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COMPUTING GENERALIZED INVERSES USING MATRIX FACTORIZATIONS

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Abstract. A full-rank representation of $A^{(2)}_{T,S}$ inverse of a given constant matrix A which is based on the SVD decomposition and SVD-like decompositions of an appropriate matrix W is presented. The notion of thin generalized inverses, corresponding to the notion of thin SVD decomposition, is introduced. Numerical examples which illustrate theoretical investigations are presented.

1. Introduction

Computation of generalized inverses by means of various matrix decompositions has been extensively investigated in the scientific literature.

Also, for the sake of completeness, we restate main known results about the representations of various classes of generalized inverses and the SVD decomposition.

First of all, it is necessary to mention known representations of inner inverses based on the Singular Value Decomposition (SVD). We restart these results from [1]:

Let the SVD of $A \in \mathbb{C}_r^{m\times n}$ be

$$A = U \begin{bmatrix} \Sigma_r & O \\ O & O \end{bmatrix} V^*, \tag{1.1}$$

where $U^*U = I_m$ and $V^*V = I_n$ and

$$\Sigma_r = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix} = \operatorname{diag}\{\sigma_1, \ldots, \sigma_r\},\ \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0. \tag{1.2}$$

Then the following statements are valid:

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a) The set of $\{1\}$-inverses of A is defined by

$$A\{1\} = V \begin{bmatrix} \Sigma_r^{-1} & X \\ Y & Z \end{bmatrix} U^*, \ X, Y, Z \text{ are arbitrary of appropriate sizes.} \tag{1.3}$$

In particular, representation (1.5) gives analogous representations for several classes of inner inverses [1]:

b) The relation $Z = Y\Sigma_r X$ between X, Y, Z produces a representation of the general $\{1,2\}$–inverse from (1.5);

c) $X = O$ gives the general $\{1,3\}$–inverse;

d) $Y = O$ gives the general $\{1,4\}$–inverse;

e) the MoorePenrose inverse is defined by the relation $X = Y = Z = O$.

The weighted Moore–Penrose inverse $A^{\dagger}_{M,N}$ can be expressed from the (M,N) weighted generalized singular value decomposition $(MN-SVD)$ [17, 18]. Let M, N be Hermitian positive definite matrices of order m and n, respectively. Let the weighted generalized SVD of $A \in \mathbb{C}_r^{m\times n}$ be of the form

$$A = U \begin{bmatrix} \Sigma_r & O \\ O & O \end{bmatrix} V^*, \tag{1.4}$$

where $U^*MU = I_m$ and $V^*N^{-1}V = I_n$, $\Sigma_r = \operatorname{diag}(\sigma_1, \dots, \sigma_r)$, $\sigma_i = \sqrt{\lambda_i}$ and $\lambda_1 \geq \cdots \geq \lambda_r$ are the nonzero eigenvalues of $N^{-1}A^*MA = A^{\sharp}A$. Then the following representation of the weighted Moore-Penrose inverse $A^{\dagger}_{M,N}$ is valid:

$$A^{\dagger}_{M,N} = N^{-1}V \begin{bmatrix} \Sigma_r^{-1} & O \\ O & O \end{bmatrix} U^*M. \tag{1.5}$$

A fast computational method for computing the Moore–Penrose inverse $A^{\dagger}$ based on the QR decomposition of the matrix A is introduced in [11]. The QR decomposition is assumed to be defined as in Theorem 3.3.11 from [21]. The analogous QR decomposition for complex matrices is used from [4] More precisely, if $AP = QR$ is a QR factorization of A, where P is a permutation matrix $Q \in \mathbb{R}^{n\times n}$ is orthogonal and $R = \begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix}$, $R_{11} \in \mathbb{R}^{n\times n}$ is nonsingular and upper triangular, then $A^{\dagger} = PR^{\dagger}Q^*$.

An extension of this representation to the set of outer inverses with prescribed range and null space is presented in [16]. We restate these results for the sake of completeness.

Lemma 1.1. [16] *Assume that the matrix $A \in \mathbb{C}_r^{m\times n}$ is given. Let us consider an arbitrary matrix $W \in \mathbb{C}_s^{n\times m}$, $s \leq r$. Suppose that the QR factorization of W is of the form*

$$WP = QR, \tag{1.6}$$

where P is an $m\times m$ permutation matrix, $Q\in\mathbb{C}^{n\times n}$, $Q^*Q=I_n$ and $R\in\mathbb{C}_s^{n\times m}$ is an upper trapezoidal matrix. Assume that P is chosen so that Q and R can be partitioned as

$$(1.7)\qquad Q=\begin{bmatrix} Q_1 & Q_2 \end{bmatrix},\quad R=\begin{bmatrix} R_{11} & R_{12}\\ O & O\end{bmatrix}=\begin{bmatrix} R_1\\ O\end{bmatrix},$$

where Q_1 consists of the first s columns of the matrix Q and $R_{11}\in\mathbb{C}^{s\times s}$ is nonsingular.

If A has a $\{2\}$-inverse $A^{(2)}_{\mathcal{R}(W),\mathcal{N}(W)}$, then:

(a) $R_1P^*AQ_1$ *is an invertible matrix;*

(b) $A^{(2)}_{\mathcal{R}(W),\mathcal{N}(W)}=Q_1(R_1P^*AQ_1)^{-1}R_1P^*$;

(c) $A^{(2)}_{\mathcal{R}(W),\mathcal{N}(W)}=A^{(2)}_{\mathcal{R}(Q_1),\mathcal{N}(R_1P^*)}$;

(d) $A^{(2)}_{\mathcal{R}(W),\mathcal{N}(W)}=Q_1(Q_1^*WAQ_1)^{-1}Q_1^*W$;

(e) $A^{(2)}_{\mathcal{R}(W),\mathcal{N}(W)}\in A\{2\}_s$.

An algorithm for symbolic computation of $A^{(2)}_{T,S}$ inverses based on the QDR decomposition is presented in [15]. An efficient algorithm, based on the LDL^* factorization, for computing $\{1,2,3\}$, $\{1,2,4\}$ inverses and the Moore-Penrose inverse of a given rational matrix is developed in [14]. Recently, the canonical form for the DMP inverse $A^DAA^\dagger$ of a square matrix A based on the Hartwig-Spindelbck decomposition is presented in [8].

