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ON SOME NEW HADAMARD-LIKE INEQUALITIES FOR COORDINATED s-CONVEX FUNCTIONS

M. Emin Özdemir, Mevlüt Tunç* and Ahmet Ocak Akdemir

Abstract. In this paper, we prove some new integral inequalities of Hadamard-like type for s-convex functions in the second sense on the co-ordinates.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a < b$. The following double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is well known in the literature as Hadamard's inequality. Both inequalities hold in the reversed direction if f is concave. In [2], Alomari and Darus defined s-convex functions on the co-ordinates as following:

Definition 1.1. Consider the bi-dimensional interval $\Delta = [a,b] \times [c,d]$ in $[0,\infty)^2$ with $a < b$ and $c < d$. The mapping $f : \Delta \to \mathbb{R}$ is $s-$convex in the second sense on Δ if

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda^s f(x,y) + (1-\lambda)^s f(z,w) \tag{1.2}$$

hold for all $(x,y),(z,w) \in \Delta$ with $\lambda \in [0,1]$.and for some fixed $s \in (0,1]$.

A function $f : \Delta \to \mathbb{R}$ is s-convex in the second sense on Δ is called co-ordinated s-convex in the second sense on Δ if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are $s-$convex in the second sense for all $y \in [c,d]$ and $x \in [a,b]$ with some fixed $s \in (0,1]$. Moreover, in [2], Alomari and Darus established the following inequalities of Hadamard's type for coordinated s-convex functions in the second sense on a rectangle from the plane $\mathbb{R}^2$.

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Theorem 1.1. *Suppose that $f : \Delta = [a,b] \times [c,d] \subset [0,\infty)^2 \to [0,\infty)$ is $s-$convex function in the second sense on the co-ordinates on Δ. Then one has the inequalities;*

$$\begin{aligned}
(1.3)\qquad & 4^{s-1} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
\leq\ & 2^{s-2}\left[\frac{1}{b-a}\int_a^b f(x, \frac{c+d}{2}dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\
\leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \\
\leq\ & \frac{1}{2(s+1)}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx + \frac{1}{b-a}\int_a^b f(x,d)\,dx \right. \\
& \left. +\frac{1}{d-c}\int_c^d f(a,y)\,dy + \frac{1}{d-c}\int_c^d f(b,y)\,dy\right] \\
\leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{(s+1)^2}.
\end{aligned}$$

Similar results can be found in ([1]-[13]).

However, Özdemir et.al.[7] established the following lemma for twice partial differentiable mapping on $\Delta = [a,b] \times [c,d]$.

Lemma 1.1. *Let $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ be a twice partial differentiable mapping on $\Delta = [a,b] \times [c,d]$. If $\frac{\partial^2 f}{\partial t \partial \lambda} \in L(\Delta)$, then the following equality holds:*

$$\begin{aligned}
& \frac{1}{(b-a)(d-c)}\left[A - (x-a)\int_c^d f(a,v)\,dv - (b-x)\int_c^d f(b,v)\,dv \right. \\
& \left. -(d-y)\int_a^b f(u,d)\,du - (y-c)\int_a^b f(u,c)\,du + \int_a^b\int_c^d f(u,v)\,dudv\right] \\
=\ & \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\int_0^1\int_0^1 (t-1)(\lambda-1)\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)a, \lambda y+(1-\lambda)c)\,d\lambda dt \\
& +\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}\int_0^1\int_0^1 (t-1)(1-\lambda)\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)a, \lambda y+(1-\lambda)d)\,d\lambda dt \\
& +\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\int_0^1\int_0^1 (1-t)(\lambda-1)\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)b, \lambda y+(1-\lambda)c)\,d\lambda dt \\
& +\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\int_0^1\int_0^1 (1-t)(1-\lambda)\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)b, \lambda y+(1-\lambda)d)\,d\lambda dt
\end{aligned}$$

where

$$A = \frac{(x-a)(y-c)f(a,c)+(x-a)(d-y)f(a,d)}{(b-a)(d-c)} + \frac{(b-x)(y-c)f(b,c)+(b-x)(d-y)f(b,d)}{(b-a)(d-c)}.$$

The main purpose of this paper is to prove some new inequalities of Hadamard-type for $s-$convex functions in the second sense on the co-ordinates.

2. Main Results

Theorem 2.1. *Let $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ be a partial differentiable mapping on $\Delta=[a,b]\times[c,d]$ and $\frac{\partial^2 f}{\partial t\partial\lambda}\in L(\Delta)$. If $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|$ is a $s-$convex function in the second sense on the co-ordinates on Δ, for some fixed $s\in(0,1]$, then the following inequality holds;*

(2.1)

$$\begin{aligned}
&\Bigg|\frac{1}{(b-a)(d-c)}\Bigg[A-(x-a)\int_c^d f(a,v)\,dv-(b-x)\int_c^d f(b,v)\,dv \\
&-(d-y)\int_a^b f(u,d)\,du-(y-c)\int_a^b f(u,c)\,du+\int_a^b\int_c^d f(u,v)\,dudv\Bigg]\Bigg| \\
\leq\ & \frac{1}{(b-a)(d-c)(s+2)^2} \\
&\Bigg[\left(\frac{\left((x-a)^2+(b-x)^2\right)\left((y-c)^2+(d-y)^2\right)}{(s+1)^2}\right)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right| \\
&+\left(\frac{(x-a)^2\left((y-c)^2+(d-y)^2\right)}{(s+1)}\right)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right| \\
&+\left(\frac{(b-x)^2\left((y-c)^2+(d-y)^2\right)}{(s+1)}\right)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,y)\right| \\
&+\left(\frac{(y-c)^2\left((x-a)^2+(b-x)^2\right)}{(s+1)}\right)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right| \\
&+\left(\frac{(d-y)^2\left((x-a)^2+(b-x)^2\right)}{(s+1)}\right)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,d)\right|
\end{aligned}$$

$$+(x-a)^2(y-c)^2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|+(x-a)^2(d-y)^2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|$$
$$+(b-x)^2(y-c)^2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|+(b-x)^2(d-y)^2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\Bigg],$$

where A is as above.

