FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 28, No 3 (2013), 323–334

A COMMON FIXED POINT THEOREM OF THREE (ψ,φ)-WEAKLY CONTRACTIVE MAPPING IN G-METRIC SPACES

Rashwan A. Rashwan and Samera M. Saleh

Abstract. In this paper, we establish a common fixed point for three mappings under (ψ,φ)-weakly contractive condition in G-metric spaces. Our results generalize and improve many recent fixed point theorems in the literature. We also provide an example to support our results.

1. Introduction and preliminaries

Banach contraction principle is one of the core subject that has been studied. It has so many different generalizations with different approaches. One of the remarkable generalizations, known as Φ-contraction, was given by Boyd and Wong [7] in 1969. In 1997, Alber and Guerre-Delabriere [6], introduced the notion of a weak φ-contraction which generalizes Boyd and Wong results, so Banachs result. Recently, inspired from the notion of weak φ-contractions, a new concept of (ψ,φ)-contractions was introduced. Khan et al. [12] initiated the use of a control function in metric fixed point theory, which they called an altering distance function. This function and its generalizations have been used in fixed point problems in metric and generalized metric spaces (see e.g. [8], [10], [11] ,[15], [16], [17] and [18]).

Mustafa and Sims [13], [14] generalized the concept of a metric in which the real number is assigned to every triplet of an arbitrary set which called G-metric space. Afterwards Mustafa, Sims and others authors introduced and developed several fixed point theorems for mappings satisfying different contractive conditions in G-metric spaces, also extend known theorems in metric spaces to G-metric spaces. Shatanawi obtained fixed points of ϕ-maps in G-metric spaces [17]. Ding and Karapinar [9] obtained some fixed point theorems for Meir-Keeler type contractions in partially ordered G-metric spaces. The study of unique common fixed points of mappings satisfying strict contractive conditions has been at the center of rigorous research activity. Study of common fixed point theorems in G-metric spaces was initiated by Abbas and Rhoades [1], see also ([2]-[5]).

Received October 27, 2012.; Accepted October 12, 2013.
2010 *Mathematics Subject Classification.* Primary 47H10; Secondary 54H25

In this paper, we establish a common fixed point theorem for three mappings satisfying generalized (ψ, φ)-weakly contractive condition. Also we give an example satisfying all requirements of our results.

Consistent with [14], the following definitions and results will be needed in the sequel. Now onwards, N will denote the set of natural numbers.

Definition 1.1. [14] Let X be a nonempty set and let $G : X^3 \to [0, \infty)$ be a function satisfying:

(G_1) $G(x, y, z) = 0$ if $x = y = z$,

(G_2) $0 < G(x, x, y)$, for all $x, y \in X$, with $x \neq y$,

(G_3) $G(x, x, y) \leq G(x, y, z), \forall x, y, z \in X$, with $z \neq y$,

(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$, (symmetry in all three variables),

(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z), \forall x, y, z, a \in X$, (rectangle inequality).

Then the function G is called a G-metric on X, and the pair (X, G) is called a G-metric space.

Definition 1.2. [14] Let (X, G) be a G-metric space, a sequence (x_n) is said to be

(i) G-convergent if for every $\varepsilon > 0$, there exists an $x \in X$, and $k \in N$ such that for all $m, n \geq k, G(x, x_n, x_m) < \varepsilon$.

(ii) G-Cauchy if for every $\varepsilon > 0$, there exists an $k \in N$ such that for all $m, n, p \geq k, G(x_m, x_n, x_p) < \varepsilon$, that is $G(x_m, x_n, x_p) \to 0$ as $m, n, p \to \infty$.

(iii) A space (X, G) is said to be G-complete if every G-Cauchy sequence in (X, G) is G-convergent.

Lemma 1.1. *[14] Let (X, G) be a G-metric space. Then the following are equivalent:*

(i) (x_n) *is convergent to* x,

(ii) $G(x_n, x_n, x) \to 0$ *as* $n \to \infty$,

(iii) $G(x_n, x, x) \to 0$ *as* $n \to \infty$,

(iv) $G(x_n, x_m, x) \to 0$ *as* $n, m \to \infty$,

Lemma 1.2. *[14] Let (X, G) be a G-metric space. Then the following are equivalent:*

(i) The sequence (x_n) *is G-Cauchy,*

(ii) for every $\varepsilon > 0$, *there exists* $k \in \mathbf{N}$ *such that* $G(x_n, x_m, x_m) < \varepsilon$ *for* $m, n \geq k$.