In the present paper we develop a numerical algorithm for computing $A^{(2)}_{T,S}$ inverses which is based on the full rank representation of an appropriately chosen matrix W arising from its SVD decomposition. An analogous representation of the outer inverse corresponding to the thin SVD decomposition of W is investigated. This kind of generalized inverse is called the *thin* outer $A^{(2)}_{T,S}$ inverse of A.

The rest of the paper is organized as follows. The second section restates some familiar concepts and notations. A numerical algorithm for computing $A^{(2)}_{T,S}$ inverses which is based on the SVD decomposition of an appropriately chosen matrix W is developed in the third section. Generalized inverses arising from the thin SVD factorization of the matrix W is investigated in sections 4 and 5. These generalized inverses are called *thin* outer generalized inverses with prescribed range and null space. Particularly, we investigate the *thin* Moore-Penrose inverse, *thin* weighted Moore-Penrose inverse and *thin* Drazin inverse in the fifth section.

2. Preliminaries

Following the usual notation, by $\mathbb{R}_r^{m\times n}$ id denoted the set of all real $m\times n$ matrices of rank r, by I we denote the unit matrix of an appropriate order and O denotes the zero matrix of an appropriate order. Furthermore A^T, $\mathcal{R}(A)$, rank(A) and $\mathcal{N}(A)$ denote the transpose, the range, the rank and the null space of $A \in \mathbb{R}^{m\times n}$, respectively.

If $A \in \mathbb{R}_r^{m\times n}$, T is a subspace of $\mathbb{R}^n$ of dimension $t \le r$ and S is a subspace of $\mathbb{R}^m$ of dimension $m - t$, then A has a $\{2\}$-inverse X such that $\mathcal{R}(X) = T$ and $\mathcal{N}(X) = S$ if and only if $AT \oplus S = \mathbb{R}^m$, in which case X is unique and it is denoted by $A^{(2)}_{T,S}$. The outer generalized inverses with prescribed range and null-space are of the special importance in matrix theory. The $\{2\}$-inverses have application in the iterative methods for solving the nonlinear equations [1, 10] as well as in statistics [6, 7]. In particular, outer inverses play an important role in stable approximations of ill-posed problems and in linear and nonlinear problems involving rank-deficient generalized inverse [9, 25]. On the other hand, it is well known that the Moore-Penrose inverse and the weighted Moore-Penrose inverse $A^\dagger, A^\dagger_{M,N}$, the Drazin and the group inverse $A^D, A^\#$, as well as the Bott-Duffin inverse $A^{(-1)}_{(L)}$ and the generalized Bott-Duffin inverse $A^{(\dagger)}_{(L)}$ can be presented by a unified approach, as generalized inverses $A^{(2)}_{T,S}$ for appropriate choice of matrices T and S. For example, the next is valid for a rectangular matrix A [1]:

$$A^\dagger = A^{(2)}_{\mathcal{R}(A^T),\mathcal{N}(A^T)}, \quad A^\dagger_{M,N} = A^{(2)}_{\mathcal{R}(A^\sharp),\mathcal{N}(A^\sharp)}, \tag{2.1}$$

where M, N are positive definite matrices of appropriate orders and $A^\sharp = N^{-1}A^TM$. For a given square matrix A the next identities are satisfied [1, 3, 20]:

$$A^D = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}(A^k)}, \quad A^\# = A^{(2)}_{\mathcal{R}(A),\mathcal{N}(A)}, \tag{2.2}$$

where $k = \text{ind}(A)$. If A is a L-positive semi-definite matrix and L is a subspace of $\mathbb{C}^n$ which satisfies $AL \oplus L^\perp = \mathbb{C}^n$, $S = \mathcal{R}(P_LA)$, then the next identities are satisfied [3, 20, 23]:

$$A^{(-1)}_{(L)} = A^{(2)}_{L,L^\perp}, \quad A^{(\dagger)}_{(L)} = A^{(2)}_{S,S^\perp}. \tag{2.3}$$

For any matrix A of the order $m \times n$ consider the following matrix equations in X, where $*$ denotes conjugate and transpose:

$$(1) \quad AXA = A \quad (2) \quad XAX = X \quad (3) \quad (AX)^* = AX \quad (4) \quad (XA)^* = XA.$$

In the case $m = n$ we also consider equations

$$(5) \quad AX = XA \qquad (1^k) \quad A^{k+1}X = A^k.$$

For a sequence $\mathcal{S}$ of elements from the set $\{1, 2, 3, 4, 5, 1^k\}$, the set of matrices obeying the equations with corresponding numbers contained in $\mathcal{S}$ is denoted by $A\{\mathcal{S}\}$. A matrix from $A\{\mathcal{S}\}$ is called an $\mathcal{S}$-inverse of A. The matrix $X = A^\dagger$ is said to be the Moore-Penrose inverse of A satisfies equations (1)–(4). The group inverse $A^\#$ is the unique $\{1, 2, 5\}$ inverse of A, and exists if and only if the index of A is equal to 1:

$\mathrm{ind}(A) = \min_k\{k|\,\mathrm{rank}(A^{k+1}) = \mathrm{rank}(A^k)\} = 1$. A matrix X is said to be the Drazin inverse of A if it satisfies the matrix equations (1^k) (for some positive integer k), (2) and (5) and it is denoted by $X = A^D$. In the case $\mathrm{ind}(A) = 1$, the Drazin inverse of A is equal to the group inverse of A, i.e. $A^D = A^\#$. If A is nonsingular, it is easily seen that $\mathrm{ind}(A) = 0$ and $A^D = A^{-1}$.

The rank of generalized inverse X is important, and it will be convenient to consider the subset $A\{i, j, k\}_s$ of $A\{i, j, k\}$, consisting $\{i, j, k\}$-inverses of rank s (see [1]).

3. SVD rank factorizations and outer inverses

There exist a number of full-rank representations for different generalized inverses of prescribed rank as well as for the generalized inverses with prescribed range and kernel. For the sake of completeness, in Proposition 3.1, we restate the general full–rank representations of outer inverses with prescribed range and null space.

Proposition 3.1. [13] *Let $A \in \mathbb{C}_r^{m\times n}$, T be a subspace of $\mathbb{C}^n$ of dimension $s \le r$ and let S be a subspace of $\mathbb{C}^m$ of dimensions $m - s$. In addition, suppose that $W \in \mathbb{C}^{n\times m}$ satisfies $\mathcal{R}(W) = T, \mathcal{N}(W) = S$. Let W has an arbitrary full-rank decomposition, that is $W = FG$. If A has a $\{2\}$-inverse $A^{(2)}_{T,S}$, then:*

(1) *GAF is an invertible matrix;*

(2) $A^{(2)}_{T,S} = F(GAF)^{-1}G = A^{(2)}_{\mathcal{R}(F),\mathcal{N}(G)}$.