Proof. From Lemma 1.1 and using the property of modulus, we have

$$\begin{aligned}
&\left|\frac{1}{(b-a)(d-c)}\left[A-(x-a)\int_c^d f(a,v)\,dv-(b-x)\int_c^d f(b,v)\,dv\right.\right.\\
&\left.\left.-(d-y)\int_a^b f(u,d)\,du-(y-c)\int_a^b f(u,c)\,du+\int_a^b\int_c^d f(u,v)\,dudv\right]\right|\\
\leq\;&\frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\\
&\times\int_0^1\int_0^1|(t-1)(\lambda-1)|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)a,\lambda y+(1-\lambda)c)\right|d\lambda dt\\
&+\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}\\
&\times\int_0^1\int_0^1|(t-1)(1-\lambda)|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)a,\lambda y+(1-\lambda)d)\right|d\lambda dt\\
&+\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\\
&\times\int_0^1\int_0^1|(1-t)(\lambda-1)|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)b,\lambda y+(1-\lambda)c)\right|d\lambda dt\\
&+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\\
&\times\int_0^1\int_0^1|(1-t)(1-\lambda)|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)b,\lambda y+(1-\lambda)d)\right|d\lambda dt.
\end{aligned}$$

Since $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ is coordinated $s-$convex in the second sense, for some fixed $s\in(0,1]$, we can write

$$\left|\frac{1}{(b-a)(d-c)}\left[A-(x-a)\int_c^d f(a,v)\,dv-(b-x)\int_c^d f(b,v)\,dv\right.\right.$$

$$\left.\left.-(d-y)\int_a^b f(u,d)\,du-(y-c)\int_a^b f(u,c)\,du+\int_a^b\int_c^d f(u,v)\,dudv\right]\right|$$

$$\leq \quad \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\int_0^1 |(\lambda-1)|$$

$$\times\left[\int_0^1 (t-1)\,t^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,\lambda y+(1-\lambda)c)\right|dt\right.$$

$$\left.+\int_0^1 (t-1)(1-t)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,\lambda y+(1-\lambda)c)\right|dt\right]d\lambda$$

$$+\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}\int_0^1 |(1-\lambda)|$$

$$\times\left[\int_0^1 (t-1)\,t^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,\lambda y+(1-\lambda)d)\right|dt+\int_0^1 (t-1)(1-t)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,\lambda y+(1-\lambda)d)\right|dt\right]d\lambda$$

$$+\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\int_0^1 |(\lambda-1)|$$

$$\times\left[\int_0^1 (1-t)\,t^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,\lambda y+(1-\lambda)c)\right|dt\right.$$

$$\left.+\int_0^1 (1-t)(1-t)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,\lambda y+(1-\lambda)c)\right|dt\right]d\lambda$$

$$+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\int_0^1 |(1-\lambda)|$$

$$\times\left[\int_0^1 (1-t)\,t^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,\lambda y+(1-\lambda)d)\right|dt\right.$$

$$+\int_0^1 (1-t)(1-t)^s \left|\frac{\partial^2 f}{\partial t \partial \lambda}(b, \lambda y + (1-\lambda) d)\right| dt \Bigg] d\lambda.$$

By computing these integrals, we obtain

$$\left|\frac{1}{(b-a)(d-c)}\left[A-(x-a)\int_c^d f(a,v)\,dv-(b-x)\int_c^d f(b,v)\,dv\right.\right.$$
$$\left.\left.-(d-y)\int_a^b f(u,d)\,du-(y-c)\int_a^b f(u,c)\,du+\int_a^b\int_c^d f(u,v)\,dudv\right]\right|$$
$$\leq \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\int_0^1 |\lambda-1|\left[\frac{-1}{(s+1)(s+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,\lambda y+(1-\lambda)c)\right|\right.$$
$$\left.-\frac{1}{s+2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,\lambda y+(1-\lambda)c)\right|\right]d\lambda$$
$$+\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}\int_0^1 |1-\lambda|\left[\frac{-1}{(s+1)(s+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,\lambda y+(1-\lambda)d)\right|\right.$$
$$\left.-\frac{1}{s+2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,\lambda y+(1-\lambda)d)\right|\right]d\lambda$$
$$+\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\int_0^1 |\lambda-1|\left[\frac{-1}{(s+1)(s+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,\lambda y+(1-\lambda)c)\right|\right.$$
$$\left.-\frac{1}{s+2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,\lambda y+(1-\lambda)c)\right|\right]d\lambda$$
$$+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\int_0^1 |1-\lambda|\left[\frac{-1}{(s+1)(s+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,\lambda y+(1-\lambda)d)\right|\right.$$
$$\left.-\frac{1}{s+2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,\lambda y+(1-\lambda)d)\right|\right]d\lambda.$$

