors are:

$$T_4 = \begin{bmatrix} 3 & 4 & 0 & 0 \\ 3 & 7 & 0 & 0 \\ 3 & 0 & 0 & 1 \\ 3 & 3 & 0 & 1 \end{bmatrix}, \quad \rho(T_4) = 9, \quad \boldsymbol{x}_\rho = (1, 1.5, 0.4375, 0.9375) > 0.$$

Taking into account the previous results, we can state the following assertion.

Theorem 4.1 *The ranges of the lower bounds of the R-order of convergence of the single-step method* (4.2) ($\lambda = 3$) *is*

$$\Omega^{(3)} = (7, 9).$$

Since the increased convergence is attained without any additional calculations we conclude that the inclusion methods (4.2) possess a high computational efficiency.

5. Numerical example

The presented algorithms (2.4), (2.5), (4.1), and (4.2) have been tested in solving many polynomial equations. To provide the enclosure of the zeros in the second and third iteration that produce very small disks, we have used the programming package *Mathematica* with multiple precision arithmetic. In realization of all methods we used only the centered inversion, that is $\mathrm{INV}_1 = \mathrm{INV}_2 = ()^{I_c}$.

Example 1. To find the circular inclusion approximations to the multiple zeros of the polynomial

$$f(z) = z^9 - 8z^8 + 25z^7 - 34z^6 + 64z^4 - 76z^3 + 8z^2 + 48z - 32,$$

we implemented the interval methods (2.4), (2.5) (for $\lambda = 1, 2, 3$) and (4.2) (for $\lambda = 1, 2, 3$). The exact zeros of f are

$$\zeta_1 = -1, \; z_2 = 2, \; \zeta_3 = 1 + i, \; \zeta_4 = 1 - i$$

of the respective multiplicities $\mu_1 = 2$, $\mu_2 = 3$, $\mu_3 = \mu_4 = 2$. The initial disks were selected to be $Z_i^{(0)} = \{z_i^{(0)}; 0.5\}$, with the centers

$$z_1^{(0)} = -1.1 + 0.2i, \quad z_2^{(0)} = 2.1 - 0.2i, \quad z_3^{(0)} = 0.8 + 1.2i, \quad z_4^{(0)} = 0.9 - 1.2i.$$

The maximal radii of the inclusion disks produced in the first three iterative steps are given in Table 1, where the denotation $A(-q)$ means $A \times 10^{-q}$.

Table 1: The maximal radii of inclusion disks

	$r^{(1)}$	$r^{(2)}$	$r^{(3)}$
(2.4)	1.89(−2)	2.48(−9)	9.34(−39)
(4.1)	6.03(−3)	3.38(−12)	7.57(−50)
(2.5), $\lambda = 1$	2.69(−2)	3.18(−11)	1.81(−60)
(4.2), $\lambda = 1$	8.43(−3)	3.27(−14)	1.28(−69)
(2.5), $\lambda = 2$	2.77(−2)	3.41(−14)	1.05(−86)
(4.2), $\lambda = 2$	9.55(−3)	3.48(−16)	4.76(−96)
(2.5), $\lambda = 3$	2.76(−2)	7.21(−15)	3.96(−105)
(4.2), $\lambda = 3$	9.71(−3)	9.72(−17)	4.16(−114)

Example 2. We implemented the same interval methods as in Example 1 to find inclusion disks of multiple zeros of the polynomial

$$\begin{aligned} f(z) \;=\;& z^{13} - 8z^{12} + 27z^{11} - 50z^{10} + 51z^9 - 12z^8 - 51z^7 + 102z^6 - 104z^5 \\ & +48z^4 + 20z^3 - 56z^2 + 48z - 32. \end{aligned}$$

The exact zeros of f are

$$\zeta_1 = -1, \; z_{2,3} = 1 \pm i, \; \zeta_{4,5} = \pm i, \; \zeta_6 = 2$$

of the multiplicity $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = 2$, $\mu_6 = 3$, respectively. We have taken the following initial disks $Z_i^{(0)} = \{z_i^{(0)}; 0.5\}$, with the centers

$$\begin{aligned} &z_1^{(0)} = -1.1 + 0.2, \quad && z_2^{(0)} = 1.1 + 0.9i, \quad && z_3^{(0)} = 0.9 - 1.1i, \\ &z_4^{(0)} = 0.1 + 0.9i, \quad && z_5^{(0)} = 0.1 - 1.2i, \quad && z_6^{(0)} = 2.2 - 0.1i. \end{aligned}$$

The maximal radii of the inclusion disks are given in Table 2.