Our main strategy can be explained as follows. Since the thin SVD of W is a possible approach to derive its full rank factorization, it is possible to apply full–rank representation from Proposition 3.1. This strategy immediately generates a SVD full-rank representation of outer inverses of A with prescribed range and null space. According to Proposition 3.1, the matrix A is of the order $m \times n$ and the matrix W is of the order $n \times m$. Assume that the rank of A is equal to r. Also, it is known result that the rank s of an arbitrary outer inverse of A satisfies $0 \le s \le r$. Since the case $s = 0$ corresponds the zero outer inverse $X = O$, in the sequel we assume the condition $0 < s \le r$.

Suppose that the SVD factorization of W is of the general form

$$W = U\Sigma V^*, \tag{3.1}$$

where $U \in \mathbb{C}_s^{n\times n}$ and $V \in \mathbb{C}_s^{m\times m}$ are column-orthogonal and $\Sigma \in \mathbb{C}_s^{n\times m}$ is a diagonal matrix with the singular values of W in descent order on the main diagonal. Suppose that the nonzero singular values of W are ordered as

$$\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_s. \tag{3.2}$$

Unfortunately, SVD decomposition (3.1) is not a full-rank factorization of W. In order to derive a full-rank factorization of W we must consider a thin SVD

decomposition arising from (3.1). Assume also that the matrices $U \in \mathbb{C}^{n\times n}$, $V \in \mathbb{C}^{m\times m}$ and $\Sigma \in \mathbb{C}^{n\times m}$ are partitioned in the blocks

$$U = \begin{bmatrix} U_s & U_R \end{bmatrix}, \quad V = \begin{bmatrix} V_s & V_R \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_s & O \\ O & O \end{bmatrix}. \tag{3.3}$$

The blocks in (3.3) have the following meaning. The block U_s consists of the first s columns of the matrix U, V_s contains the first s columns of V and $\Sigma_s = diag\{\sigma_1, \dots, \sigma_s\}$ consists of the first s rows and the first s columns from Σ. Also, U_R denotes the remaining $n - s$ columns of U and V_R denotes the remaining $n - s$ columns of V.

Lemma 3.1. *Assume that the matrix $A \in \mathbb{C}_r^{m\times n}$ is given. Let us choose an arbitrary matrix $W \in \mathbb{C}_s^{n\times m}$, $0 < s \leq r$. Let* (3.1) *be the SVD decomposition of W. Let the matrices $U \in \mathbb{C}_s^{n\times n}$, $V \in \mathbb{C}_s^{m\times m}$ and $\Sigma \in \mathbb{C}_s^{n\times m}$ are partitioned as in* (3.3).

Then

$$A^{(2)}_{\mathcal{R}(W),\mathcal{N}(W)} = U_s(\Sigma_s V_s^* A U_s)^{-1}\Sigma_s V_s^* \tag{3.4}$$

$$= A^{(2)}_{\mathcal{R}(U_s),\mathcal{N}(V_s^*)} \tag{3.5}$$

$$= U_s(U_s^* W A U_s)^{-1} U_s^* W. \tag{3.6}$$

Proof. It is easy to verify that

$$W = U_s(\Sigma_s V_s^*) = \sum_{i=1}^{s} \sigma_i u_i v_i^* \tag{3.7}$$

is a full–rank factorization of W. Then (3.4) follows from Proposition 3.1. Further, (3.5) is implied by the equalities $\mathcal{R}(W) = \mathcal{R}(U_s)$, $\mathcal{N}(W) = \mathcal{N}(V_s^*)$.

The equality (3.6) can be derived from $U_s^* U_s = I_s$, which implies $\Sigma_s V_s^* = U_s^* W$. □

Remark 3.1. *Computation of the outer inverse by means of* (3.4) *requires computation of the matrix inverse. More efficient method for computing* (3.4) *is to solve a set of equations* (*see* [20])

$$\Sigma_s V_s^* A U_s X = \Sigma_s V_s^* \tag{3.8}$$

with respect to unknown matrix $X \in \mathbb{C}^{n\times m}$ and then compute the matrix product

$$A^{(2)}_{\mathcal{R}(U_s),\mathcal{N}(V_s^*)} = U_s X. \tag{3.9}$$

In accordance with the representations introduced in Lemma 3.1 and the comment stated in Remark 3.1 it is possible to state the following Algorithm 3..1.

Using the results of Lemma 3.1 and taking into account (2.1)–(2.3), we get the following particular results.

Algorithm 3..1 *SVD* method for computing outer inverse.

Require: $A \in \mathbb{C}_r^{m,n}$.
1: Choose $G \in \mathbb{C}_s^{m,n}$, $0 < s \le r$.
2: Find *SVD* decomposition (3.1).
3: Generate matrices U_s and V_s as in (3.3).
4: Solve the matrix equation (3.8).
5: Return $A^{(2)}_{\mathcal{R}(U_s),\mathcal{N}(V_s^*)} = A^{(2)}_{\mathcal{R}(W),\mathcal{N}(W)}$ defined in (3.9).

Corollary 3.1. *For a given matrix $A\in\mathbb{C}_r^{m\times n}$ the following statements are valid.*

$$A^{(2)}_{\mathcal{R}(U_s),\mathcal{N}(V_s^*)} = \begin{cases} A^{\dagger}, & W = A^*; \\ A^{\dagger}_{M,N}, & W = A^{\sharp}; \\ A^{\#}, & W = A; \\ A^{D}, & W = A^k,\ k \ge \operatorname{ind}(A); \\ A^{(-1)}_{(L)}, & \mathcal{R}(W) = L,\ \mathcal{N}(W) = L^{\perp}; \\ A^{(\dagger)}_{(L)}, & \mathcal{R}(W) = S,\ \mathcal{N}(W) = S^{\perp}. \end{cases} \tag{3.10}$$

