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COMPACT OPERATORS ON QUATERNIONIC HILBERT SPACES

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Abstract. In this paper, compact operators on quaternionic Hilbert space are introduced and some properties of this class of operators are studied.

1. Introduction and Preliminaries

The field of quaternions, which will be denoted by $\mathbb{H}$ throughout this paper, contains all elements of the form $\mathbf{q} = x_0 + x_1 i + x_2 j + x_3 k$, where x_0, x_1, x_2 and x_3 are real numbers and i, j, k are the so-called imaginary units with the following multiplication rules:

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad \text{and} \quad ki = -ik = j. \tag{1.1}$$

Therefore, we see that multiplication is not commutative in $\mathbb{H}$. The quaternionic conjugate of $\mathbf{q}$ is defined by $\overline{\mathbf{q}} = x_0 - x_1 i - x_2 j - x_3 k$, and the absolute value of $\mathbf{q}$ is $|\mathbf{q}| = \sqrt{{x_0}^2 + {x_1}^2 + {x_2}^2 + {x_3}^2}$. It is easy to see that for any two quaternions $\mathbf{q}_1$ and $\mathbf{q}_2$, $\overline{\mathbf{q}_1\mathbf{q}_2} = \overline{\mathbf{q}_2}\ \overline{\mathbf{q}_1}$. For more information about the properties of the quaternions one may see [1] and [5].

Throughout this paper H stands for a linear vector space over $\mathbb{H}$ under left scalar multiplication. If there exists a function $\langle .,. \rangle : \mathsf{H} \times \mathsf{H} \longrightarrow \mathbb{H}$, such that for every $f, g, h \in \mathsf{H}$ and $\mathbf{q} \in \mathbb{H}$ the following properties hold:

(i) $\overline{\langle f, g \rangle} = \langle g, f \rangle$,

(ii) $\langle f, f \rangle > 0$ unless $f = 0$,

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(iii) $\langle f, g+h\rangle = \langle f, g\rangle + \langle f, h\rangle$,

(iv) $\langle \mathbf{q} f, g\rangle = \mathbf{q}\langle f, g\rangle$,

then we call it an inner product. The quaternionic norm of an element $f \in \mathsf{H}$ is defined by $\|f\| = \sqrt{\langle f, f\rangle}$ which satisfies all properties of a norm especially Cauchy-Schwartz inequality (see [5], Proposition 2.2). The parallelogram law is also proved readily for this norm, i.e. for each $f, g \in \mathsf{H}$,

$$\|f+g\|^2 + \|f-g\|^2 = 2(\|f\|^2 + \|g\|^2). \tag{1.2}$$

Now, the term "left quaternionic Hilbert space" refers to the latter normed space $(\mathsf{H}, \|.\|)$, whenever it is a separable Hilbert space. Similarly, the notion of a right quaternionic Hilbert space has been defined (see [5] or [8]). We will focus on left quaternionic Hilbert spaces although it is easy to express and prove all claims for the right version.

It is well-known that a complex Hilbert space and its dual are isometrically isomorphic, this is a direct result of the Riesz representation theorem. In quaternionic case, the Riesz representation theorem is also valid (see [10] and [9]). In more details, if $h : \mathsf{H} \longrightarrow \mathbb{H}$ is a functional on a left quaternionic Hilbert space H, i.e. h is left linear, namely, for $\mathbf{p} \in \mathbb{H}$ and $x, y \in \mathsf{H}$,

$$h(\mathbf{p}x + y) = \mathbf{p}h(x) + h(y),$$

and bounded, which means $\|h\| := \sup\{|h(x)|, \|x\| = 1, x \in \mathsf{H}\}$ is finite, then there exists a unique vector $x' \in \mathsf{H}$ such that $\| h \| = \| x' \|$ and

$$h(x) = \langle x, x'\rangle. \tag{1.3}$$

We will refer to (1.3) as the Riesz representation theorem.