Lemma 1.3. *[14] Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Definition 1.3. [14] A G metric space X is symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Proposition 1.1. *[14] Every G-metric space (X, G) will define a metric space (X, d_G) by*

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \qquad \forall x, y \in X.$$

Proposition 1.2. *[14] Let (X, G) be a G-metric space. Then for any x, y, z, and $a \in X$, it follows that*

(i) if $G(x, y, z) = 0$ then $x = y = z$,

(ii) $G(x, y, z) \le G(x, x, y) + G(x, x, z)$,

(iii) $G(x, y, y) \le 2G(x, x, y)$,

(iv) $G(x, y, z) \le G(x, a, z) + G(a, y, z)$,

(v) $G(x, y, z) \le \frac{2}{3}(G(x, y, a) + G(x, a, z) + G(a, y, z))$,

(vi) $G(x, y, z) \le G(x, a, a) + G(y, a, a) + G(z, a, a)$,

Definition 1.4. [12] A function $\psi : [0,\infty) \to [0,\infty)$ is called altering distance function if

(i) ψ is increasing and continuous,

(ii) $\psi(t) = 0$ if and only if $t = 0$.

2. Main results

First we state the following Lemma.

Lemma 2.1. *Let f, g and h be self maps on a G-metric space X satisfying*

$$\psi(G(fx, gy, hz)) \le \psi(M(x, y, z)) - \varphi(M(x, y, z)), \tag{2.1}$$

where

$$M(x, y, z) = \max\{G(x,y,z), G(x, y, gy), G(y, z, hz), G(z, x, fx), \alpha G(fx, x, gy) + (1-\alpha)G(y, gy, hz)\},$$

for all $x, y, z \in X$, where $0 < \alpha < 1$, ψ is an altering distance function, and $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function with $\varphi(t) = 0$ if and only if $t = 0$. Then any fixed point of f is a fixed point of g and h and conversely.

Proof. Suppose that $p \in X$ is such that $fp = p$. We claim that $p = gp = hp$. If it is not then $p \neq gp$ or $p \neq hp$. In the case $p \neq gp$ and $p \neq hp$, we have

$$\psi(G(p, gp, hp)) \leq \psi(M(p, p, p)) - \varphi(M(p, p, p)),$$

where

$$\begin{aligned} M(p, p, p) = \max\{&G(p, p, p), G(p, p, gp), G(p, p, hp), G(p, p, fp), \\ &\alpha G(fp, p, gp) + (1 - \alpha)G(p, gp, hp)\} \\ = \max\{&0, G(p, p, gp), G(p, p, hp), \alpha G(p, p, gp) + (1 - \alpha)G(p, gp, hp)\}. \end{aligned}$$

If $M(p, p, p) = G(p, p, gp)$ then

$$\begin{aligned} \psi(G(p, gp, hp) &\leq \psi(G(p, p, gp)) - \varphi(G(p, p, gp)) \\ &\leq \psi(G(p, hp, gp)) - \varphi(G(p, p, gp)). \end{aligned}$$

Hence $gp = p$ is a contradiction. Similarly if $M(p, p, p) = G(p, p, hp)$ or $M(p, p, p) = \alpha G(p, p, gp) + (1 - \alpha)G(p, gp, hp)$. Following the similar arguments to those given above, we obtain a contradiction for $p \neq gp$ and $p = hp$ or for $p = gp$ and $p \neq hp$. Hence in all the cases, we conclude that $p = gp = hp$, The same conclusion holds if $p = gp$ or $p = hp$. □

Theorem 2.1. *Let (X, G) be a complete G-metric space and f, g and h be self maps on X satisfying inequality (2.1) for all $x, y, z \in X$, where $0 < \alpha < 1$, ψ is an altering distance function, and $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function with $\varphi(t) = 0$ if and only if $t = 0$.. Then f, g, and h have a unique common fixed point in X.*