Example 3.1. *In this example we consider the matrix*

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 1 & 3 & 4 & 6 & 2 \\ 2 & 3 & 4 & 5 & 3 \\ 3 & 4 & 5 & 6 & 4 \\ 4 & 5 & 6 & 7 & 6 \\ 6 & 6 & 7 & 7 & 8 \end{bmatrix}$$

and choose the corresponding matrix

$$W = \begin{bmatrix} -3 & -2 & 0 & 0 & -9 & 0 \\ -7 & 6 & 0 & 0 & -21 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & 5 & 0 & 0 & 21 & 0 \\ -4 & 0 & 0 & 0 & -12 & 0 \end{bmatrix}.$$

SVD decomposition of W is given by

$$U=\begin{bmatrix} 0.270344 & 0.251279 & -0.588348 & 0.719457 & 0. \\ 0.631629 & -0.736802 & -0.196116 & -0.140382 & 0. \\ 0. & 0. & 0. & 0. & 1. \\ -0.630777 & -0.627665 & 0. & 0.456241 & 0. \\ 0.360665 & 0.00425894 & 0.784465 & 0.504498 & 0. \end{bmatrix},$$

$$\Sigma=\begin{bmatrix} 35.0715 & 0. & 0. & 0. & 0. & 0. \\ 0. & 8.06173 & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. \end{bmatrix},$$

$$V=\begin{bmatrix} -0.316227 & -0.00085836 & 0. & -0.948683 & 0. & 0. \\ 0.00271437 & -0.999996 & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 1. \\ 0. & 0. & 0. & 0. & 1. & 0. \\ -0.94868 & -0.00257508 & 0. & 0.316228 & 0. & 0. \\ 0. & 0. & 1. & 0. & 0. & 0. \end{bmatrix}.$$

Further, we have

$$U_2=\begin{bmatrix} 0.270344 & 0.251279 & -0.588348 & 0.719457 & 0. \\ 0.631629 & -0.736802 & -0.196116 & -0.140382 & 0. \\ 0. & 0. & 0. & 0. & 1. \\ -0.630777 & -0.627665 & 0. & 0.456241 & 0. \\ 0.360665 & 0.00425894 & 0.784465 & 0.504498 & 0. \end{bmatrix},$$

$$\Sigma_2=\begin{bmatrix} 35.0715 & 0. \\ 0. & 8.06173 \end{bmatrix},$$

$$V_2=\begin{bmatrix} -0.316227 & -0.00085836 \\ 0.00271437 & -0.999996 \\ 0. & 0. \\ 0. & 0. \\ -0.94868 & -0.00257508 \\ 0. & 0. \end{bmatrix}.$$

Outer inverse of A with prescribed range and null space is, according to (3.4)*, equal to*

$$A^{(2)}_{\mathcal{R}(U_s),\mathcal{N}(V_s^*)}=A^{(2)}_{\mathcal{R}(W),\mathcal{N}(W)}=\begin{bmatrix} 0.0249788 & -0.152625 & 0. & 0. & 0.0749365 & 0. \\ 0.0808637 & -0.222904 & 0. & 0. & 0.242591 & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. \\ -0.0575783 & 0.360288 & 0. & 0. & -0.172735 & 0. \\ 0.03895 & -0.170195 & 0. & 0. & 0.11685 & 0. \end{bmatrix}.$$

The authors of the paper [16] proved that the the full–rank representation of the outer inverse $A^{(2)}_{\mathcal{R}(W),\mathcal{N}(W)}$ defined in Proposition 3.1 is invariant with respect to the choice of the full–rank factorization of W. In the following Corollary 3.2 we show that the same statements holds for the SVD full-rank decomposition of W.

Corollary 3.2. *Assume that all assumptions of Lemma 3.1 are valid and $W = FG$ is an arbitrary full–rank factorization of W. Then*

$$U_s(\Sigma_s V_s^* A U_s)^{-1}\Sigma_s V^* = F(GAF)^{-1}G \tag{3.11}$$

Proof. Applying Lemma 4.1 and Proposition 3.1, part (2), we conclude

$$U_s(\Sigma_s V^* A U_s)^{-1}\Sigma_s V^* = A^{(2)}_{\mathcal{R}(W),\mathcal{N}(W)}$$

Now, using $\mathcal{R}(W) = \mathcal{R}(F)$ and $\mathcal{N}(W) = N(G)$ we immediately derive

$$\begin{aligned} U_s(\Sigma_s V^* A U_s)^{-1}\Sigma_s V^* &= A^{(2)}_{\mathcal{R}(F),\mathcal{N}(G)} \\ &= F(GAF)^{-1}G, \end{aligned}$$

which completes the proof. □

4. TSVD rank factorizations and outer inverses

Our assumption is that the nonzero singular values of W are ordered as in (3.2). and (3.7) is a full-rank factorization of W. But, the situation when the matrix $\Sigma_s V_s^* A U_s$ is ill-conditioned frequently occurs. In these cases, it is impossible to compute the inverse $(\Sigma_s V_s^* A U_s)^{-1}$. Therefore, computation of outer inverse by the representation (3.4) is jeopardized. We propose a solution based on further truncations of the SVD decomposition of W.

Lemma 4.1. *Assume that the matrix $A \in \mathbb{C}_r^{m\times n}$ is given. Let us consider an arbitrary matrix $W \in \mathbb{C}_s^{n\times m}$, $s \le r$. Suppose that the nonzero singular values of W are ordered as in* (3.2). *Let us choose an integer $0 < t \le s$ and a small real number $\varepsilon > 0$ such that the following inequalities hold:*

$$\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_t \gg \varepsilon \ge \sigma_{t+1} \ge \cdots \ge \sigma_s.$$

Assume also that U and V are partitioned as

$$U = \begin{bmatrix} U_t & U_{[t,s]} & U_R \end{bmatrix}, \quad V = \begin{bmatrix} V_t & V_{[t,s]} & V_R \end{bmatrix}, \tag{4.1}$$

where U_t (resp. V_t) consists of the first t columns $u_1,\dots,u_t$ of the matrix U (resp. of the first t columns $v_1,\dots,v_t$ of the matrix V), $U_{[t,s]}$ (resp. $V_{[t,s]}$) consists of the columns $u_{t+1},\dots,u_s$ from U (resp. of the columns $v_{t+1},\dots,v_s$ from V) and U_R (resp. V_R) consists of the last $n-s$ columns of the matrix U (resp. V). Also, the matrix Σ can be partitioned as

$$\Sigma = \begin{bmatrix} \Sigma_t & O & O \\ O & \Sigma_{[t,s]} & O \\ O & O & O \end{bmatrix},$$

where $\Sigma_t = \operatorname{diag}\{\sigma_1, \ldots, \sigma_t\}$ *and* $\Sigma_{[t,s]} = \operatorname{diag}\{\sigma_{t+1}, \ldots, \sigma_s\}$.