Using coordinated s–convexity in the second sense of $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|$ again and computing all integrals, we obtain

$$\left|\frac{1}{(b-a)(d-c)}\left[A-(x-a)\int_c^d f(a,v)\,dv-(b-x)\int_c^d f(b,v)\,dv\right.\right.$$

$$-(d-y)\int_a^b f(u,d)\,du-(y-c)\int_a^b f(u,c)\,du+\int_a^b\int_c^d f(u,v)\,dudv\Bigg|\Bigg|$$

$$\leq \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}$$

$$\times\left\{\int_0^1|\lambda-1|\left[\frac{-1}{(s+1)(s+2)}\left(\lambda^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|+(1-\lambda)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|\right)\right]d\lambda\right.$$

$$\left.+\int_0^1|\lambda-1|\left[-\frac{1}{s+2}\left(\lambda^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right|+(1-\lambda)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|\right)\right]d\lambda\right\}$$

$$+\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}$$

$$\times\left\{\int_0^1|1-\lambda|\left[\frac{-1}{(s+1)(s+2)}\left(\lambda^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|+(1-\lambda)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,d)\right|\right)\right]d\lambda\right.$$

$$\left.+\int_0^1|1-\lambda|\left[-\frac{1}{s+2}\left(\lambda^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right|+(1-\lambda)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|\right)\right]d\lambda\right\}$$

$$+\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}$$

$$\times\left\{\int_0^1|\lambda-1|\left[\frac{-1}{(s+1)(s+2)}\left(\lambda^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|+(1-\lambda)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|\right)\right]d\lambda\right.$$

$$\left.+\int_0^1|\lambda-1|\left[-\frac{1}{s+2}\left(\lambda^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,y)\right|+(1-\lambda)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|\right)\right]d\lambda\right\}$$

$$+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}$$

$$\times\left\{\int_0^1|1-\lambda|\left[\frac{-1}{(s+1)(s+2)}\left(\lambda^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|+(1-\lambda)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,d)\right|\right)\right]d\lambda\right.$$

$$\left.+\int_0^1|1-\lambda|\left[-\frac{1}{s+2}\left(\lambda^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,y)\right|+(1-\lambda)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right)\right]\right\}$$

Computing the above integrals,

$$\left|\frac{1}{(b-a)(d-c)}\left[A-(x-a)\int_c^d f(a,v)\,dv-(b-x)\int_c^d f(b,v)\,dv\right.\right.$$
$$\left.\left.-(d-y)\int_a^b f(u,d)\,du-(y-c)\int_a^b f(u,c)\,du+\int_a^b\int_c^d f(u,v)\,dudv\right]\right|$$
$$\leq \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}$$
$$\times\left[\frac{1}{(s+1)^2(s+2)^2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|+\frac{1}{(s+1)(s+2)^2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|\right.$$
$$\left.+\frac{1}{(s+1)(s+2)^2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right|+\frac{1}{(s+2)^2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|\right\}$$
$$+\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}$$
$$\times\left\{\frac{1}{(s+1)^2(s+2)^2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|+\frac{1}{(s+1)(s+2)^2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,d)\right|\right.$$
$$\left.+\frac{1}{(s+1)(s+2)^2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right|+\frac{1}{(s+2)^2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|\right\}$$
$$+\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}$$
$$\times\left\{\frac{1}{(s+1)^2(s+2)^2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|+\frac{1}{(s+1)(s+2)^2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|\right.$$
$$\left.+\frac{1}{(s+1)(s+2)^2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,y)\right|+\frac{1}{(s+2)^2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|\right\}$$
$$+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}$$
$$\times\left\{\frac{1}{(s+1)^2(s+2)^2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|+\frac{1}{(s+1)(s+2)^2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,d)\right|\right.$$
$$\left.+\frac{1}{(s+1)(s+2)^2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,y)\right|+\frac{1}{(s+2)^2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right\}.$$

Thus

$$\left|\frac{1}{(b-a)(d-c)}\left[A-(x-a)\int_c^d f(a,v)\,dv-(b-x)\int_c^d f(b,v)\,dv\right.\right.$$

$$-(d-y)\int_a^b f(u,d)\,du-(y-c)\int_a^b f(u,c)\,du+\int_a^b\int_c^d f(u,v)\,dudv\Bigg|\Bigg|$$

$$\leq \Bigg[\left(\frac{\left((x-a)^2+(b-x)^2\right)\left((y-c)^2+(d-y)^2\right)}{(b-a)(d-c)(s+1)^2(s+2)^2}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|$$

$$+\left(\frac{(x-a)^2\left((y-c)^2+(d-y)^2\right)}{(b-a)(d-c)(s+1)(s+2)^2}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(a,y)\right|$$

$$+\left(\frac{(b-x)^2\left((y-c)^2+(d-y)^2\right)}{(b-a)(d-c)(s+1)(s+2)^2}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(b,y)\right|$$

$$+\left(\frac{(y-c)^2\left((x-a)^2+(b-x)^2\right)}{(b-a)(d-c)(s+1)(s+2)^2}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(x,c)\right|$$

$$+\left(\frac{(d-y)^2\left((x-a)^2+(b-x)^2\right)}{(b-a)(d-c)(s+1)(s+2)^2}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(x,d)\right|$$

$$+\frac{(x-a)^2(y-c)^2}{(b-a)(d-c)(s+2)^2}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)(s+2)^2}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|$$

$$+\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)(s+2)^2}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)(s+2)^2}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\Bigg],$$

which completes the proof. □

Corollary 2.1. *In Theorem 2.1,*

(1) if we choose $x=a$, $y=c$, we obtain the following inequality;

$$\frac{1}{(b-a)(d-c)}\left|f(b,d)-(b-a)\int_c^d f(b,v)\,dv-(d-c)\int_a^b f(u,d)\,du+\int_a^b\int_c^d f(u,v)\,dudv\right|$$

$$\leq \frac{(b-a)(d-c)}{(s+2)^2}$$

$$\times\left[\frac{1}{(s+1)^2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|+\frac{1}{s+1}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|+\frac{1}{s+1}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right].$$