Proof. Let x_0 be an arbitrary point of X. Define x_n by $fx_{3n} = x_{3n+1}$, $gx_{3n+1} = x_{3n+2}$, $hx_{3n+2} = x_{3n+3}$, $n = 0, 1, 2, \cdots$. If $x_n = x_{n+1}$ for some n, with $n = 3m$, then $p = x_{3m}$ is a fixed point of f and by Lemma 2.1 p is a common fixed point for f, g and h. The same holds if $n = 3m + 1$ or $n = 3m + 2$. Now, we assume that $x_n \neq x_{n+1}$ for all $n \in N$. Then from (2.1) we have

$$\begin{aligned} \psi(G(x_{3n+1}, x_{3n+2}, x_{3n+3})) &= \psi(G(fx_{3n}, gx_{3n+1}, hx_{3n+2})) \\ &\leq \psi(M(x_{3n}, x_{3n+1}, x_{3n+2})) - \varphi(M(x_{3n}, x_{3n+1}, x_{3n+2})), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} M(x_{3n}, x_{3n+1}, x_{3n+2}) = \max\{&G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n}, x_{3n+1}, gx_{3n+1}), \\ &G(x_{3n+1}, x_{3n+2}, hx_{3n+2}), G(x_{3n+2}, x_{3n}, fx_{3n}), \\ &\alpha G(x_{3n}, fx_{3n}, gx_{3n+1}) + (1 - \alpha)G(x_{3n+1}, gx_{3n+1}, hx_{3n+2})\}, \end{aligned}$$

yields,

$$\begin{aligned} M(x_{3n}, x_{3n+1}, x_{3n+2}) = \max\{&G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n}, x_{3n+1}, x_{3n+2}), \\ &G(x_{3n+1}, x_{3n+2}, x_{3n+3}), G(x_{3n+2}, x_{3n}, x_{3n+1}), \\ &\alpha G(x_{3n}, x_{3n+1}, x_{3n+2}) + (1 - \alpha)G(x_{3n+1}, x_{3n+2}, x_{3n+3})\}. \end{aligned}$$

Therefore

$$M(x_{3n},x_{3n+1},x_{3n+2}) = \max\{G(x_{3n},x_{3n+1},x_{3n+2}), G(x_{3n+1},x_{3n+2},x_{3n+3})\}.$$

If for some $n \in N$, $M(x_{3n},x_{3n+1},x_{3n+2}) = G(x_{3n+1},x_{3n+2},x_{3n+3})$, then from (2.2) we obtain

$$\psi(G(x_{3n+1},x_{3n+2},x_{3n+3})) \le \psi(G(x_{3n+1},x_{3n+2},x_{3n+3})) - \varphi(G(x_{3n+1},x_{3n+2},x_{3n+3})).$$

Hence $\varphi(G(x_{3n+1},x_{3n+2},x_{3n+3})) = 0$, implies that $x_{3n+1} = x_{3n+2} = x_{3n+3}$, which is a contradiction. Thus $M(x_{3n},x_{3n+1},x_{3n+2}) = G(x_{3n},x_{3n+1},x_{3n+2})$ for each $n \in N$ and (2.2) becomes

$$\begin{aligned}\psi(G(x_{3n+1},x_{3n+2},x_{3n+3})) &\le \psi(G(x_{3n},x_{3n+1},x_{3n+2})) - \varphi(G(x_{3n},x_{3n+1},x_{3n+2}))\\ &\le \psi(G(x_{3n},x_{3n+1},x_{3n+2})).\end{aligned}$$

Since ψ is a nondecreasing function, then

$$G(x_{3n+1},x_{3n+2},x_{3n+3}) \le G(x_{3n},x_{3n+1},x_{3n+2}) = M(x_{3n},x_{3n+1},x_{3n+2}). \tag{2.3}$$

Similarly, we can be shown that

$$G(x_{3n+2},x_{3n+3},x_{3n+4}) \le G(x_{3n+1},x_{3n+2},x_{3n+3}) = M(x_{3n+1},x_{3n+2},x_{3n+3}), \tag{2.4}$$

and

$$G(x_{3n+3},x_{3n+4},x_{3n+5}) \le G(x_{3n+2},x_{3n+3},x_{3n+4}) = M(x_{3n+2},x_{3n+3},x_{3n+4}). \tag{2.5}$$