Table 2: The maximal radii of inclusion disks

	$r^{(1)}$	$r^{(2)}$	$r^{(3)}$
(2.4)	2.53(−1)	1.22(−7)	3.90(−33)
(4.1)	4.29(−2)	5.60(−10)	3.04(−42)
(2.5), $\lambda = 1$	1.44(−1)	1.44(−9)	1.45(−49)
(4.2), $\lambda = 1$	4.14(−2)	1.04(−10)	7.58(−56)
(2.5), $\lambda = 2$	1.21(−1)	8.18(−12)	7.09(−73)
(4.2), $\lambda = 2$	3.55(−2)	7.05(−13)	1.30(−79)
(2.5), $\lambda = 3$	1.20(−1)	1.59(−12)	2.23(−87)
(4.2), $\lambda = 3$	3.58(−2)	2.25(−13)	5.67(−93)

Example 3. We implemented the same interval methods to find inclusion disks of multiple zeros of the polynomial

$$\begin{aligned} f(z) &= z^{18} + (2-2i)z^{17} - 14z^{16} - (18-26i)z^{15} + (80-12i)z^{14} + (26-118i)z^{13} \\ &-(238-136i)z^{12} + (146+182i)z^{11} + (307-476i)z^{10} - (380-160i)z^{9} \\ &+(236+320i)z^{8} + (32-712i)z^{7} - (804-880i)z^{6} + (512+96i)z^{5} - (80+832i)z^{4} \\ &-(1024-1152i)z^{3} - (448-256i)z^{2} - (1024-512i)z - (768-1024i). \end{aligned}$$

The exact zeros of f are

$$\zeta_1 = -1,\ z_2 = -2,\ \zeta_{3,4} = 1 \pm i,\ \zeta_{5,6} = \pm i,\ \zeta_7 = 2,\ \zeta_8 = -2+i$$

of the multiplicity $\mu_1 = 2, \mu_2 = 3, \mu_3 = \mu_4 = \mu_5 = \mu_6 = 2, \mu_7 = 3, \mu_8 = 2$ respectively. We have taken the initial disks $Z_i^{(0)} = \{z_i^{(0)}; 0.4\}$, with the centers

$$\begin{aligned} &z_1^{(0)} = -1.2+0.1i, \quad z_2^{(0)} = -2.2-0.1i, \quad z_3^{(0)} = 1.1+1.2i, \quad z_4^{(0)} = 0.9-1.1i, \\ &z_5^{(0)} = -0.1+0.8i, \quad z_6^{(0)} = 0.1-1.1i, \quad z_7^{(0)} = 2.2-0.1i, \quad z_8^{(0)} = -2.2+0.9i. \end{aligned}$$

The maximal radii of the inclusion disks are given in Table 3.

Table 3: The maximal radii of inclusion disks

	$r^{(1)}$	$r^{(2)}$	$r^{(3)}$
(2.4)	9.47(−2)	3.91(−7)	8.87(−31)
(4.1)	2.55(−2)	4.76(−9)	1.73(−38)
(2.5), $\lambda = 1$	1.64(−1)	8.96(−8)	3.10(−42)
(4.2), $\lambda = 1$	1.45(−1)	6.98(−9)	3.22(−48)
(2.5), $\lambda = 2$	2.32(−1)	8.34(−10)	1.04(−62)
(4.2), $\lambda = 2$	2.32(−1)	2.95(−11)	7.04(−67)
(2.5), $\lambda = 3$	2.37(−1)	7.57(−10)	5.98(−70)
(4.2), $\lambda = 3$	2.37(−1)	1.21(−10)	2.15(−75)

From the numerical examples presented in Tables 1, 2 and 3, and a lot of other numerical experiments, we can conclude that the convergence rate of the considered methods matches well the theoretical convergence speed of these methods given in Theorems 3.2 and 4.1. Enormously small disks obtained in the third iteration are not required in practice, but we have presented them to point out the convergence rate and the growing accuracy of inclusion methods with corrections as the number of iteration steps increases.

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