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ON ESTIMATES FOR THE DUNKL TRANSFORM IN THE SPACE $L^2(\mathbb{R}^d, w_k(x)dx)$

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Abstract. Two useful estimates are proved for the Dunkl transform in $L^2(\mathbb{R}^d, w_k(x)dx)$, where w_k is a weight function invariant under the action of an associated reflection group, on certain classes of functions characterized by the generalized continuity modulus.

1. Introduction and preliminaries

In [2], Abilov et al. proved two useful estimates for the Fourier transform in the space of square integrable functions on certain classes of functions characterized by the generalized continuity modulus, using a translation operator .
In this paper, we prove two useful estimates in certain classes of functions characterized by a generalized continuity modulus and connected with the Dunkl transform in the space $L^2(\mathbb{R}^d, w_k(x)dx)$ analogs of the statements proved in [2]. For this purpose, we use a generalized spherical mean operator.

We consider the Dunkl operators D_i; $1 \leq i \leq d$, on $\mathbb{R}^d$, which are the differential-difference operators introduced by C.F. Dunkl in [4]. These operators are very important in pure mathematics and in physics. The theory of Dunkl operators provides generalizations of various multivariable analytic structures, among others we cite the exponential function, the Fourier transform and the translation operator. For more details about these operators see [3, 4, 5]. The Dunkl Kernel E_k has been introduced by C.F.Dunkl in [6]. This Kernel is used to define the Dunkl transform

Let R be a root system in $\mathbb{R}^d$, W the corresponding reflection group, R_+ a positive subsystem of R (see[3, 5, 7, 9, 10]) and k a non-negative and W-invariant function defined on R.

The Dunkl operators is defined for $f \in C^1(\mathbb{R}^d)$ by

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$$\mathrm{D}_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in \mathrm{R}_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle},\ x \in \mathbb{R}^d\ (1 \le j \le d).$$

Here $\langle , \rangle$ is the usual Euclidean scalar product on $\mathbb{R}^d$ with the associated norm $|.|$ and σ_α the reflection with respect to the hyperplane H_α orthogonal to α, and

$$\alpha_j = \langle \alpha, e_j\rangle, (e_1, e_2, ..., e_d)$$

being the canonical basis of $\mathbb{R}^d$.

The weight function w_k defined by

$$w_k(x) = \prod_{\zeta \in \mathrm{R}_+} |\langle \zeta, x\rangle|^{2k(\alpha)},\ x \in \mathbb{R}^d,$$

where w_k is W-invariant and homogeneous of degree 2γ where

$$\gamma = \gamma(\mathrm{R}) = \sum_{\zeta \in \mathrm{R}_+} k(\zeta) \ge 0.$$

The Dunkl Kernel E_k on $\mathbb{R}^d \times \mathbb{R}^d$ has been introduced by C.F. Dunkl in [6]. For $y \in \mathbb{R}^d$ the function $x \mapsto E_k(x, y)$ is the unique solution on $\mathbb{R}^d$ of

$$\begin{cases} \mathrm{D}_j u(x, y) = y_j u(x, y) & \text{for} \quad 1 \le j \le d \\ u(0, y) = 1 & \text{for all} \quad y \in \mathbb{R}^d \end{cases}$$

E_k is called the Dunkl Kernel.

Proposition 1.1. *[3]*
Let $z, w \in \mathbb{C}$ and $\lambda \in \mathbb{C}$. Then

1. $E_k(z, 0) = 1.$
2. $E_k(z, w) = E_k(w, z).$
3. $E_k(\lambda z, w) = E_k(z, \lambda w).$
4. *For all $\nu = (\nu_1, ..., \nu_d) \in \mathbb{N}^d,\ x \in \mathbb{R}^d,\ z \in \mathbb{C}^d$, we have*

$$|\partial_z^\nu E_k(x, z)| \le |x|^{|\nu|} exp(|x||Re(z)|),$$

where

$$\partial_z^\nu = \frac{\partial^{|\nu|}}{\partial z_1^{\nu_1} ... \partial z_d^{\nu_d}},\ |\nu| = \nu_1 + ... + \nu_d.$$

In particular

$$|\partial_z^{\nu} E_k(ix,z)| \leq |x|^{\nu},$$

for all $x, z \in \mathbb{R}^d$.