From (2.3),(2.4) and (2.5) we get

$$G(x_{n+1},x_{n+2},x_{n+3}) \le G(x_n,x_{n+1},x_{n+2}) = M(x_n,x_{n+1},x_{n+2}). \tag{2.6}$$

Therefore we conclude that $\{G(x_n,x_{n+1},x_{n+2}),\ n \in N\}$ is a nonincreasing sequence of positive real numbers. Hence there exists $\delta \ge 0$ such that

$$\lim_{n\to\infty} G(x_n,x_{n+1},x_{n+2}) = \delta. \tag{2.7}$$

Letting $n \to \infty$, in (2.6), we get

$$\lim_{n\to\infty} M(x_n,x_{n+1},x_{n+2}) = \delta. \tag{2.8}$$

Letting $n \to \infty$, in (2.2) and using (2.7) and (2.8) and the continuity of ψ and φ, we get $\psi(\delta) \le \psi(\delta) - \varphi(\delta) \le \psi(\delta)$ and hence $\varphi(\delta) = 0$. This gives us

$$\lim_{n\to\infty} G(x_n,x_{n+1},x_{n+2}) = 0. \tag{2.9}$$

Moreover, from G_3 in Definition 1.1 we obtain

$$\lim_{n\to\infty} G(x_n,x_{n+1},x_{n+1}) = 0. \tag{2.10}$$

Now, we show that the sequence $\{x_n\}$ is a G-Cauchy sequence in X. Suppose that $\{x_n\}$ is not. Then there exist $\varepsilon > 0$, and subsequences $\{x_{m(k)}\}$, $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ and $m(k) = 3r$ and $n(k) = 3s + 1$, where r and s are nonnegative integers such that

$$G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \geq \varepsilon. \tag{2.11}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (2.11), i.e.

$$G(x_{m(k)}, x_{n(k)-1}, x_{n(k)-1}) < \varepsilon. \tag{2.12}$$

From rectangle inequality and (2.12), we have

$$\begin{aligned} \varepsilon \leq G(x_{m(k)}, x_{n(k)}, x_{n(k)}) &\leq G(x_{m(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) \\ &< \varepsilon + G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}). \end{aligned} \tag{2.13}$$

Letting $k \to \infty$ in (2.13) and by (2.10) we conclude that

$$\lim_{k\to\infty} G(x_{m(k)}, x_{n(k)}, x_{n(k)}) = \varepsilon. \tag{2.14}$$

Again, by the triangle inequality we have

$$\begin{aligned} G(x_{m(k)}, x_{n(k)}, x_{n(k)+1}) &\leq G(x_{m(k)}, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{n(k)}, x_{n(k)+1}) \\ &\leq G(x_{m(k)}, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+2}), \end{aligned} \tag{2.15}$$

and

$$G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \leq G(x_{m(k)}, x_{n(k)}, x_{n(k)+1}). \tag{2.16}$$

Letting $k \to \infty$, in (2.15) and (2.16) and using (2.9) and (2.14) we have

$$\lim_{k\to\infty} G(x_{m(k)}, x_{n(k)}, x_{n(k)+1}) = \varepsilon. \tag{2.17}$$

On the other hand, we have

$$G(x_{m(k)}, x_{n(k)+1}, x_{n(k)+1}) \leq G(x_{m(k)}, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}), \tag{2.18}$$

and

$$G(x_{n(k)}, x_{n(k)+1}, x_{m(k)}) \leq G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) + G(x_{n(k)+1}, x_{n(k)+1}, x_{m(k)}). \tag{2.19}$$

Letting $k \to \infty$ in (2.18) and (2.19) and using (2.10), (2.14) and (2.17) we find that

$$\lim_{k\to\infty} G(x_{m(k)}, x_{n(k)+1}, x_{n(k)+1}) = \varepsilon. \tag{2.20}$$

Similarly, we have

$$\begin{aligned} G(x_{m(k)+1}, x_{n(k)}, x_{n(k)+1}) &\leq G(x_{m(k)+1}, x_{m(k)}, x_{m(k)}) + G(x_{m(k)}, x_{n(k)}, x_{n(k)+1}) \\ &\leq 2G(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}) + G(x_{m(k)}, x_{n(k)}, x_{n(k)+1}), \end{aligned} \tag{2.21}$$

and

$$G(x_{m(k)}, x_{n(k)}, x_{n(k)+1}) \le G(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}) + G(x_{m(k)+1}, x_{n(k)}, x_{n(k)+1}). \tag{2.22}$$