(2) if we choose $x=b$, $y=d$, we obtain the following inequality;

$$\frac{1}{(b-a)(d-c)}\left|f(a,c)-(b-a)\int_c^d f(a,v)\,dv-(d-c)\int_a^b f(u,c)\,du+\int_a^b\int_c^d f(u,v)\,dudv\right|$$

$$\leq \frac{(b-a)(d-c)}{(s+2)^2}$$

$$\times\left[\frac{1}{(s+1)^2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|+\frac{1}{s+1}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|+\frac{1}{s+1}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|\right].$$

(3) if we choose $x=a$, $y=d$, we obtain the following inequality;

$$\frac{1}{(b-a)(d-c)}\left|f(b,c)-(b-a)\int_c^d f(b,v)\,dv-(d-c)\int_a^b f(u,c)\,du+\int_a^b\int_c^d f(u,v)\,dudv\right|$$
$$\leq \frac{(b-a)(d-c)}{(s+2)^2}$$
$$\times\left[\frac{1}{(s+1)^2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|+\frac{1}{s+1}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|+\frac{1}{s+1}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|\right]$$

(4) if we choose $x=b$, $y=c$, we obtain the following inequality;

$$\frac{1}{(b-a)(d-c)}\left|f(a,d)-(b-a)\int_c^d f(a,v)\,dv-(d-c)\int_a^b f(u,d)\,du+\int_a^b\int_c^d f(u,v)\,dudv\right|$$
$$\leq \frac{(b-a)(d-c)}{(s+2)^2}$$
$$\times\left[\frac{1}{(s+1)^2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|+\frac{1}{s+1}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|+\frac{1}{s+1}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|\right]$$

Remark 2.1. From sum of four inequalities above, we obtain;

(2.2)
$$\left|f(b,d)-(b-a)\int_c^d f(b,v)\,dv-(d-c)\int_a^b f(u,d)\,du+\int_a^b\int_c^d f(u,v)\,dudv\right|$$
$$+\left|f(a,c)-(b-a)\int_c^d f(a,v)\,dv-(d-c)\int_a^b f(u,c)\,du+\int_a^b\int_c^d f(u,v)\,dudv\right|$$
$$+\left|f(b,c)-(b-a)\int_c^d f(b,v)\,dv-(d-c)\int_a^b f(u,c)\,du+\int_a^b\int_c^d f(u,v)\,dudv\right|$$
$$+\left|f(a,d)-(b-a)\int_c^d f(a,v)\,dv-(d-c)\int_a^b f(u,d)\,du+\int_a^b\int_c^d f(u,v)\,dudv\right|$$
$$\leq \frac{(b-a)^2(d-c)^2}{(s+1)^2}\times$$
$$\left[\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|\right].$$

Corollary 2.2. *In Theorem 2.1, if we choose $x=\frac{a+b}{2}$, $y=\frac{c+d}{2}$ in (2.1), we obtain the following inequality;*

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4(b-a)(d-c)}-\frac{1}{2(d-c)}\int_c^d f(a,v)\,dv-\frac{1}{2(d-c)}\int_c^d f(b,v)\,dv\right.\\
&\left.-\frac{1}{2(b-a)}\int_a^b f(u,d)\,du-\frac{1}{2(b-a)}\int_a^b f(u,c)\,du+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dudv\right|\\
\leq\;& \frac{(b-a)(d-c)}{4(s+2)^2}\left[\frac{1}{(s+1)^2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right.\\
&+\frac{1}{2(s+1)}\left\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{c+d}{2}\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{c+d}{2}\right)\right|\right\}\\
&+\frac{1}{2(s+1)}\left\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{a+b}{2},c\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{a+b}{2},d\right)\right|\right\}\\
&\left.+\frac{1}{4}\left\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right\}\right].
\end{aligned}$$

Theorem 2.2. *Let $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ be a partial differentiable mapping on $\Delta=[a,b]\times[c,d]$ and $\frac{\partial^2 f}{\partial t\partial\lambda}\in L(\Delta)$. If $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q$, $q>1$, is a s−convex function in the second sense on the co-ordinates on Δ, for some fixed $s\in(0,1]$, then the following inequality holds;*

(2.3)
$$\begin{aligned}
&\left|\frac{1}{(b-a)(d-c)}\left[A-(x-a)\int_c^d f(a,v)\,dv-(b-x)\int_c^d f(b,v)\,dv\right.\right.\\
&\left.\left.-(d-y)\int_a^b f(u,d)\,du-(y-c)\int_a^b f(u,c)\,du+\int_a^b\int_c^d f(u,v)\,dudv\right]\right|\\
\leq\;& \frac{1}{(p+1)^{\frac{2}{p}}}\frac{1}{(s+1)^{\frac{2}{q}}}\times\\
&\left\{\frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q\right)^{\frac{1}{q}}\right.\\
&+\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q\right)^{\frac{1}{q}}\\
&+\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right)^{\frac{1}{q}}\Bigg\},$$

where $p^{-1}+q^{-1}=1$.

Proof. From Lemma 1.1, we have

$$\left|\frac{1}{(b-a)(d-c)}\left[A-(x-a)\int_c^d f(a,v)\,dv-(b-x)\int_c^d f(b,v)\,dv\right.\right.$$
$$\left.\left.-(d-y)\int_a^b f(u,d)\,du-(y-c)\int_a^b f(u,c)\,du+\int_a^b\int_c^d f(u,v)\,dudv\right]\right|$$
$$\leq\frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\int_0^1\int_0^1|(t-1)(\lambda-1)|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)a,\lambda y+(1-\lambda)c)\right|d\lambda dt$$
$$+\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}\int_0^1\int_0^1|(t-1)(1-\lambda)|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)a,\lambda y+(1-\lambda)d)\right|d\lambda dt$$
$$+\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\int_0^1\int_0^1|(1-t)(\lambda-1)|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)b,\lambda y+(1-\lambda)c)\right|d\lambda dt$$
$$+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\int_0^1\int_0^1|(1-t)(1-\lambda)|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)b,\lambda y+(1-\lambda)d)\right|d\lambda dt.$$