The Dunkl transform is defined for

$$f \in L^1_k(\mathbb{R}^d) = L^1(\mathbb{R}^d, w_k(x)dx)$$

by

$$\widehat{f}(\xi) = c_k^{-1} \int_{\mathbb{R}^d} f(x) E_k(-i\xi, x) w_k(x) dx,$$

where the constant c_k is given by

$$c_k = \int_{\mathbb{R}^d} e^{\frac{-|z|^2}{2}} w_k(z) dz.$$

The Dunkl transform shares several properties with its counterpart in the classical case, we mention here in particular that Parseval Theorem holds in

$$L^2_k = L^2_k(\mathbb{R}^d) = L^2_k(\mathbb{R}^d, w_k(x)dx),$$

when both f and $\widehat{f}$ are in $L^1_k(\mathbb{R}^d)$, we have the inversion formula

$$f(x) = \int_{\mathbb{R}^d} \widehat{f}(\xi) E_k(ix, \xi) w_k(\xi) d\xi, \; x \in \mathbb{R}^d.$$

The Dunkl Laplacian Δ_k is defined by

$$\Delta_k = \sum_{i=1}^{d} D_i^2.$$

In $L^2_k(\mathbb{R}^d)$, consider the generalized spherical mean operator defined in [8] by

$$M_h f(x) = \frac{1}{d_k} \int_{\mathbb{S}^{d-1}} \tau_x(f)(hy) d\eta_k(y), \quad (x \in \mathbb{R}^d, h > 0)$$

where τ_x Dunkl translation operator (see [10, 11]), η is the normalized surface measure on the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$ and set $d\eta_k(y) = w_k(x)d\eta(y)$, η_k is a W-invariant measure on S^{d-1} and $d_k = \eta_k(S^{d-1})$.

We see that $M_h f \in L^2_k(\mathbb{R}^d)$ whenever $f \in L^2_k(\mathbb{R}^d)$ and

$$\|\mathrm{M}_h f\|_{\mathrm{L}^2_k} \leq \|f\|_{\mathrm{L}^2_k},$$

for all $h > 0$.

For $p \geq -\frac{1}{2}$, we introduce the normalized Bessel function of the first kind j_p defined by

$$j_p(z) = \Gamma(p+1)\sum_{j=0}^{\infty}\frac{(-1)^j(\frac{z}{2})^{2j}}{j!\Gamma(j+p+1)},\ z \in \mathbb{C}.$$

The first and higher order finite differences of $f(x)$ are defined as follows

$$\mathrm{Z}_h f(x) = (\mathrm{M}_h - \mathrm{I})f(x),$$

where I is the identity operator in $\mathrm{L}^2_k(\mathbb{R}^d)$.

$$\mathrm{Z}_h^m f(x) = \mathrm{Z}_h(\mathrm{Z}_h^{m-1}f(x)) = (\mathrm{M}_h - \mathrm{I})^m f(x) = \sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}\mathrm{M}_h^i f(x),$$

where

$$\mathrm{M}_h^0 f(x) = f(x),\ \ \mathrm{M}_h^i f(x) = \mathrm{M}_h(\mathrm{M}_h^{i-1}f(x))$$

for $i = 1, 2, ..., m$ and $m = 1, 2, ...$

The m^{th} order generalized modulus of continuity of function $f \in \mathrm{L}^2_k(\mathbb{R}^d)$ is defined as