Under the assumption that $\Sigma_t V_t^* A U_t$ *is invertible, i.e.* $\operatorname{rank}(\Sigma_t V_t^* A U_t) = t$, *we get*

$$U_t(\Sigma_t V_t^* A U_t)^{-1}\Sigma_t V_t^* = A^{(2)}_{\mathcal{R}(W_{(t)}),\mathcal{N}(W_{(t)})} = A^{(2)}_{\mathcal{R}(U_t),\mathcal{N}(V_t^*)} \tag{4.2}$$

$$= U_t(U_t^* W_{(t)} A U_t)^{-1} U_t^* W_{(t)}, \tag{4.3}$$

where the matrix $W_{(t)}$ *is defined by*

$$W_{(t)} = U_t(\Sigma_t V_t^*) = \sum_{i=1}^{t} \sigma_i u_i v_i^*. \tag{4.4}$$

Proof. We have that (4.4) is a full-rank factorization of the matrix $W_{(t)}$ of rank t whose singular values are ordered as $\sigma_1 \geq \cdots \geq \sigma_t$. According to Proposition 3.1 we have

$$U_t(\Sigma_t V_t^* A U_t)^{-1}\Sigma_t V_t^* = A^{(2)}_{\mathcal{R}(W_{(t)}),\mathcal{N}(W_{(t)})} = A^{(2)}_{\mathcal{R}(U_t),\mathcal{N}(\Sigma_t V_t^*)}.$$

The rest of the proof of the statements (4.2) follows from invertibility of matrices Σ_t.

Finally, (4.3) can be derived from $\Sigma_t V_t^* = U_t^* U_t \Sigma_t V_t^* = U_t^* W_{(t)}$. □

A commendable method for computing (4.2) is to solve a set of equations (see [20])

$$V_t^* A U_t X = V_t^* \tag{4.5}$$

and then compute $A^{(2)}_{\mathcal{R}(U_t),\mathcal{N}(V_t^*)}$ as the matrix product

$$A^{(2)}_{\mathcal{R}(U_t),\mathcal{N}(V_t^*)} = U_t X. \tag{4.6}$$

Remark 4.1. *The number* $t \leq s$ *in Lemma 4.1 is proposed to avoid relatively small singular values of* W *and improve numerical stability of representations* (3.4) *and* (3.6). *Full–rank representation of* $W_{(t)}$ *defined in* (4.4) *we denote by the TSVD full–rank factorization of* $W_{(t)}$ *in the case* $t < s$ *and by the TSVD full–rank factorization of* W *in the case* $t = s$.

The number $t \leq s$ is chosen so that $W_{(t)}$ has numerical rank equal to t. Clearly, we have $W_{(s)} = W$.

Example 4.1. *In this example we consider the matrices* A *and* W *from* (3.1). *But, in this case we use thin SVD decomposition of* W *associated with the largest singular value of* A

(i.e. defined by $t = 1$). This gives

$$U_1 = \begin{bmatrix} 0.270344 \\ 0.631629 \\ 0. \\ -0.630777 \\ 0.360665 \end{bmatrix}, \quad \Sigma_1 = \begin{bmatrix} 35.0715 \end{bmatrix}, \quad V_1 = \begin{bmatrix} -0.316227 \\ 0.00271437 \\ 0. \\ 0. \\ -0.94868 \\ 0. \end{bmatrix}.$$

Outer inverse of A with prescribed range and null space is, according to (3.4)*, equal to*

$$A^{(2)}_{\mathcal{R}(U_1),\mathcal{N}(V_1^*)} = A^{(2)}_{\mathcal{R}(W_{(1)}),\mathcal{N}(W_{(1)})}$$
$$= \begin{bmatrix} 0.0505971 & -0.000434307 & 0. & 0. & 0.151791 & 0. \\ 0.118214 & -0.00101471 & 0. & 0. & 0.354643 & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. \\ -0.118055 & 0.00101334 & 0. & 0. & -0.354165 & 0. \\ 0.0675014 & -0.000579407 & 0. & 0. & 0.202504 & 0. \end{bmatrix},$$

where

$$W_{(1)} = \begin{bmatrix} -2.99826 & 0.025736 & 0. & 0. & -8.99478 & 0. \\ -7.0051 & 0.0601292 & 0. & 0. & -21.0153 & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. \\ 6.99566 & -0.0600481 & 0. & 0. & 20.987 & 0. \\ -3.99997 & 0.0343343 & 0. & 0. & -11.9999 & 0. \end{bmatrix}.$$

is the matrix determined by the largest singular value $\sigma_1 = 35.0715$ of W.

5. Properties of thin TSVD representation of outer inverses

Our particular interest in this section is to investigate properties of the thin outer inverse of A in the particular case $W = A^*$. This choice of the matrix W produces the thin Moore-Penrose inverse of A, which will be denoted by $(A^\dagger)_t$. Truncated SVD representation of the Moore–Penrose inverse has been widely investigated in the scientific literature. For example, the most frequently it has been used in the image restoration [5]. Now, we investigate properties of the generalized inverse $(A^\dagger)_t$. The main problem with which we have to face is: which kind of the generalized inverse of A is the thin Moore-Penrose inverse of A?

Definition 5.1. *Outer generalized inverse $A^{(2)}_{\mathcal{R}(U_t),\mathcal{N}(V_t^*)}$ is called* **thin** *outer inverse corresponding to the outer inverse $A^{(2)}_{\mathcal{R}(U_s),\mathcal{N}(V_s^*)}$, $0 < t \le s \le r$. On the other hand, $A^{(2)}_{\mathcal{R}(U_s),\mathcal{N}(V_s^*)}$ is called* **extended** *generalized inverse corresponding to the outer inverse $A^{(2)}_{\mathcal{R}(U_t),\mathcal{N}(V_t^*)}$.*

5.1. Thin Moore-Penrose inverse

Corollary 5.1. *In the case $W = A^*$, the representation* (4.2) *produces the following approximation of the Moore–Penrose inverse (called thin Moore-Penrose inverse of A):*

$$(A^\dagger)_t = U_t(\Sigma_t^*)^{-1}V_t^* \tag{5.1}$$

$$= \sum_{i=1}^{t} \frac{1}{\overline{\sigma_i}} u_i v_i^* \tag{5.2}$$

$$= A^{(2)}_{\mathcal{R}(U_t),\mathcal{N}(V_t^*)}, \tag{5.3}$$

where u_i and v_i are left and right singular vectors of A^* and $\overline{\sigma_i}$ is the conjugate of σ_i. Also, $(A^\dagger)_t$ is the outer inverse of A of rank t.