By using the well known Hölder inequality for double integrals, then one has:

$$\left|\frac{1}{(b-a)(d-c)}\left[A-(x-a)\int_c^d f(a,v)\,dv-(b-x)\int_c^d f(b,v)\,dv\right.\right. \tag{2.4}$$
$$\left.\left.-(d-y)\int_a^b f(u,d)\,du-(y-c)\int_a^b f(u,c)\,du+\int_a^b\int_c^d f(u,v)\,dudv\right]\right|$$
$$\leq\frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1|(t-1)(\lambda-1)|^p\,d\lambda dt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)a,\lambda y+(1-\lambda)c)\right|^q d\lambda dt\right)^{\frac{1}{q}}$$

$$+\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1|(t-1)(1-\lambda)|^p\,d\lambda dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)a,\lambda y+(1-\lambda)d)\right|^q d\lambda dt\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1|(1-t)(\lambda-1)|^p\,d\lambda dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)b,\lambda y+(1-\lambda)c)\right|^q d\lambda dt\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1|(1-t)(1-\lambda)|^p\,d\lambda dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)b,\lambda y+(1-\lambda)d)\right|^q d\lambda dt\right)^{\frac{1}{q}}$$

Since $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q$, $q>1$, is s-convex function in the second sense on the co-ordinates on Δ, for some fixed $s\in(0,1]$, we know that for $t\in[0,1]$

$$\begin{aligned}
&\left|\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)a,\lambda y+(1-\lambda)c)\right|^q\\
\leq\ & t^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,\lambda y+(1-\lambda)c)\right|^q+(1-t)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,\lambda y+(1-\lambda)c)\right|^q\\
\leq\ & t^s\left(\lambda^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+(1-\lambda)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|^q\right)\\
&+(1-t)^s\left(\lambda^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right|^q+(1-\lambda)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q\right).
\end{aligned}$$

Hence it follows that

(2.5)

$$\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)a,\lambda y+(1-\lambda)c)\right|^q d\lambda dt\right)^{\frac{1}{q}}$$

$$\leq \left(\int_0^1\int_0^1 \left\{ t^s\lambda^s \left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q + t^s(1-\lambda)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|^q \right.\right.$$

$$\left.\left. +(1-t)^s\lambda^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right|^q + (1-t)^s(1-\lambda)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q\right\} dtd\lambda\right)^{\frac{1}{q}}$$

$$= \frac{1}{(s+1)^{\frac{2}{q}}}\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q\right)^{\frac{1}{q}}$$

A similar way for other integral, since $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q$, $q>1$, is co-ordinated s-convex function in the second sense on Δ, we get

(2.6)

$$\left(\int_0^1\int_0^1 \left|\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)a,\lambda y+(1-\lambda)d)\right|^q dsdt\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{(s+1)^{\frac{2}{q}}}\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,d)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q\right)^{\frac{1}{q}},$$

(2.7)

$$\left(\int_0^1\int_0^1 \left|\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)b,\lambda y+(1-\lambda)c)\right|^q dsdt\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{(s+1)^{\frac{2}{q}}}\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,y)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q\right)^{\frac{1}{q}},$$

(2.8)

$$\left(\int_0^1\int_0^1 \left|\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)b,\lambda y+(1-\lambda)d)\right|^q dsdt\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{(s+1)^{\frac{2}{q}}}\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,d)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,y)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right)^{\frac{1}{q}}.$$

By the (2.5)-(2.8), we get the inequality (2.3). □

Corollary 2.3. *In Theorem 2.2,*

(1) if we choose $x=a$, $y=c$, or $x=b$, $y=d$, we obtain the following inequality:

(2.9)

$$\begin{aligned}&\frac{1}{(b-a)(d-c)}\left|f(b,d)-(b-a)\int_c^d f(b,v)\,dv-(d-c)\int_a^b f(u,d)\,du\right.\\&\quad\left.+\int_a^b\int_c^d f(u,v)\,dudv\right|\\&\leq \frac{(b-a)(d-c)}{(p+1)^{\frac{2}{p}}(s+1)^{\frac{2}{q}}}\\&\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right)^{\frac{1}{q}}.\end{aligned}$$

(2) if we choose $x=b$, $y=d$, we obtain the following inequality;

$$\begin{aligned}&\frac{1}{(b-a)(d-c)}\left|f(a,c)-(b-a)\int_c^d f(a,v)\,dv-(d-c)\int_a^b f(u,c)\,du\right.\\&\quad\left.+\int_a^b\int_c^d f(u,v)\,dudv\right|\\&\leq \frac{(b-a)(d-c)}{(p+1)^{\frac{2}{p}}(s+1)^{\frac{2}{q}}}\\&\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q\right)^{\frac{1}{q}}.\end{aligned}\tag{2.10}$$

(3) if we choose $x=a$, $y=d$, we obtain the following inequality;

$$\begin{aligned}&\frac{1}{(b-a)(d-c)}\left|f(b,c)-(b-a)\int_c^d f(b,v)\,dv-(d-c)\int_a^b f(u,c)\,du\right.\\&\quad\left.+\int_a^b\int_c^d f(u,v)\,dudv\right|\\&\leq \frac{(b-a)(d-c)}{(p+1)^{\frac{2}{p}}(s+1)^{\frac{2}{q}}}\\&\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q\right)^{\frac{1}{q}}.\end{aligned}\tag{2.11}$$

(4) if we choose $x=b$, $y=c$, we obtain the following inequality;