It is said that $T : \mathsf{H} \longrightarrow \mathsf{H}$ is a left linear operator if for all $f, g \in \mathsf{H}$ and $\mathbf{p} \in \mathbb{H}$,

$$T(\mathbf{p}f + g) = \mathbf{p}Tf + Tg.$$

Such an operator is called bounded if there exists $K \geq 0$ such that for all $f \in \mathsf{H}$,

$$\|Tf\| \leq K\|f\|.$$

As in the complex case, the norm of a bounded left linear operator T is defined by

$$\|T\| = \sup\left\{\frac{\|Tf\|}{\|f\|}, 0 \neq f \in \mathsf{H}\right\}. \tag{1.4}$$

The set of all bounded left linear operators on H is denoted by $\mathfrak{B}(\mathsf{H})$, which is a complete normed space with the norm defined by (1.4) (see [5]; Proposition 2.11, for more properties of $\mathfrak{B}(\mathsf{H})$). For every $T \in \mathfrak{B}(\mathsf{H})$, there exists a unique operator $T^* \in \mathfrak{B}(\mathsf{H})$, which is called the adjoint of T, such that, for all $f, g \in \mathbb{H}$, $\langle Tf, g\rangle = \langle f, T^*g\rangle$. Among many properties of the adjunction, stated and proved in Theorem 2.15 and Remark 2.16 of [5], we remind of $\|T\| = \|T^*\|$. Also, the following theorem of [4] will be needed later.

Theorem 1. *[4] If $T \in \mathcal{B}(V_H^L)$, is a self-adjoint operator, then*

$$\|T\| = \sup\{|\langle Tf, f\rangle|; \|f\| = 1, f \in V_H^L\}.$$

It is emphasised that, the adjoint operation is not an involution on $\mathfrak{B}(\mathsf{H})$, as the equality $(\mathbf{q}T)^* = \overline{\mathbf{q}}T^*$ holds only for $\mathbf{q} \in \mathbb{R}$ (see [10]). In the next section, we define compact operators on a left quaternionic Hilbert space and investigate some properties analogous to the compact operators on complex Hilbert spaces.

2. Compact operators on quaternionic Hilbert spaces

Similar to the complex case, we define a compact operator on a left quaternionic Hilbert space H, an operator $T : \mathsf{H} \longrightarrow \mathsf{H}$, for which $\overline{T(B)}$ is a compact set of H, where B is a bounded set of H. The set of all compact operators on H will be denoted by $\mathfrak{B}_0(\mathsf{H})$. Clearly, $\mathfrak{B}_0(\mathsf{H}) \subseteq \mathfrak{B}(\mathsf{H})$.

It is well-known that in metric spaces compactness and sequentially compactness are equivalent, therefore, we have the next lemma.

Lemma 1. $T \in \mathfrak{B}_0(\mathsf{H})$ *is a compact operator if and only if for each bounded sequence* $\{x_n\} \subseteq \mathsf{H}$, *the sequence* $\{Tx_n\}$ *has a convergent subsequence.*

Lemma 1 will be used repeatedly in the proof of Theorem 2. The following lemma is also needed to prove the same theorem and is the quaternionic version of Arzela-Ascoli theorem, whose proof is exactly the same as its complex version (see, Theorem 7.25 of [7]).

Lemma 2. *For a compact metric space (K, d), let $\{f_n\}$ be a pointwise bounded and equicontinuous sequence of quaternionic valued functions on K, then*

(i) $\{f_n\}$ is uniformly bounded on K,

(ii) $\{f_n\}$ contains a uniformly convergent subsequence.

In Lemma 2, an equicontinuous sequence of quaternionic valued functions $\{f_n\}$ on a metric space (X, d) satisfies the following condition

$$\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in X, \forall n \, (d(x, y) < \delta \Longrightarrow |f_n(x) - f_n(y)| < \epsilon) . \tag{2.1}$$

By $|.|$, in (2.1) we mean the absolute value in quaternionic sense.