Proof. Since (4.4) is the $TSVD$ full-rank factorization of $W = A^*$, in conjunction with

$$A = V\Sigma^* U^* = V_t \Sigma_t^* U_t^*,$$

we have

$$(A^\dagger)_t = U_t \left(\Sigma_t V_t^* A U_t\right)^{-1} \Sigma_t V_t^*$$
$$= U_t \left(\Sigma_t V_t^* (V_t \Sigma_t^* U_t^*) U_t\right)^{-1} \Sigma_t V_t^*.$$

Identity (5.1) can be derived using $V_t^* V_t = U_t^* U_t = I_t$:

$$(A^\dagger)_t = U_t \left(\Sigma_t \Sigma_t^*\right)^{-1} \Sigma_t V_t^* = U_t(\Sigma_t^*)^{-1} V_t^*.$$

Now, (5.2) follows from (5.1) and $\Sigma_t^* = \overline{\Sigma_t}$.

It is not difficult to verify that $(A^\dagger)_t$ is the outer inverse of A of rank t which satisfies

$$\mathcal{R}((A^\dagger)_t) = \mathcal{R}(U_t), \quad \mathcal{N}((A^\dagger)_t) = \mathcal{N}(\Sigma_t V_t^*) = \mathcal{N}(V_t^*).$$

Therefore, (5.3) is verified and the proof is complete. □

The truncated SVD expansion (5.1) of the generalized inverse $(A^\dagger)_t$, which is truncated with respect to $(A^\dagger)_r = A^\dagger$, is widely used in the image restoration so far. These investigations correspond to the particular choice $W_{(t)} = (A^*)_{(t)}$.

This means that approximations of the Moore-Penrose inverse used in the image restoration are in essence outer inverses.

Example 5.1. *In this example we consider the matrix A from Example 3.1. The corresponding matrix W is given by $W = A^T$, announcing our intention to calculate $A^\dagger$. SVD* *decomposition of $W = A^T$ is given by*

$$U = \begin{bmatrix} 0.31292 & 0.493469 & -0.619628 & -0.15694 & 0.5 \\ 0.393065 & 0.024053 & 0.00953435 & -0.771253 & -0.5 \\ 0.484763 & -0.150838 & -0.420949 & 0.561297 & -0.5 \\ 0.564908 & -0.620255 & 0.208213 & -0.0530158 & 0.5 \\ 0.439847 & 0.59029 & 0.628824 & 0.250348 & 0. \end{bmatrix},$$

$$\Sigma = \begin{bmatrix} 25.311 & 0. & 0. & 0. & 0. & 0. \\ 0. & 4.12375 & 0. & 0. & 0. & 0. \\ 0. & 0. & 0.575726 & 0. & 0. & 0. \\ 0. & 0. & 0. & 0.133129 & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. \end{bmatrix},$$

$$V = \begin{bmatrix} 0.207531 & -0.436901 & -0.69778 & 0.170736 & 0.353553 & 0.353553 \\ 0.304228 & -0.625323 & 0.403151 & -0.322313 & 0.353553 & -0.353553 \\ 0.31165 & -0.212103 & 0.0574693 & 0.77755 & -0.353553 & -0.353553 \\ 0.398391 & -0.13045 & -0.279507 & -0.496128 & -0.707107 & 0. \\ 0.50251 & 0.094348 & 0.475742 & 0.110686 & 0. & 0.707107 \\ 0.596673 & 0.589221 & -0.206916 & -0.0631291 & 0.353553 & -0.353553 \end{bmatrix}.$$

By the attempt to compute the Moore-Penrose inverse in the form (3.4)*, which yields $A^\dagger = U(\Sigma V^* A U)^{-1}\Sigma V^*$, ends with a unsuccessful attempt to invert (almost) singular matrix*

$$\Sigma V^* A U = \begin{bmatrix} 640.646 & 0. & 0. & 4.2633\times 10^{-14} & 0. \\ -2.4869\times 10^{-14} & 17.0053 & 0. & -1.3323\times 10^{-15} & -8.8818\times 10^{-16} \\ -1.7930\times 10^{-14} & -1.9151\times 10^{-15} & 0.331461 & 1.1796\times 10^{-16} & 2.7756\times 10^{-16} \\ 8.4514\times 10^{-15} & -2.1580\times 10^{-15} & -3.7643\times 10^{-16} & 0.0177234 & -2.0817\times 10^{-17} \\ 0. & 0. & 0. & 0. & 0. \end{bmatrix}.$$

A possible solution of the problem is usage of a low rank approximation of W by means of truncated SVD. Then we use the thin SVD full-rank factorization of the order $t = 3$ to find outer inverse which approximates $A^\dagger$. The SVD based on 3 greatest singular values of

A^T is defined by

$$U_3 = \begin{bmatrix} 0.31292 & 0.493469 & -0.619628 \\ 0.393065 & 0.024053 & 0.00953435 \\ 0.484763 & -0.150838 & -0.420949 \\ 0.564908 & -0.620255 & 0.208213 \\ 0.439847 & 0.59029 & 0.628824 \end{bmatrix},$$

$$\Sigma_3 = \begin{bmatrix} 25.311 & 0. & 0. \\ 0. & 4.12375 & 0. \\ 0. & 0. & 0.575726 \end{bmatrix},$$

$$V_3 = \begin{bmatrix} 0.207531 & -0.436901 & -0.69778 \\ 0.304228 & -0.625323 & 0.403151 \\ 0.31165 & -0.212103 & 0.0574693 \\ 0.398391 & -0.13045 & -0.279507 \\ 0.50251 & 0.094348 & 0.475742 \\ 0.596673 & 0.589221 & -0.206916 \end{bmatrix}.$$

In this case, $\Sigma_3 V_3^* A U_3$ is pretty regular matrix:

$$\Sigma_3 V_3^* A U_3 = \begin{bmatrix} 1.00357 & 0.993266 & 2.01625 & 2.98963 & 4.00231 & 5.99868 \\ 2.01753 & 2.96691 & 3.07984 & 3.94906 & 5.01136 & 5.99352 \\ 2.98724 & 4.02408 & 3.9419 & 5.03707 & 5.99173 & 7.00472 \\ 4.00121 & 5.99773 & 5.00549 & 5.9965 & 7.00078 & 6.99955 \\ 0.99431 & 2.01074 & 2.97409 & 4.01654 & 5.99631 & 8.0021 \end{bmatrix}.$$