Letting $k \to \infty$ in (2.21) and (2.22) and using (2.10) and (2.17) we have

$$\lim_{k\to\infty} G(x_{m(k)+1}, x_{n(k)}, x_{n(k)+1}) = \varepsilon. \tag{2.23}$$

Further, we have

$$G(x_{m(k)}, x_{n(k)+1}, x_{n(k)+1}) \le G(x_{m(k)}, x_{m(k)+1}, x_{n(k)+1}), \tag{2.24}$$

and

$$\begin{aligned} G(x_{m(k)}, x_{m(k)+1}, x_{n(k)+1}) &\le G(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}) + G(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)+1}) \\ &\le G(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}) + G(x_{m(k)+1}, x_{n(k)}, x_{n(k)+1}), \end{aligned} \tag{2.25}$$

Letting $k \to \infty$ in (2.24) and (2.25) and using (2.10), (2.20) and (2.23) we get

$$\lim_{k\to\infty} G(x_{m(k)}, x_{m(k)+1}, x_{n(k)+1}) = \varepsilon. \tag{2.26}$$

Also,

$$G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}) \le G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)}), \tag{2.27}$$

and

$$G(x_{m(k)+1}, x_{n(k)}, x_{n(k)+1}) \le G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}) + G(x_{n(k)+1}, x_{n(k)+1}, x_{n(k)}), \tag{2.28}$$

Letting $k \to \infty$ in (2.27) and (2.28) and using (2.10) and (2.23) we have

$$\lim_{k\to\infty} G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}) = \varepsilon. \tag{2.29}$$

Further, we get

$$\begin{aligned} G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+2}) &\le G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}) + G(x_{n(k)+1}, x_{n(k)+1}, x_{n(k)+2}) \\ &\le G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}) + G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+2}), \end{aligned} \tag{2.30}$$

and

$$G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}) \le G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+2}), \tag{2.31}$$

Letting $k \to \infty$ in (2.30) and (2.31) and using (2.9) and (2.29) we have

$$\lim_{k\to\infty} G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+2}) = \varepsilon. \tag{2.32}$$

Setting $x = x_{m(k)}$, $y = x_{n(k)}$ and $z = x_{n(k)+1}$ in (2.1) we conclude that

$$\begin{aligned} \psi(G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+2})) &= \psi(G(fx_{m(k)}, gx_{n(k)}, hx_{n(k)+1})) \\ &\le \psi(M(x_{m(k)}, x_{n(k)}, x_{n(k)+1})) - \varphi(M(x_{m(k)}, x_{n(k)}, x_{n(k)+1})), \end{aligned}$$

where

$$\begin{aligned} M(x_{m(k)}, x_{n(k)}, x_{n(k)+1}) &= \max\{G(x_{m(k)}, x_{n(k)}, x_{n(k)+1}), G(x_{m(k)}, x_{n(k)}, gx_{n(k)}), \\ &\quad G(x_{n(k)}, x_{n(k)+1}, hx_{n(k)+1}), G(x_{n(k)+1}, x_{m(k)}, fx_{m(k)}), \\ &\quad \alpha G(fx_{m(k)}, x_{m(k)}, gx_{n(k)}) + (1-\alpha)G(x_{n(k)}, gx_{n(k)}, hx_{n(k)+1})\} \\ &= \max\{G(x_{m(k)}, x_{n(k)}, x_{n(k)+1}), G(x_{m(k)}, x_{n(k)}, x_{n(k)+1}), \\ &\quad G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+2}), G(x_{n(k)+1}, x_{m(k)}, x_{m(k)+1}), \\ &\quad \alpha G(x_{m(k)+1}, x_{m(k)}, x_{n(k)+1}) + (1-\alpha)G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+2})\}. \end{aligned}$$

Taking the limits as $k \to \infty$, and using (2.9), (2.17), (2.26) and (2.32) we obtain

$$\lim_{k\to\infty} M(x_{m(k)}, x_{n(k)}, x_{n(k)+1}) = \max\{\varepsilon, 0, \alpha\varepsilon\} = \varepsilon. \tag{2.33}$$