Theorem 2. $\mathfrak{B}_0(\mathsf{H})$ *is a cloesd biideal of* $\mathfrak{B}(\mathsf{H})$ *and is cloesd under adjunction.*

Proof. For an arbitrary $\mathbf{q} \in \mathbb{H}$ and $T \in \mathfrak{B}_0(\mathsf{H})$, the left scalar multiplication $\mathbf{q}T$ is defined by $(\mathbf{q}T)(f) = \mathbf{q}T(f)$ and it is easily seen that $\mathbf{q}T \in \mathfrak{B}_0(\mathsf{H})$. Lemma 1 helps us to prove readily that $\mathfrak{B}_0(\mathsf{H})$ is closed under summation and composition from both left and right by a bounded left linear operator. Again, using Lemma 1 and following the proof of Theorem 8.1-5 of [6], closedness of $\mathfrak{B}_0(\mathsf{H})$ under the norm topology of $\mathfrak{B}(\mathsf{H})$ is proved. Therefore, $\mathfrak{B}_0(\mathsf{H})$ is a closed biideal of $\mathfrak{B}(\mathsf{H})$. To show that $\mathfrak{B}_0(\mathsf{H})$ is closed under adjunction, take $T \in \mathfrak{B}_0(\mathsf{H})$ and a bounded subset of H, say B. Put $M = \sup\{\|y\|, y \in B\}$, which is finite, and for a sequence $\{y_n\}$ of B, define $\theta_{y_n} : \overline{T(B)} \longrightarrow \mathbb{H}$ by $\theta_{y_n}(x) = \langle x, y_n \rangle$. For each n and $x, z \in \overline{T(B)}$, we see that

$$|\theta_{y_n}(x) - \theta_{y_n}(z)| = |\langle x - z, y_n \rangle| \leq M\|x - z\| \tag{2.2}$$

and $|\theta_{y_n}(x)| \leq M'M$, where $M' = \sup\{\|x\|, x \in \overline{T(B)}\}$. Hence, $\{\theta_{y_n}\}$ is a uniformly bounded equicontinuous sequence of functionals in $C(\overline{T(B)})$. Applying Lemma 2, we obtain a subsequence $\{\theta_{y_{n_k}}\}$ which uniformly converges to a functional $f \in C(\overline{T(B)})$. Using the Riesz representation theorem, $f = \theta_y$ for some $y \in \mathsf{H}$ and $\|\theta_y\| = \|y\|$. Now, for each $x \in B$,

$$\theta_{y_{n_k}}(Tx) = \langle Tx, y_{n_k} \rangle = \langle x, T^* y_{n_k} \rangle = \theta_{T^* y_{n_k}}(x).$$

So $\{\theta_{T^* y_{n_k}}\}$ is also a uniformly convergent sequence on B and $\|\theta_{T^* y_{n_k}}\| = \|T^* y_{n_k}\|$ implies that $\{T^* y_{n_k}\}$ is a convergent sequence, too. Finally, Lemma 1, guarantees the compactness of the adjoint operator T^* on H. $\square$

In a complex Hilbert space, the linear operator $T - \lambda I$ is the main tool for dealing with the spectral theory of bounded linear operator T. Although, for a left quaternionic Hilbert space H, if $T \in \mathfrak{B}(\mathsf{H})$ and $\mathbf{q} \in \mathbb{H}$, then $T - \mathbf{q}I$ is not even a left linear operator. In order to study in this direction, Colombo [2] and then, Ghiloni et al. [5] have provided a replacement for the operator $T - \mathbf{q}I$ and generalized equivalently the notions of spectrum and resolvent set of a left linear operator (see Section 4.7 of [2] and Section 4 of [5]). We follow the work by Ghiloni et al. [5] to study quaternionic version of some spectral properties of compact operators. First, we recall the next definition from [5].

Definition 1. *[5] Let H be a left quaternionic Hilbert space and T be a left linear operator on H. For $\boldsymbol{q} \in \mathbb{H}$, the associated operator $\Delta\boldsymbol{q}(T)$ is defined by:*

$$\Delta\boldsymbol{q}(T) = T^2 - (\boldsymbol{q} + \overline{\boldsymbol{q}})T + |\boldsymbol{q}|^2 I.$$

The spherical resolvent set of T is the set $\rho_S(T) \subset \mathbb{H}$ consisting all quaternions $\boldsymbol{q}$ satisfying all the following conditions:

(a) $\ker(\Delta\boldsymbol{q}(T)) = \{0\}$.