Corresponding outer inverse of A, which approximates $A^\dagger$, is given by

$$A^{(2)}_{\mathcal{R}(U_3),\mathcal{N}(V_3^*)} = A^{(2)}_{\mathcal{R}(W_{(3)}),\mathcal{N}(W_{(3)})}$$

$$= \begin{bmatrix} 0.701273 & -0.504961 & -0.08338 & 0.290136 & -0.494517 & 0.30058 \\ -0.01088 & 0.007753 & 0.004554 & 0.0007971 & 0.016233 & 0.009276 \\ 0.530146 & -0.266069 & -0.0282922 & 0.216767 & -0.341672 & 0.141164 \\ -0.182008 & 0.246646 & 0.0596421 & -0.0725721 & 0.169078 & -0.15014 \\ -0.821067 & 0.356107 & 0.0378239 & -0.317035 & 0.541856 & -0.131286 \end{bmatrix},$$

where $W_{(3)}$ is the following matrix (close to $W = W_{(4)} = A^T$):

$$W_{(3)} = U_3 \Sigma_3 V_3^T = \begin{bmatrix} 1.00357 & 0.993266 & 2.01625 & 2.98963 & 4.00231 & 5.99868 \\ 2.01753 & 2.96691 & 3.07984 & 3.94906 & 5.01136 & 5.99352 \\ 2.98724 & 4.02408 & 3.9419 & 5.03707 & 5.99173 & 7.00472 \\ 4.00121 & 5.99773 & 5.00549 & 5.9965 & 7.00078 & 6.99955 \\ 0.99431 & 2.01074 & 2.97409 & 4.01654 & 5.99631 & 8.0021 \end{bmatrix}.$$

It is easy to verify that X satisfies Penrose equations (2), (3) *and* (4).

Further reduction by means of the thin SVD full-rank factorization based on 2 greatest *singular values of A^T produced another $\{2,3,4\}$-inverse of A. Thin SVD is defined by*

$$U_2 = \begin{bmatrix} 0.31292 & 0.493469 \\ 0.393065 & 0.024053 \\ 0.484763 & -0.150838 \\ 0.564908 & -0.620255 \\ 0.439847 & 0.59029 \end{bmatrix},$$

$$\Sigma_2 = \begin{bmatrix} 25.311 & 0. \\ 0. & 4.12375 \end{bmatrix},$$

$$V_2 = \begin{bmatrix} 0.207531 & -0.436901 \\ 0.304228 & -0.625323 \\ 0.31165 & -0.212103 \\ 0.398391 & -0.13045 \\ 0.50251 & 0.094348 \\ 0.596673 & 0.589221 \end{bmatrix}.$$

Corresponding $\{2,3,4\}$-inverse of A, which approximates $A^\dagger$, is given by

$$A^{(2)}_{\mathcal{R}(U_2),\mathcal{N}(V_2^*)} = A^{(2)}_{\mathcal{R}(W_{(2)}),\mathcal{N}(W_{(2)})}$$
$$= \begin{bmatrix} -0.0497162 & -0.0710683 & -0.0215285 & -0.010685 & 0.0175027 & 0.077886 \\ 0.00067447 & 0.00107709 & 0.00360258 & 0.00542589 & 0.008354 & 0.012703 \\ 0.0199557 & 0.0286997 & 0.0137271 & 0.0124017 & 0.00617316 & -0.010125 \\ 0.0703463 & 0.100845 & 0.0388582 & 0.0285126 & -0.00297557 & -0.075308 \\ -0.0589334 & -0.0842245 & -0.0249456 & -0.01175 & 0.0222378 & 0.0947124 \end{bmatrix},$$

where $W_{(2)}$ is the following matrix (far out of $W = W_{(4)} = A^T$):

$$W_{(2)} = U_2\Sigma_2 V_2^T = \begin{bmatrix} 0.754644 & 1.13708 & 2.03675 & 2.88992 & 4.17203 & 5.92487 \\ 2.02136 & 2.96469 & 3.07952 & 3.95059 & 5.00875 & 5.99465 \\ 2.81813 & 4.12179 & 3.95583 & 4.96933 & 6.10703 & 6.95457 \\ 4.08485 & 5.9494 & 4.9986 & 6.03 & 6.94375 & 7.02436 \\ 1.24693 & 1.86479 & 2.95328 & 4.11773 & 5.82408 & 8.07701 \end{bmatrix}.$$

5.2. Thin weighted Moore-Penrose inverse

The thin outer inverse corresponding to the weighted Moore-Penrose inverse is denoted by $(A^\dagger_{M,N})_t$. This generalized inverse corresponds to the $(M,N) - SVD$ of the matrix $W_{(t)} = (N^{-1}A^*M)_{(t)} = (A^\sharp)_{(t)}$.

5.3. Thin Drazin inverse

In addition, it is known that in some case the Drazin inverse solution $A^D b$ can be used appropriately in solving linear systems and restricted matrix equations.

In the monograph [2, Page 123] it is showed that the Drazin inverse solution $A^D b$ solves the real singular linear system $Ax = b$ if and only if $b \in \mathcal{R}(A^k)$. Also, $A^D b$ is the unique solution of $Ax = b$ provided that $x \in \mathcal{R}(A^k)$ [2, Page 123]. It is also known result that the Drazin inverse solution represents the minimal P-norm solution of the linear system $Ax = b$, where P is invertible matrix such that $P^{-1}AP$ is the Jordan canonical form of A and $\|x\|_P = \|P^{-1}x\|_2$ [22]. The restricted matrix equation

$$AXB = D, \mathcal{R}(X) \subset \mathcal{R}(A^k), \mathcal{N}(X) \supset \mathcal{N}(A^k),$$
$$k = \max\{\mathrm{ind}(A), \mathrm{ind}(B)\}$$

has a unique solution $X = A^D D B^D$ [19].

These investigations are corresponding to the choice $W = A^k$, $k \geq \mathrm{ind}(A)$.

The thin Drazin inverse $A^{(2)}_{\mathcal{R}(U_t),\mathcal{N}(V_t^*)} = (A^D)_t$ arises from the $TSVD$ full-rank representation of the matrix $W_{(t)} = (A^k)_{(t)}$, $t \leq \mathrm{rank}(A^k)$.