This gives that

$$\psi(\varepsilon) \le \psi(\varepsilon) - \varphi(\varepsilon).$$

Hence $\varepsilon = 0$, which is a contradiction. Therefore $\{x_n\}$ is a G-Cauchy sequence. By G-completeness of X, there exists $u \in X$ such that $\{x_n\}$ converges to u as $n \to \infty$. Now, we prove that $fu = u$. By (2.1) we have

$$\begin{aligned} \psi(G(fu, x_{3n+2}, x_{3n+3})) &= \psi(G(fu, gx_{3n+1}, hx_{3n+2})) \\ &\le \psi(M(u, x_{3n+1}, x_{3n+2})) - \varphi(M(u, x_{3n+1}, x_{3n+2})), \end{aligned} \tag{2.34}$$

where

$$\begin{aligned} M(u, x_{3n+1}, x_{3n+2}) &= \max\{G(u, x_{3n+1}, x_{3n+2}), G(u, x_{3n+1}, gx_{3n+1}), \\ &\quad G(x_{3n+1}, x_{3n+2}, hx_{3n+2}), G(x_{3n+2}, u, fu), \\ &\quad \alpha G(fu, u, gx_{3n+1}) + (1-\alpha)G(x_{3n+1}, gx_{3n+1}, hx_{3n+2})\} \\ &= \max\{G(u, x_{3n+1}, x_{3n+2}), G(u, x_{3n+1}, x_{3n+2}), \\ &\quad G(x_{3n+1}, x_{3n+2}, x_{3n+3}), G(x_{3n+2}, u, fu), \\ &\quad \alpha G(fu, u, x_{3n+2}) + (1-\alpha)G(x_{3n+1}, x_{3n+2}, x_{3n+3})\}. \end{aligned}$$

Letting $n \to \infty$ we have

$$\lim_{n\to\infty} M(u, x_{3n+1}, x_{3n+2}) = \max\{0, G(u, u, fu), \alpha G(fu, u, u)\} = G(fu, u, u).$$

On taking the limit as $n \to \infty$ in (2.34), we have

$$\psi(G(fu, u, u) \le \psi(G(fu, u, u)) - \varphi(G(fu, u, u))). \tag{2.35}$$

Hence $fu = u$. Similarly, it can be shown that $gu = u$ and $hu = u$. To prove the uniqueness, suppose that v is another common fixed point of f, g and h, hence

$$\begin{aligned} \psi(G(u, v, v)) &= \psi(G(fu, gv, hv)) \\ &\le \psi(M(u, v, v)) - \varphi(M(u, v, v)), \end{aligned} \tag{2.36}$$

where

$$\begin{aligned} M(u, v, v) &= \max\{G(u, v, v), G(u, v, gv), \\ &\qquad G(v, v, hv), G(v, u, fu), \\ &\qquad \alpha G(fu, u, gv) + (1-\alpha)G(v, gv, hv)\} \\ &= \max\{G(u, v, v), G(v, u, u)\}. \end{aligned}$$

Also,

$$\begin{aligned} \psi(G(v, u, u)) &= \psi(G(fv, gu, hu)) \\ &\le \psi(M(v, u, u)) - \varphi(M(v, u, u)), \end{aligned} \tag{2.37}$$

where

$$\begin{aligned} M(v, u, u) &= \max\{G(v, u, u), G(v, u, gu), G(u, u, hu), G(u, v, fv), \\ &\qquad \alpha G(fv, v, gu) + (1-\alpha)G(u, gu, hu)\} \\ &= \max\{G(v, u, u), G(u, v, v)\}. \end{aligned}$$

From (2.36) and (2.37)

$$\begin{aligned} \psi(\max\{G(v, u, u), G(u, v, v)\}) &= \max\{(\psi G(v, u, u), \psi G(u, v, v))\} \\ &\le \psi(\max\{G(v, u, u), G(u, v, v)\}) \\ &\quad - \varphi(\max\{G(v, u, u), G(u, v, v)\}), \end{aligned} \tag{2.38}$$

Hence $\varphi(\max\{G(v, u, u), G(u, v, v)\}) = 0$, that means $u = v$ is a contradiction. Thus u is a unique common fixed point of f, g and h. □

Corollary 2.1. *Let (X, G) be a complete G-metric space and $f, g, h : (X, G) \to (X, G)$ satisfying the following inequality*

$$\begin{aligned} G(fx, gy, hz) \le \lambda \max\{&G(x,y,z), G(x, y, gy), G(y, z, hz), G(z, x, fx), \\ &\alpha G(fx, x, gy) + (1-\alpha)G(y, gy, hz)\}, \end{aligned}$$

for all $x, y, z \in X$, where $0 < \lambda, \alpha < 1$, Then f, g, and h have a unique common fixed point in X.