(b) *$Ran(\Delta\boldsymbol{q}(T))$ is dense in H.*

(c) *$\Delta\boldsymbol{q}(T)^{-1} : Ran(\Delta\boldsymbol{q}(T)) \to D(T^2)$ is bounded.*

The complement of $\rho_S(T)$ in $\mathbb{H}$ is defined to be the *spherical spectrum* $\sigma_S(T)$ of T. Ghiloni et al. [5] have introduced a partition for $\sigma_S(T)$ as follows:

(i) The *spherical point spectrum* of T:

$$\sigma_{pS} = \{\mathbf{q} \in \mathbb{H}; \ker(\Delta_{\mathbf{q}}(T)) \neq \{0\}\}.$$

(ii) The *spherical residual spectrum* of T:

$$\sigma_{rS}(T) = \{\mathbf{q} \in \mathbb{H}; \ker(\Delta_{\mathbf{q}}(T)) = \{0\}, \overline{Ran(\Delta_{\mathbf{q}}(T))} \neq \mathsf{H}\}.$$

(iii) The spherical continuous spectrum of T:

$$\sigma_{cS}(T) = \{\mathbf{q} \in \mathbb{H}; \ker(\Delta_{\mathbf{q}}(T)) = \{0\}, \overline{Ran(\Delta_{\mathbf{q}}(T))} = \mathsf{H}, \Delta_{\mathbf{q}}(T)^{-1} \notin \mathsf{H}\}.$$

The *spherical spectral radious* of T, denoted by $r_S(T)$, is defined by

$$r_S(T) = \sup\{|\mathbf{q}| \in \mathbb{R}^+; \mathbf{q} \in \sigma_S(T)\}.$$

The *eigenvector* of T with *eigenvalue* $\mathbf{q}$ is an element $u \in \mathsf{H} - \{0\}$, for which $Tu = u\mathbf{q}$.

The following proposition summerizes some properties of $\Delta_{\mathbf{q}}(T)$ that can be proved easily.

Proposition 1. *Let $T \in \mathfrak{B}(\mathsf{H})$, then*

(i) $\Delta\boldsymbol{q}(T) \in \mathfrak{B}(\mathsf{H})$.

(ii) *$(\Delta\boldsymbol{q}(T))^* = \Delta\boldsymbol{q}(T^*)$, and if T is self adjoint then so is $\Delta\boldsymbol{q}(T)$.*

(iii) If T is a normal operator, i.e. $TT^* = T^*T$, then $\Delta_{\boldsymbol{q}}(T)$ is normal, and

$$\ker \Delta_{\boldsymbol{q}}(T) = \ker \Delta_{\boldsymbol{q}}(T^*).$$

Now, we prove the quaternionic version of Proposition 4.13 of [3], which is stated in the next theorem.

Theorem 3. *If $T \in \mathfrak{B}_0(\mathsf{H})$ and $0 \neq \boldsymbol{q} \in \sigma_{pS}(T)$, then $\ker(\Delta_{\boldsymbol{q}}(T))$ is finite dimensional.*

Proof. According to Proposition 2.6 of [5], every quaternionic Hilbert space admits an orthonormal Hilbert basis. Let $\ker(\Delta_{\mathbf{q}}(T))$, as a subspace of H, contain an infinite orthonormal sequence $\{e_n\}$. Compactness of T and Lemma 1 imply the existence of a subsequence $\{e_{n_k}\}$ for which $\{Te_{n_k}\}$ is convergent. Hence, $\{Te_{n_k}\}$ is a Cauchy sequence and by Theorem 2, we conclude $\{T^2 e_{n_k}\}$ and $\{(\mathbf{q}+\overline{\mathbf{q}})T(e_{n_k})\}$ are also Cauchy sequences. But for $n_j \neq n_k$,

$$\begin{aligned} \|T^2(e_{n_k}) - (\mathbf{q}+\overline{\mathbf{q}})T(e_{n_k}) - T^2(e_{n_j}) + (\mathbf{q}+\overline{\mathbf{q}})T(e_{n_j})\|^2 &= \||\mathbf{q}|^2(e_{n_j} - e_{n_k})\|^2 \\ &= 2|\mathbf{q}|^4 > 0. \end{aligned}$$