$$\Omega_m(f,\delta) = \sup_{0<h\leq\delta}\|\mathrm{Z}_h^m f(x)\|_{L^2_k}.$$

Let $\mathrm{W}^{r,m}_{2,\phi}(\Delta_k)$ denote the class of functions $f \in \mathrm{L}^2_k(\mathbb{R}^d)$ that have generalized derivatives satisfying the estimate

$$\Omega_m(\Delta_k^r f,\delta) = O(\phi(\delta^m)),\ \delta \longrightarrow 0$$

i.e:

$$\mathrm{W}^{r,m}_{2,\phi}(\Delta_k) = \{f \in L^2_k(\mathbb{R}^d)\ /\ \Delta_k^r f \in L^2_k(\mathbb{R}^d) \qquad \text{and} \quad \Omega_m(\Delta_k^r f,\delta) = O(\phi(\delta^m)),\ \delta \longrightarrow 0\},$$

where $\phi(t)$ is any nonnegative function given on $[0,\infty)$. For the Dunkl Laplacian Δ_k, we have $\Delta_k^0 f = f$, $\Delta_k^r f = \Delta_k(\Delta_k^{r-1}f), r = 1, 2, ...$

In view in [8]:

$$(\widehat{\mathrm{M}_h f})(\xi) = j_{\gamma+\frac{d}{2}-1}(h|\xi|)\widehat{f}(\xi)$$

i.e

$$M_h f(x) = \int_{\mathbb{R}^d} j_{\gamma+\frac{d}{2}-1}(h|\xi|)\widehat{f}(\xi)E_k(ix,\xi)w_k(\xi)d\xi$$

and

$$f(x) = \int_{\mathbb{R}^d} \widehat{f}(\xi)E_k(ix,\xi)w_k(\xi)d\xi.$$

We have

$$M_h f(x) - f(x) = \int_{\mathbb{R}^d} (j_{\gamma+\frac{d}{2}-1}(h|\xi|) - 1)\widehat{f}(\xi)E_k(ix,\xi)w_k(\xi)d\xi. \tag{1.1}$$

Invoking Parseval's identity (1.1) gives

$$\|M_h f(x) - f(x)\|^2_{L^2_k} = \int_{\mathbb{R}^d} |j_{\gamma+\frac{d}{2}-1}(h|\xi|) - 1|^2|\widehat{f}(\xi)|^2 w_k(\xi)d\xi.$$

In view [3, 5] we can write

$$\widehat{(D_j f)}(y) = iy_j\widehat{f}(y);\ j = 1,\dots,d;\ y \in \mathbb{R}^d. \tag{1.2}$$

From formula (1.2), we obtain

$$\widehat{Z^m_h \Delta^r_k} f(\xi) = |\xi|^{2r}(j_{\gamma+\frac{d}{2}-1}(h|\xi|) - 1)^m\widehat{f}(\xi).$$

By Parseval's identity we have

$$\|Z^m_h \Delta^r_k f(x)\|^2_{L^2_k} = \int_{\mathbb{R}^d} |\xi|^{4r}|j_{\gamma+\frac{d}{2}-1}(h|\xi|) - 1|^{2m}|\widehat{f}(\xi)|^2 w_k(\xi)d\xi. \tag{1.3}$$

2. Estimates for the Dunkl transform

Theorem 2.1. *Given k,ϕ,r,m and $f \in W^{r,m}_{2,\phi}(\Delta_k)$, then there exists constants $C,C' > 0$ such that the following inequality holds, for all $N > 0$*

$$\int_{|\xi|\geq N} |\widehat{f}(\xi)|^2 w_k(\xi)d\xi \leq CN^{-4r}(\phi(C'N^{-m}))^2.$$