Example 5.2. *Let us consider the following matrix*

$$A = \begin{bmatrix} 2 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -2 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 2 \end{bmatrix}$$

of index $\mathrm{ind}(A) = 3$. *Therefore, the Drazin inverse corresponds to the choice* $W = A^3$, *which is equal to*

$$\begin{bmatrix} 4.48 & 1.664 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ -8.32 & -2.176 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ -0.4 & 2.08 & 4. & -4. & 0. & 0. & 0. & 0. & -5. & 3. & 1.8 & 1. \\ 1.6 & -1.32 & -4. & 4. & 0. & 0. & 0. & 0. & 3. & -1. & -1. & -1. \\ -1. & -1. & -1. & 3. & 3.2 & 1.6 & -3.2 & -3.2 & 0. & 0. & -1.76 & 6.8 \\ -1. & -1. & -1. & 3. & 3.2 & 1.6 & -3.2 & -3.2 & 0. & 0. & -2.4 & 2. \\ 0. & 0.8 & 2. & -3.6 & -3.2 & -1.92 & 3.2 & 3.2 & -1. & 0. & 2.4 & -4. \\ 0. & 0. & 0. & 2.4 & 4.8 & 2.88 & -4.8 & -4.8 & 0. & 0. & -3.2 & 4. \\ 6.8 & -1.76 & 0. & 0. & 0. & 0. & 0. & 0. & 4. & -4. & -0.32 & 1.6 \\ -2. & 2.4 & 0. & 0. & 0. & 0. & 0. & 0. & -4. & 4. & -0.32 & 1.6 \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -2.176 & -8.32 \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1.664 & 4.48 \end{bmatrix}.$$

Each of the matrices $\Sigma V^* AU, \Sigma V_{10}^* AU_{10}, \Sigma V_9^* AU_9$ *is almost singular. Finally, the matrix* $\Sigma V_8^* AU_8$ *is pretty regular, so that in this case we get the following thin Drazin inverse of* A*:*

$$(A^D)_8 = U_t \left(\Sigma_8 V_8^* A U_8\right)^{-1} \Sigma_8 V_8^* =$$

$$\left[\begin{array}{cccccc}
0.25 & -0.25 & -4.9267\times10^{-16} & -3.7470\times10^{-16} & -6.1453\times10^{-16} & -6.9389\times10^{-18} \\
1.25 & 1.25 & 1.0963\times10^{-15} & 4.3021\times10^{-16} & 6.2450\times10^{-16} & -5.2736\times10^{-16} \\
-1.66406 & -0.992187 & 0.25 & -0.25 & 7.4160\times10^{-16} & -1.2490\times10^{-16} \\
-1.19531 & -0.679687 & -0.25 & 0.25 & 1.1501\times10^{-15} & -8.3267\times10^{-16} \\
-2.76367 & -1.04492 & -1.875 & -1.25 & -1.25 & 1.25 \\
-2.76367 & -1.04492 & -1.875 & -1.25 & -1.25 & 1.25 \\
14.1094 & 6.30078 & 6.625 & 3.375 & 5. & -3. \\
-19.3242 & -8.50781 & -9.75 & -5.25 & -7.5 & 4.5 \\
-0.625 & -0.3125 & 1.7418\times10^{-15} & 9.1295\times10^{-16} & 1.6011\times10^{-15} & -1.7618\times10^{-17} \\
-1.25 & -0.9375 & 4.4409\times10^{-16} & 1.3878\times10^{-16} & 5.8027\times10^{-16} & 2.7756\times10^{-16} \\
5.3620\times10^{-15} & 2.4746\times10^{-15} & 3.2734\times10^{-15} & 1.7816\times10^{-15} & 2.6092\times10^{-15} & -2.0713\times10^{-15} \\
-7.2511\times10^{-16} & -3.1442\times10^{-16} & -4.4994\times10^{-16} & -6.4488\times10^{-16} & -4.6320\times10^{-16} & 5.9241\times10^{-16}
\end{array}\right.$$

$$\left.\begin{array}{cccccc}
5.5815\times10^{-16} & 5.7723\times10^{-16} & 7.6328\times10^{-16} & 2.7756\times10^{-16} & 1.1670\times10^{-15} & 1.8906\times10^{-15} \\
-5.4297\times10^{-16} & -5.8460\times10^{-16} & -6.8001\times10^{-16} & -1.8596\times10^{-15} & -1.6757\times10^{-15} & -3.5497\times10^{-15} \\
-8.2312\times10^{-16} & -8.4221\times10^{-16} & -0.0625 & -0.0625 & -1.1068\times10^{-15} & 0.15625 \\
-7.3639\times10^{-16} & -5.8807\times10^{-16} & -0.0625 & 0.1875 & 0.6875 & 1.34375 \\
1.25 & 1.25 & 1.48437 & 2.57812 & 3.32031 & 6.64062 \\
1.25 & 1.25 & 1.48437 & 2.57812 & 4.57031 & 8.51562 \\
-5. & -5. & -4.1875 & -8.5 & -10.5078 & -22.4609 \\
7.5 & 7.5 & 6.375 & 12.5625 & 15.9766 & 33.7891 \\
-1.9552\times10^{-15} & -2.1147\times10^{-15} & 0.25 & -0.25 & -0.875 & -1.625 \\
-1.1675\times10^{-15} & -0.25 & 0.25 & -0.875 & -1.625 & \\
-2.2417\times10^{-15} & -1.9262\times10^{-15} & -3.2431\times10^{-15} & -3.9570\times10^{-15} & 1.25 & 1.25 \\
5.0234\times10^{-16} & 3.9961\times10^{-16} & 7.1514\times10^{-16} & 1.0192\times10^{-15} & -0.25 & 0.25
\end{array}\right].$$

In the considered case, we have the coincidence $(A^D)_8 = A^D$*.*

6. Conclusion

A full-rank SVD representation of outer inverses with prescribed range and null space is presented. Corresponding numerical algorithm for computing $A^{(2)}_{\mathcal{R}(W),\mathcal{N}(W)}$ is derived using the SVD decomposition of an appropriately chosen matrix W. An analogous representation of the outer inverse corresponding to the Using a thin SVD decomposition of W we derive full-rank SVD representations of the corresponding outer inverse of A. This kind of generalized inverse is called the *thin* outer inverse of A with prescribed range and null space.

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