Proof. The result follows by taking $\psi(t) = t$ and $\varphi(t) = t - \lambda t$ where $\psi, \varphi : [0, \infty) \to [0, \infty)$ in Theorem 2.1. □

Corollary 2.2. *Let (X, G) be a complete G-metric space. Let f be a self mapping on X satisfying the following*

$$\psi(G(fx, fy, fz)) \le \psi(M(x, y, z)) - \varphi(M(x, y, z)),$$

where

$$\begin{aligned} M(x, y, z) = \max\{&G(x,y,z), G(x, y, fy), G(y, z, fz), G(z, x, fx), \\ &\alpha G(fx, x, fy) + (1-\alpha)G(y, fy, fz)\}, \end{aligned}$$

for all $x, y, z \in X$, where $0 < \alpha < 1$, ψ is an altering distance function, and $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function with $\varphi(t) = 0$ if and only if $t = 0$. Then f has a unique fixed point in X.

Motivated by Abbas et al. [4] we give the following example which satisfying the hypotheses of Theorem 2.1.

Example 2.1. Let $X = \{0, 1, 2, 3\}$ be a set with G-metric defined by

(x, y, z)	$G(x, y, z)$
$(0,0,0),(1,1,1),(2,2,2),(3,3,3),$	0
$(0,0,2),(0,2,0),(2,0,0),(0,2,2),(2,0,2),(2,2,0),$	1
$(0,0,3),(0,3,0),(0,0,1),(0,1,0),(1,0,0),(0,1,1),$ $(1,0,1),(1,1,0),(3,0,0),(0,3,3),(3,0,3),(3,3,0),$ $(1,1,2),(1,2,1),(2,1,1),(1,2,2),(2,1,2),(2,2,1),$ $(1,1,3),(1,3,1),(3,1,1),(1,3,3),(3,1,3),(3,3,1),$ $(2,2,3),(2,3,2),(3,2,2),(2,3,3),(3,2,3),(3,3,2),$	3
$(0,1,2),(0,1,3),(0,2,1),(0,2,3),(0,3,1),(0,3,2),$ $(1,0,2),(1,0,3),(1,2,0),(1,2,3),(1,3,0),(1,3,2),$ $(2,0,1),(2,0,3),(2,1,0),(2,1,3),(2,3,0),(2,3,1),$ $(3,0,1),(3,0,2),(3,1,0),(3,1,2),(3,2,0),(3,2,1),$	3

For f, g and h are self mappings of X defined by

x	$f(x)$	$g(x)$	$h(x)$
0	0	0	0
1	0	2	2
2	0	0	0
3	2	0	2

It is clearly that X is a complete G-metric space. We define $\psi, \varphi : [0, \infty) \to [0, \infty)$ by $\psi(t) = t^3$ and $\varphi(t) = \frac{t}{4}$. Then ψ and φ have the properties mentioned in Theorem 2.1. We find that $G(fx, gy, hz) \in \{0, 1\}$. If $G(fx, gy, hz) = 0$ then (2.1) holds.
On otherwise one find that if $G(fx, gy, hz) = 1$ then $M(x, y, z) = 3$. Hence

$$\psi(G(fx, gy, hz)) = 1 \leq \psi(M(x, y, z)) - \varphi(M(x, y, z)) = 27 - \frac{3}{4}.$$

Then condition (2.1) satisfied for all $x, y, z \in X$. Hence all hypotheses of Theorem 2.1 are satisfied and 0 is the unique common fixed point of f, g and h.

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Rashwan A. Rashwan
Assiut University
Faculty of Science
Department of Mathematics
P. O. Box

Assiut, Egypt
rr_rashwan54@yahoo.com

Samera M. Saleh
Assiut University
Faculty of Science
Department of Mathematics
Assiut, Egypt
samirasaleh2007@yahoo.com