The last equality obtained from orthonormality of $\{e_n\}$ and the parallelogram law (1.2). This contradiction shows that $\ker(\Delta_{\mathbf{q}}(T))$ must be finite dimensional. $\square$

Theorem 4. *If $T \in \mathfrak{B}_0(\mathsf{H})$ and $\inf\{\|\Delta_{\boldsymbol{q}}(T)h\|; \|h\| = 1\} = 0$, for a non-zero $\boldsymbol{q} \in \mathbb{H}$, then $\boldsymbol{q} \in \sigma_{pS}(T)$.*

Proof. To obtain the result, we follow the analogous procedure as in the proof of Proposition 4.14 of [3]. By the assumption, there is a sequence of unit members of H, say $\{h_n\}$, such that $\|\Delta_{\mathbf{q}}(T)h_n\| \to 0$. Since T is a compact operator, by Lemma 1, there is a subsequence $\{h_{n_k}\}$ and a vector $f \in \mathsf{H}$ such that $\|Th_{n_k} - f\| \to 0$. Boundedness of T implies that $\|T^2 h_{n_k} - Tf\| \to 0$ and $\|(\mathbf{q}+\overline{\mathbf{q}})Th_{n_k} - (\mathbf{q}+\overline{\mathbf{q}})f\| \to 0$. But,

$$h_{n_k} = |\mathbf{q}|^{-2}(T^2 h_{n_k} - (\mathbf{q}+\overline{\mathbf{q}})Th_{n_k} - \Delta_{\mathbf{q}}(T)h_{n_k}) \to |\mathbf{q}|^{-2}(Tf - (\mathbf{q}+\overline{\mathbf{q}})f),$$

so $\|Tf - (\mathbf{q}+\overline{\mathbf{q}})f\| = |\mathbf{q}|^2$ and hence, $f \neq 0$. Also, $Th_{n_k} \to |\mathbf{q}|^{-2}(T^2 f - (\mathbf{q}+\overline{\mathbf{q}})Tf)$. Since, $Th_{n_k} \to f$, we must have $|\mathbf{q}|^{-2}(T^2 f - (\mathbf{q}+\overline{\mathbf{q}})Tf) = f$ or $T^2 f - (\mathbf{q}+\overline{\mathbf{q}})Tf - |\mathbf{q}|^2 f = 0$, that is $0 \neq f \in \ker \Delta_{\mathbf{q}}(T)$, so $\mathbf{q} \in \sigma_{pS}(T)$. $\square$

Corollary 1. *Let $T \in \mathfrak{B}_0(\mathsf{H})$, $0 \neq \boldsymbol{q} \notin \sigma_{pS}(T)$, and $\overline{\boldsymbol{q}} \notin \sigma_{pS}(T^*)$. Then $\Delta_{\boldsymbol{q}}(T)^{-1}$ is bounded and $Ran\Delta_{\boldsymbol{q}}(T) = \mathsf{H}$.*

Proof. Since $\mathbf{q} \notin \sigma_{pS}(T)$, Theorem 4 implies that there is a constant $c > 0$ such that $\|\Delta_{\mathbf{q}}(T)h\| \geqslant c\|h\|$, for all $h \in \mathsf{H}$. If $f \in \overline{Ran\Delta_{\mathbf{q}}(T)}$, then there is a sequence $\{h_n\}$ in H such that $\Delta_{\mathbf{q}}(T)h_n \to f$. Thus,

$$\|h_n - h_m\| \leqslant c^{-1}\|\Delta_{\mathbf{q}}(T)h_n - \Delta_{\mathbf{q}}(T)h_m\|,$$