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\left|f(a,d)-(b-a)\int_c^d f(a,v)\,dv-(d-c)\int_a^b f(u,d)\,du\right. \\
&\left.+\int_a^b\int_c^d f(u,v)\,dudv\right| \\
&\leq \frac{(b-a)(d-c)}{(p+1)^{\frac{2}{p}}(s+1)^{\frac{2}{q}}} \\
&\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}\tag{2.12}$$

(5) if we choose $x=\frac{a+b}{2}$, $y=\frac{c+d}{2}$, we obtain the following inequality;

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4(b-a)(d-c)}-\frac{1}{2(d-c)}\int_c^d f(a,v)\,dv\right. \\
&-\frac{1}{2(d-c)}\int_c^d f(b,v)\,dv. \\
&-\frac{1}{2(b-a)}\int_a^b f(u,d)\,du-\frac{1}{2(b-a)}\int_a^b f(u,c)\,du \\
&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dudv\right| \\
&\leq \frac{(b-a)(d-c)}{16(p+1)^{\frac{2}{p}}(s+1)^{\frac{2}{q}}}\times \\
&\left\{\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{a+b}{2},c\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q\right)^{\frac{1}{q}}\right. \\
&+\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{a+b}{2},d\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q\right)^{\frac{1}{q}} \\
&+\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{a+b}{2},c\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q\right)^{\frac{1}{q}} \\
&\left.+\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{a+b}{2},d\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Remark 2.2. From sum of (2.9)-(2.12), we obtain;

(2.13)
$$\left|f(a,c)-(b-a)\int_c^d f(a,v)\,dv-(d-c)\int_a^b f(u,c)\,du+\int_a^b\int_c^d f(u,v)\,dudv\right|$$
$$+\left|f(a,d)-(b-a)\int_c^d f(a,v)\,dv-(d-c)\int_a^b f(u,d)\,du+\int_a^b\int_c^d f(u,v)\,dudv\right|$$
$$+\left|f(b,c)-(b-a)\int_c^d f(b,v)\,dv-(d-c)\int_a^b f(u,c)\,du+\int_a^b\int_c^d f(u,v)\,dudv\right|$$
$$+\left|f(b,d)-(b-a)\int_c^d f(b,v)\,dv-(d-c)\int_a^b f(u,d)\,du+\int_a^b\int_c^d f(u,v)\,dudv\right|$$
$$\leq \frac{4(b-a)^2(d-c)^2}{(p+1)^{\frac{2}{p}}(s+1)^{\frac{2}{q}}}\times$$
$$\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right)^{\frac{1}{q}}.$$

Theorem 2.3. *Let $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ be a partial differentiable mapping on $\Delta=[a,b]\times[c,d]$ and $\frac{\partial^2 f}{\partial t\partial\lambda}\in L(\Delta)$. If $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q$, $q\geq 1$, is a $s-$convex function in the second sense on the co-ordinates on Δ, for some fixed $s\in(0,1]$, then the following inequality holds;*

(2.14)
$$\left|\frac{1}{(b-a)(d-c)}\left[A-(x-a)\int_c^d f(a,v)\,dv-(b-x)\int_c^d f(b,v)\,dv\right.\right.$$
$$\left.\left.-(d-y)\int_a^b f(u,d)\,du-(y-c)\int_a^b f(u,c)\,du+\int_a^b\int_c^d f(u,v)\,dudv\right]\right|$$
$$\leq \frac{2^{2-\frac{2}{q}}}{(s+1)^{\frac{2}{q}}(s+2)^{\frac{2}{q}}}\times$$
$$\left\{\frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\left\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+(s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|^q\right.\right.$$
$$\left.+(s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right|^q+(s+1)^2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q\right\}^{\frac{1}{q}}$$
$$+\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}\left\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+(s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,d)\right|^q\right.$$

$$+(s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right|^q+(s+1)^2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q\Bigg\}^{\frac{1}{q}}$$
$$+\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\left\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+(s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|^q\right.$$
$$+(s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,y)\right|^q+(s+1)^2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q\Bigg\}^{\frac{1}{q}}$$
$$+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\left\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+(s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,d)\right|^q\right.$$
$$\left.+(s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,y)\right|^q+(s+1)^2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\Bigg\}^{\frac{1}{q}}\right\}.$$

Proof. From Lemma 1.1, we have

$$\left|\frac{1}{(b-a)(d-c)}\left[A-(x-a)\int_c^d f(a,v)\,dv-(b-x)\int_c^d f(b,v)\,dv\right.\right.$$
$$\left.\left.-(d-y)\int_a^b f(u,d)\,du-(y-c)\int_a^b f(u,c)\,du+\int_a^b\int_c^d f(u,v)\,dudv\right]\right|$$
$$\leq\frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\int_0^1\int_0^1|(t-1)(\lambda-1)|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)a,\lambda y+(1-\lambda)c)\right|d\lambda dt$$
$$+\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}\int_0^1\int_0^1|(t-1)(1-\lambda)|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)a,\lambda y+(1-\lambda)d)\right|d\lambda dt$$
$$+\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\int_0^1\int_0^1|(1-t)(\lambda-1)|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)b,\lambda y+(1-\lambda)c)\right|d\lambda dt$$
$$+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\int_0^1\int_0^1|(1-t)(1-\lambda)|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)b,\lambda y+(1-\lambda)d)\right|d\lambda dt.$$