Proof. In the terms of $j_p(x)$, we have (see [1])

$$1 - j_p(x) = O(1), \ x \geq 1., \tag{2.1}$$

$$1 - j_p(x) = O(x^2), \ 0 \leq x \leq 1., \tag{2.2}$$

$$\sqrt{hx} J_p(hx) = O(1), \ hx \geq 0. \tag{2.3}$$

where $J_p(x)$ is Bessel function of the first kind, which is related to $J_p(x)$ by the formula

$$j_p(x) = \frac{2^p \Gamma(p+1)}{x^p} J_p(x), \qquad \text{for all} \quad x \in \mathbb{R}^+. \tag{2.4}$$

For a given $f \in \mathrm{W}^{r,m}_{2,\phi}(\Delta_k)$ and $N > 0$ let $\mathrm{I}(N) = \int_{|\xi| \geq N} d\mu(\xi)$ where $d\mu = d\mu(\xi) = |\widehat{f}(\xi)|^2 w_k(\xi) d\xi$.

In the rest of the proof all integrals are over the $|\xi| \geq N$.

Firstly, we have

$$\mathrm{I}(N) \leq \int |j| d\mu + \int |1 - j| d\mu. \tag{2.5}$$

with $j = j_p(h|\xi|)$ and $p = \gamma + \frac{d}{2} - 1$. The parameter $h > 0$ will be chosen in an instant. In view of formulas (2.3) and (2.4), there exist a constant $C_1 > 0$ such that

$$|j| \leq C_1 (h|\xi|)^{-p-1/2}.$$

Then

$$\int |j| d\mu \leq C_1 (hN)^{-p-1/2} \mathrm{I}(N).$$

Choose a constant C_2 such that the number $C_3 = 1 - C_1 C_2^{-p-1/2}$ is positif.

Setting $h = C_2/N$ in the inequality (2.5), we have

$$C_3 \mathrm{I}(N) \leq \int |1 - j| d\mu. \tag{2.6}$$

By Hölder inequality the second term in (2.6) satisfies

$$\begin{aligned}\int |1-j|d\mu = \int |1-j|.1.d\mu &\leq \left(\int |1-j|^{2m}d\mu\right)^{1/2m}\left(\int d\mu\right)^{1-1/2m}\\ &\leq \left(\int |\xi|^{-4r}|1-j|^{2m}|\xi|^{4r}d\mu\right)^{1/2m} \mathrm{I}(N)^{1-1/2m}\\ &\leq N^{-2r/m}\left(\int |1-j|^{2m}|\xi|^{4r}d\mu\right)^{1/2m} \mathrm{I}(N)^{1-1/2m}\end{aligned}$$

In view of (1.3), we conclude that

$$\int |1-j|^{2m}|\xi|^{4r}d\mu \leq \|\mathrm{Z}_h^m \Delta_k^r f(x)\|^2_{\mathrm{L}^2_k}.$$

Therefore

$$\int |1-j|d\mu \leq N^{-2r/m}\|\mathrm{Z}_h^m \Delta_k^r f(x)\|^2_{\mathrm{L}^2_k}\mathrm{I}(N)^{1-1/2m}$$

For $f \in \mathrm{W}^{r,m}_{2,\phi}(\Delta_k)$ there exist a constant $C > 0$ such that

$$\|\mathrm{Z}_h^m \Delta_k^r f(x)\|^2_{\mathrm{L}^2_k} \leq C(\phi(h^m))^2.$$

For $h = C_2/N$, we obtain

$$C_3\mathrm{I}(N) \leq N^{-2r/m}(C\phi((C_2/N)^m)^{1/m}\mathrm{I}(N)^{1-1/2m}.$$

By raising both sides to the power $2m$ and simplifying by $\mathrm{I}(N)^{2m-1}$ we finally obtain

$$C_3^{2m}\mathrm{I}(N) \leq N^{-4r}(C\phi((C_2/N)^m)^2,$$

for all $N > 0$. The theorem is proved with $C' = C_2^m$.