and so $\{h_n\}$ is a Cauchy sequence and converges to some $h \in \mathsf{H}$. Uniqueness of limit leads to $\Delta_{\mathbf{q}}(T)h = f$. This means that $Ran\Delta_{\mathbf{q}}(T)$ is closed in H. Now, using Proposition 2.14 of [5] and the fact that $\Delta_{\mathbf{q}}(T)^* = \Delta_{\overline{\mathbf{q}}}(T^*)$, we obtain

$$Ran\Delta_{\mathbf{q}}(T) = \left(\ker \Delta_{\overline{\mathbf{q}}}(T)^*\right)^{\perp} = \mathsf{H}.$$

For $f \in \mathsf{H}$, let Af be the unique vector $h \in \mathsf{H}$ for which $\Delta_{\mathbf{q}}(T)h = f$. Thus, $\Delta_{\mathbf{q}}(T)Af = f$ for all $f \in \mathsf{H}$, and so $c\|Af\| \leqslant \|\Delta_{\mathbf{q}}(T)Af\| = \|f\|$. This shows that A is bounded. Also, $\Delta_{\mathbf{q}}(T)A\Delta_{\mathbf{q}}(T)h = \Delta_{\mathbf{q}}(T)h$, hence $\Delta_{\mathbf{q}}(T)(A\Delta_{\mathbf{q}}(T)h - h) = 0$, since $\mathbf{q} \notin \sigma_{pS}(T)$, we must have $A\Delta_{\mathbf{q}}(T)h = h$, that is $A = \Delta_{\mathbf{q}}(T)^{-1}$. $\square$

As a consequence of Corollary 1 and Definition 1, we have the next result.

Corollary 2. *If $T \in \mathfrak{B}_0(\mathsf{H})$ and $0 \neq \boldsymbol{q} \notin \sigma_{pS}(T)$, then $\boldsymbol{q} \in \rho_S(T)$.*

It is well-known that, the norm of a compact self-adjoint operator on a complex Hilbert space, is an eigenvalue of that operator (see Lemma 5.9 of [3]). In quaternionic case, we expect the validity of this claim.

Conjecture 1. *For a non zero self-adjoint opeator $T \in \mathfrak{B}_0(\mathsf{H})$, either $\pm\|T\|$ is an eigenvalue of T.*

Remark 1. By the assumptions of Conjecture 1 and Theorem 1, a sequence $\{h_n\}$ of unit vectors of H exists such that $|\langle Th_n, h_n\rangle| \to \|T\|$. Let $|\mathbf{q}| = \|T\|$, then we can find (if necessary) a subsequence of $\{h_n\}$, shown it again by $\{h_n\}$, such that $\langle Th_n, h_n\rangle \to \mathbf{q}$. Note that since T is self-adjoint, $\langle Th_n, h_n\rangle$ and so $\mathbf{q}$ are real numbers (see Theorem 3 of [4]). Since T is self-adjoint, so is T^3. If we have the extra assumption of $\langle T^3h_n, h_n\rangle \to \mathbf{q}^3$, then taking limit from the following inequality

$$\begin{aligned} 0 \leqslant \|\Delta_{\mathbf{q}}(T)h_n\|^2 &= \|T^2h_n\|^2 + 6\mathbf{q}^2\|Th_n\|^2 + \mathbf{q}^4 - 4\mathbf{q}\langle T^3h_n, h_n\rangle - 4\mathbf{q}^3\langle Th_n, h_n\rangle \\ &\leqslant 8\mathbf{q}^4 - 4\mathbf{q}\langle T^3h_n, h_n\rangle - 4\mathbf{q}^3\langle Th_n, h_n\rangle, \end{aligned}$$

we obtain $\|\Delta_{\mathbf{q}}(T)h_n\| \to 0$. Now, Theorem 4 results $\mathbf{q} \in \sigma_{pS}(T)$ and by Proposition 4.5 of [5], we conclude that $\mathbf{q}$ is an eigenvalue of T.

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