By using the well-known power mean inequality for double integrals, $f:\Delta\to\mathbb{R}$ is coordinated s-convex in the second sense on Δ, then one has:

$$\left|\frac{1}{(b-a)(d-c)}\left[A-(x-a)\int_c^d f(a,v)\,dv-(b-x)\int_c^d f(b,v)\,dv\right.\right. \tag{2.15}$$

$$
\begin{aligned}
& -(d-y)\int_a^b f(u,d)\,du-(y-c)\int_a^b f(u,c)\,du+\int_a^b\int_c^d f(u,v)\,dudv\Bigg]\Bigg] \\
\leq\ & \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1|(t-1)(\lambda-1)|\,d\lambda dt\right)^{1-\frac{1}{q}}\times \\
& \left(\int_0^1\int_0^1|(t-1)(\lambda-1)|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)a,\lambda y+(1-\lambda)c)\right|^q d\lambda dt\right)^{\frac{1}{q}} \\
& +\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1|(t-1)(1-\lambda)|\,d\lambda dt\right)^{1-\frac{1}{q}}\times \\
& \left(\int_0^1\int_0^1|(t-1)(1-\lambda)|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)a,\lambda y+(1-\lambda)d)\right|^q d\lambda dt\right)^{\frac{1}{q}} \\
& +\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1|(1-t)(\lambda-1)|\,d\lambda dt\right)^{1-\frac{1}{q}}\times \\
& \left(\int_0^1\int_0^1|(1-t)(\lambda-1)|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)b,\lambda y+(1-\lambda)c)\right|^q d\lambda dt\right)^{\frac{1}{q}} \\
& +\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1|(1-t)(1-\lambda)|\,d\lambda dt\right)^{1-\frac{1}{q}}\times \\
& \left(\int_0^1\int_0^1|(1-t)(1-\lambda)|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)b,\lambda y+(1-\lambda)d)\right|^q d\lambda dt\right)^{\frac{1}{q}}
\end{aligned}
$$

Since $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q$ is $s-$convex function in the second sense on the co-ordinates on Δ, for some fixed $s\in(0,1]$, we know that for $t\in[0,1]$

$$
\begin{aligned}
& \left|\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)a,\lambda y+(1-\lambda)c)\right|^q \\
\leq\ & t^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,\lambda y+(1-\lambda)c)\right|^q+(1-t)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,\lambda y+(1-\lambda)c)\right|^q
\end{aligned}
$$

and

$$\begin{aligned}
&\left|\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)a,\lambda y+(1-\lambda)c)\right|^q\\
\leq\ & t^s\lambda^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+t^s(1-\lambda)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|^q\\
&+(1-t)^s\lambda^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right|^q+(1-t)^s(1-\lambda)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q
\end{aligned}$$

hence, it follows that

(2.16)

$$\begin{aligned}
&\left(\int_0^1\int_0^1|(t-1)(\lambda-1)|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)a,\lambda y+(1-\lambda)c)\right|^q d\lambda dt\right)^{\frac{1}{q}}\\
&\leq\left(\int_0^1\int_0^1\left\{|(t-1)(\lambda-1)|\,t^s\lambda^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+|(t-1)(\lambda-1)|\,t^s(1-\lambda)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|^q\right.\right.\\
&+|(t-1)(\lambda-1)|(1-t)^s\lambda^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right|^q\\
&\left.\left.+|(t-1)(\lambda-1)|(1-t)^s(1-\lambda)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q\right\}dtd\lambda\right)^{\frac{1}{q}}\\
&=\left\{\frac{1}{(s+1)^2(s+2)^2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+\frac{1}{(s+1)(s+2)^2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|^q\right.\\
&\left.+\frac{1}{(s+1)(s+2)^2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right|^q+\frac{1}{(s+2)^2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q\right\}^{\frac{1}{q}}\\
&=\frac{1}{(s+1)^{\frac{2}{q}}(s+2)^{\frac{2}{q}}}\left\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+(s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|^q\right.\\
&\left.+(s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right|^q+(s+1)^2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q\right\}^{\frac{1}{q}}
\end{aligned}$$

A similar way for other integral, since $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q$ is co-ordinated $s-$convex function in the second sense on Δ, we get

$$\left(\int_0^1\int_0^1|(t-1)(1-\lambda)|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)a,\lambda y+(1-\lambda)d)\right|^q dsdt\right)^{\frac{1}{q}} \tag{2.17}$$

$$\leq \frac{1}{(s+1)^{\frac{2}{q}}(s+2)^{\frac{2}{q}}}\left\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+(s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,d)\right|^q + (s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right|^q+(s+1)^2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q\right\}^{\frac{1}{q}},$$

(2.18)
$$\left(\int_0^1\int_0^1 |(1-t)(\lambda-1)|\left|\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)b,\lambda y+(1-\lambda)c)\right|^q dsdt\right)^{\frac{1}{q}}$$
$$\leq \frac{1}{(s+1)^{\frac{2}{q}}(s+2)^{\frac{2}{q}}}\left\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+(s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|^q + (s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,y)\right|^q+(s+1)^2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q\right\}^{\frac{1}{q}},$$

(2.19)
$$\left(\int_0^1\int_0^1 |(1-t)(1-\lambda)|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(tx+(1-t)b,\lambda y+(1-\lambda)d)\right|^q dsdt\right)^{\frac{1}{q}}$$
$$\leq \frac{1}{(s+1)^{\frac{2}{q}}(s+2)^{\frac{2}{q}}}\left\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+(s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,d)\right|^q + (s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,y)\right|^q+(s+1)^2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right\}^{\frac{1}{q}}.$$

By the (2.16)-(2.19), we get the inequality (2.14). □

Corollary 2.4. *In Theorem 2.3,*

(1) if we choose $x=a$, $y=c$, or $x=b$, $y=d$, we obtain the following inequality;

(2.20)
$$\frac{1}{(b-a)(d-c)}\left|f(b,d)-(b-a)\int_c^d f(b,v)\,dv-(d-c)\int_a^b f(u,d)\,du + \int_a^b\int_c^d f(u,v)\,dudv\right|$$
$$\leq \frac{2^{2-\frac{2}{q}}(b-a)(d-c)}{(s+1)^{\frac{2}{q}}(s+2)^{\frac{2}{q}}}\left\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+(s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q + (s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+(s+1)^2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right\}^{\frac{1}{q}}.$$