□

Theorem 2.2. *Let $\phi(t) = t^{\nu}$, then*

$$\left(\int_{|\xi|\geq N} |\widehat{f(\xi)}|^2 w_k(\xi)d\xi\right)^{\frac{1}{2}} = O(N^{-2r-m\nu}) \Longleftrightarrow f \in \mathrm{W}^{r,m}_{2,\phi}(\Delta_k),$$

where, $r = 0, 1, ...; m = 1, 2, ...; 0 < \nu < 2.$

Proof. We prove sufficiency by using Theorem 2.1 let $f \in W^{r,m}_{2,t^\nu}(\Delta_k)$ then

$$\left(\int_{|\xi|\geq N} |\widehat{f}(\xi)|^2 w_k(\xi) d\xi\right)^{\frac{1}{2}} = O(N^{-2r-m\nu}).$$

To prove necessity let

$$\left(\int_{|\xi|\geq N} |\widehat{f}(\xi)|^2 w_k(\xi) d\xi\right)^{\frac{1}{2}} = O(N^{-2r-m\nu})$$

i.e

$$\int_{|\xi|\geq N} |\widehat{f}(\xi)|^2 w_k(\xi) d\xi = O(N^{-4r-2m\nu}).$$

It is easy to show, that there exists a function $f \in L^2_k(\mathbb{R}^d)$ such that $\Delta^r_k f \in L^2_k(\mathbb{R}^d)$ and

$$\Delta^r_k f(x) = \frac{(-1)^r}{c_k}\int_{\mathbb{R}^d} |\xi|^{2r} \widehat{f}(\xi) E_k(i\xi, x) w_k(\xi) d\xi. \tag{2.7}$$

From formula (2.7) and Parseval's identity, we have

$$\|Z^m_h \Delta^r_k f(x)\|^2_{L^2_k} = \int_{\mathbb{R}^d} (1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|))^{2m} |\xi|^{4r} |\widehat{f}(\xi)|^2 w_k(\xi) d\xi.$$

This integral is divided into two:

$$\int_{\mathbb{R}^d} = \int_{|\xi|<N} + \int_{|\xi|\geq N} = I_1 + I_2,$$

where $N = [h^{-1}]$. We estimate them separately.

$$I_2 = \int_{|\xi|\geq N} |\xi|^{4r} (1 - j)^{2m} d\mu(\xi),$$

with $j = j_{\gamma+\frac{d}{2}-1}(h|\xi|)$ and $d\mu(\xi) = |\widehat{f}(\xi)|^2 w_k(\xi) d\xi$.

From (2.1) and (2.2), we have the estimate

$$\begin{aligned} I_2 &\leq C\int_{|\xi|\geq N} |\xi|^{4r} d\mu(\xi) = C\sum_{l=0}^{\infty}\int_{N+l\leq|\xi|\leq N+l+l} |\xi|^{4r} d\mu(\xi) \\ &\leq C\sum_{l=0}^{\infty} a_l (u_l - u_{l+1}), \end{aligned}$$

with $a_l = (N + l + 1)^{4r}$ and $u_l = \int_{|\xi|\geq N+l} d\mu(\xi)$

For all integers $n \geq 1$, the Abel transformation shows

$$\begin{aligned}\sum_{l=0}^{n} a_l(u_l - u_{l+1}) &= a_0u_0 + \sum_{l=1}^{n}(a_l - a_{l-1})u_l - a_nu_{n+1} \\ &\leq a_0u_0 + \sum_{l=1}^{n}(a_l - a_{l-1})u_l\end{aligned}$$

because $a_nu_{n+1} \geq 0$. Moreover by the finite increments theorem, we have $a_l - a_{l-1} \leq 4r(N + l + 1)^{4r-1}$. Furthermore by the hypothesis of f there exists $C' > 0$ such that, for all $N > 0$,

$$\int_{|\xi|\geq N} d\mu(\xi) \leq C'N^{-4r-2m\nu}.$$

For $N \geq 1$, we have

$$\begin{aligned}\sum_{l=0}^{n}(a_l - a_{l-1})u_l &\leq C'\left(1 + \frac{1}{N}\right)^{4r} N^{-2m\nu} + 4rC'\sum_{l=1}^{n}\left(1 + \frac{1}{N+l}\right)^{4r-1}(N + l)^{-1-2m\nu} \\ &\leq 2^{4r}C'N^{-2m\nu} + 4r2^{4r-1}C'\sum_{l=1}^{n}(N + l)^{-1-2m\nu}.\end{aligned}$$