(2) if we choose $x = b$, $y = d$, we obtain the following inequality;

(2.21)

$$\frac{1}{(b-a)(d-c)}\left| f(a,c)-(b-a)\int_c^d f(a,v)\,dv-(d-c)\int_a^b f(u,c)\,du + \int_a^b\int_c^d f(u,v)\,dudv\right|$$

$$\leq \frac{2^{2-\frac{2}{q}}(b-a)(d-c)}{(s+1)^{\frac{2}{q}}(s+2)^{\frac{2}{q}}}\left\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q+(s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q + (s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q+(s+1)^2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q\right\}^{\frac{1}{q}}.$$

(3) if we choose $x = a$, $y = d$, we obtain the following inequality;

$$\frac{1}{(b-a)(d-c)}\left| f(b,c)-(b-a)\int_c^d f(b,v)\,dv-(d-c)\int_a^b f(u,c)\,du + \int_a^b\int_c^d f(u,v)\,dudv\right|$$

$$\leq \frac{2^{2-\frac{2}{q}}(b-a)(d-c)}{(s+1)^{\frac{2}{q}}(s+2)^{\frac{2}{q}}}\left\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q+(s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q + (s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q+(s+1)^2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q\right\}^{\frac{1}{q}}. \tag{2.22}$$

(4) if we choose $x = b$, $y = c$, we obtain the following inequality;

$$\frac{1}{(b-a)(d-c)}\left| f(a,d)-(b-a)\int_c^d f(a,v)\,dv-(d-c)\int_a^b f(u,d)\,du + \int_a^b\int_c^d f(u,v)\,dudv\right|$$

$$\leq \frac{2^{2-\frac{2}{q}}(b-a)(d-c)}{(s+1)^{\frac{2}{q}}(s+2)^{\frac{2}{q}}}\left\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+(s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q + (s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+(s+1)^2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q\right\}^{\frac{1}{q}}. \tag{2.23}$$

(5) if we choose $x=\frac{a+b}{2}$, $y=\frac{c+d}{2}$, *we obtain the following inequality;*

$$\begin{aligned}
&\Bigg|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4(b-a)(d-c)}-\frac{1}{2(d-c)}\int_c^d f(a,v)\,dv\\
&\quad-\frac{1}{2(d-c)}\int_c^d f(b,v)\,dv\\
&\quad-\frac{1}{2(b-a)}\int_a^b f(u,d)\,du-\frac{1}{2(b-a)}\int_a^b f(u,c)\,du\\
&\quad+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dudv\Bigg|\\
\leq\ &\frac{(b-a)(d-c)}{4(2(s+1)(s+2))^{\frac{2}{q}}}\times\\
&\Bigg\{\Bigg(\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+(s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{a+b}{2},c\right)\right|^q\\
&+(s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{c+d}{2}\right)\right|^q+(s+1)^2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q\Bigg)^{\frac{1}{q}}\\
&+\Bigg(\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+(s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{a+b}{2},d\right)\right|^q\\
&+(s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{c+d}{2}\right)\right|^q+(s+1)^2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q\Bigg)^{\frac{1}{q}}\\
&+\Bigg(\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+(s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{a+b}{2},c\right)\right|^q\\
&+(s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{c+d}{2}\right)\right|^q+(s+1)^2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q\Bigg)^{\frac{1}{q}}\\
&+\Bigg[\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+(s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{a+b}{2},d\right)\right|^q\\
&+(s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{c+d}{2}\right)\right|^q+(s+1)^2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\Bigg)^{\frac{1}{q}}\Bigg\}.
\end{aligned}$$

Remark 2.3. From sum of (2.20)-(2.23), we get;

$$\left|f(a,c)-(b-a)\int_c^d f(a,v)\,dv-(d-c)\int_a^b f(u,c)\,du+\int_a^b\int_c^d f(u,v)\,dudv\right|$$

$$+\left|f(a,d)-(b-a)\int_c^d f(a,v)\,dv-(d-c)\int_a^b f(u,d)\,du+\int_a^b\int_c^d f(u,v)\,dudv\right|$$

$$+\left|f(b,c)-(b-a)\int_c^d f(b,v)\,dv-(d-c)\int_a^b f(u,c)\,du+\int_a^b\int_c^d f(u,v)\,dudv\right|$$

$$+\left|f(b,d)-(b-a)\int_c^d f(b,v)\,dv-(d-c)\int_a^b f(u,d)\,du+\int_a^b\int_c^d f(u,v)\,dudv\right|$$

$$\leq \frac{4(b-a)^2(d-c)^2}{(2(s+1)(s+2))^{\frac{2}{q}}}\times$$

$$\left\{\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+(s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q+(s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+(s+1)^2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right)^{\frac{1}{q}}\right.$$

$$+\left((s+1)^2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+(s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q+(s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right)^{\frac{1}{q}}$$

$$+\left((s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q+(s+1)^2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+(s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right)^{\frac{1}{q}}$$

$$\left.+\left((s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+(s+1)^2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+(s+1)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right)^{\frac{1}{q}}\right\}.$$

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M. Emin Özdemir
Ataturk University
K.K. Education Faculty
Department of Mathematics
25240, Erzurum, Turkey
emos@atauni.edu.tr

Mevlüt Tunç
University of Kilis 7 Aralık
Faculty of Science and Arts
Department of Mathematics
79000, Kilis, Turkey
mevluttunc@kilis.edu.tr
*Corresponding Author

Ahmet Ocak Akdemir
Ibrahim Çeçen University
Faculty of Science and Arts
Department of Mathematics
04100, Ağrı,Turkey
ahmetakdemir@agri.edu.tr