Finally, by the integral comparison test we have

$$\sum_{l=1}^{n}(N + l)^{-1-2m\nu} \leq \int_{N}^{\infty} x^{-1-2m\nu}dx = \frac{1}{2m\nu}N^{-2m\nu}.$$

Letting $n \longrightarrow \infty$ we see that, for $r \geq 0$ and $m, \nu > 0$, there exists a constant $C'' > 0$ such that, for all $N \geq 1$ and for $h > 0$,

$$\mathrm{I}_2 \leq C''N^{-2m\nu}$$

Now, we estimate $\mathrm{I}_1 = \int_{|\xi|<N} |\xi|^{4r}(1 - j)^{2m}d\mu(\xi)$. For $h|\xi| \leq 1$ we use

$$|1 - j| \leq C_1{}'h^2|\xi|^2.$$

Then, we have

$$\begin{aligned} \mathrm{I}_1 &\leq C_1^{'} h^{4m} \int_{|\xi|<N} |\xi|^{4m+4r} d\mu(\xi) = C_1^{'} h^{4m} \sum_{l=0}^{N-1} \int_{l\leq|\xi|\leq l+1} |\xi|^{4m+4r} d\mu(\xi) \\ &\leq C_1^{'} h^{4m} \sum_{l=0}^{N-1} (l+1)^{4r+4m} (v_l - v_{l+1}) \end{aligned}$$

with $v_l = \int_{|\xi|\geq l} d\mu(\xi)$.
Using an Abel transformation and proceeding as with I_2 we obtain

$$\begin{aligned} \mathrm{I}_1 &\leq C_1^{'} h^{4m} \left(v_0 + \sum_{l=1}^{N-1} ((l+1)^{4r+4m} - l^{4r+4m}) v_l \right) \\ &\leq C_1^{'} h^{4m} \left(v_0 + (4r+4m) C_2' \sum_{l=1}^{N-1} (l+1)^{4r+4m-1} l^{-4r-2m\nu}) \right) \end{aligned}$$

since $v_l \leq C_2' l^{-4r-2m\nu}$ by hypothesis. From the inequality $l+1 \leq 2l$ we conclude

$$\mathrm{I}_1 \leq C_1^{'} h^{4m} \left(v_0 + C_2' \sum_{l=1}^{N-1} l^{4m-2m\nu-1} \right).$$

As a consequence of a series comparison for $\alpha \geq 1$ and $\alpha < 1$ we have the inequality,

$$\alpha \sum_{l=1}^{N-1} l^{\alpha-1} < N^{\alpha}, \ for\, \alpha > 0 \ and\ N \geq 2.$$

If $\alpha = 4m - 2m\nu > 0$ for $\nu < 2$ then we obtain

$$\mathrm{I}_1 \leq C_1^{'} h^{4m} \left(v_0 + C^{'''} N^{4m-2m\nu} \right) \leq C_1^{'} h^{4m} \left(v_0 + C^{'''} h^{-4m+2m\nu} \right)$$

since $N \leq 1/h$. If h is sufficiently small then $v_0 \leq C^{'''} h^{-4m+2m\nu}$. Then we have

$$\mathrm{I}_1 \leq C_4 h^{2m\nu}.$$

Combining the estimates for I_1 and I_2 gives

$$\|\mathrm{Z}_h^m \Delta_k^r f(x)\|_{\mathrm{L}_k^2} = O(h^{m\nu}).$$

The necessity is proved. □

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