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LEGENDRE CURVES ON THREE-DIMENSIONAL HEISENBERG GROUPS

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Abstract. The object of the present paper is to show that locally ϕ-symmetric and biharmonic Legendre curves on three-dimensional Heisenberg groups are not circles.

Keywords: Legendre curves, Heisenberg group, locally ϕ-symmetric, biharmonic, circle.

1. Introduction

In the study of contact manifolds, Legendre curves play an important role, e.g., a diffeomorphism of a contact manifold is a contact transformation if and only if it maps Legendre curves to Legendre curves. Legendre curves on contact manifolds have been studied by C. Baikoussis and D. E. Blair in the paper [1]. M Belkhelfa et al [2] have investigated Legendre curves in Riemannian and Lorentzian manifolds. In [5] slant curves, as a generalization of Legendre curves, have been studied on three-dimensional Sasakian space forms. The Heisenberg group is a Sasakian manifold with constant ϕ-sectional curvature. Heisenberg group is a unimodular Lie group with left invariant Sasakian structure.

In mathematics, the Heisenberg group, named after Werner Heisenberg, is the group of 3×3 upper triangular matrices of the form

$$M = \left\{ \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

under the operation of matrix multiplication [9]. The group becomes a Sasakian manifold of dimension three, which is a contact manifold of dimension three, with the metric $g = dx^2 + dy^2 + (dz + ydx - xdy)^2$ [5].

Sasakian manifolds have been studied by several authors [3]. The first author of the present paper has also studied Sasakian manifolds [6], [7]. The aim of the present paper is to study biharmonic Legendre curves and locally ϕ-symmetric

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Legendre curves on three-dimensional Heisenberg groups. The present paper is organized as follows:

After the introduction, we give some preliminaries in Section 2. In this section we also discuss about three-dimensional Heisenberg group. In Section 3, we study locally ϕ-symmetric Legendre curves on three-dimensional Heisenberg group. In this section it is shown that a locally ϕ-symmetric Legendre curve in three-dimensional Heisenberg group is not a circle. In Section 4, we consider biharmonic Legendre curves in three-dimensional Heisenberg group. Here we also prove that a biharmonic Legendre curve in three-dimensional Heisenberg group is not a circle.

2. Preliminaries

Let M be an almost contact metric manifold of dimension 3, that is, a $(2n+1)-$dimensional differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g). By definition, ϕ, ξ, η are tensor fields of type $(1,1)$, $(1,0)$, $(0,1)$, respectively, and g is a Riemannian metric on $\tilde{M}$ such that [3]

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

for all differentiable vector fields X, Y on M. Then also

$$\phi(\xi) = 0, \;\; \eta(\phi X) = 0, \;\; \eta(X) = g(X, \xi). \tag{2.3}$$

Let Φ be the fundamental $2-$form defined by $\Phi(X, Y) = g(X, \phi Y)$, for all differentiable vector fields X, Y on M. If the vector field ξ is a Killing vector field, then the contact manifold is called a $K-$contact manifold. In a $K-$contact manifold we always have

$$\nabla_X \xi = -\phi X, \tag{2.4}$$

for any tangent vector X of the manifold. A $K-$contact manifold is a Sasakian manifold if and only if it satisfies

$$(\nabla_X \xi) = g(X, Y)\xi - \eta(Y)X. \tag{2.5}$$

It is to be noted that every Sasakian manifold is $K-$contact, but the converse is true only for three-dimensional case.

Let M be a $3-$dimensional Riemannian manifold. Let $\gamma : I \to M$, I being an interval, be a curve in M which is parameterized by arc length, and let $\nabla_{\dot{\gamma}}$ denote the covariant differentiation along γ with respect to the Levi-Civita connection of M. We say that γ is a Frenet curve if one of the following three cases holds:

(a) γ is of osculating order 1, i.e., $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ (geodesic).

(b) γ is of osculating order 2, i.e., there exist two orthonormal vector fields $T(= \dot{\gamma})$, N and a positive function k (curvature) along γ such that $\nabla_{\dot{\gamma}} T = kN$, $\nabla_{\dot{\gamma}} N = -kT$.

(c) γ is of osculating order 3, i.e., there exist three orthonormal vectors $T(=\dot{\gamma})$, N, B and two positive functions k(curvature) and τ(torsion) along γ such that

$$\nabla_{\dot{\gamma}} T = kN, \tag{2.6}$$

$$\nabla_{\dot{\gamma}} N = -kT + \tau B, \tag{2.7}$$

$$\nabla_{\dot{\gamma}} B = -\tau N. \tag{2.8}$$

The above formulas are known as Serret-Frenet formulas. A Frenet curve for which $k =$ a positive constant and $\tau = 0$ is called a circle in M. A Frenet curve of osculating order 3 is called a helix in M if k and τ both are positive constants. The curve is called a generalized helix if $\frac{k}{\tau} =$ a constant.

A Frenet curve γ in a Riemannian manifold is said to be a Legendre curve if it is an integral curve of the contact distribution $\mathcal{D} = \ker\eta$, i.e., if $\eta(\dot{\gamma}) = 0$. For more details we refer [1], [3].

Let us consider the three-dimensional Heisenberg group

$$M = \left\{ \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

$$e_1 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}, \qquad e_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \qquad e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let $g = dx^2 + dy^2 + (dz + ydx - xdy)^2$. Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$. Let ϕ be the (1,1) tensor field defined by $\phi(e_1) = e_2$, $\phi(e_2) = -e_1$, $\phi(e_3) = 0$. Then using the linearity of ϕ and g we have

$$\eta(e_3) = 1, \qquad \phi^2 Z = -Z + \eta(Z)e_3, \qquad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W).$$

Also $d\eta(X, Y) = g(X, \phi Y)$ for any $Z, W \in \chi(M)$. Thus for $e_3 = \xi$, (ϕ, ξ, η, g) defines a contact metric structure on M. Now, by direct computations we obtain

$$[e_1, e_2] = 2e_3, \qquad [e_2, e_3] = 0, \qquad [e_1, e_3] = 0.$$

By Koszul formula

$$\begin{array}{lll} \nabla_{e_1} e_3 = -e_2, & \nabla_{e_1} e_2 = e_3, & \nabla_{e_1} e_1 = 0, \\ \nabla_{e_2} e_3 = e_1, & \nabla_{e_2} e_2 = 0, & \nabla_{e_2} e_1 = -e_3, \\ \nabla_{e_3} e_3 = 0, & \nabla_{e_3} e_2 = e_1, & \nabla_{e_3} e_1 = -e_2. \end{array}$$

From above we see that the three-dimensional manifold satisfies (2.4). Hence the manifold is a Sasakian manifold. With the help of the above results it can be verified that

$$
\begin{aligned}
&R(e_1,e_2)e_1 = 3e_2, \qquad R(e_1,e_2)e_2 = -3e_1, \qquad R(e_1,e_2)e_3 = 0,\\
&R(e_2,e_3)e_1 = 0, \qquad R(e_2,e_3)e_2 = -e_3, \qquad R(e_2,e_3)e_3 = e_2,\\
&R(e_3,e_1)e_1 = e_3, \qquad R(e_3,e_1)e_2 = 0, \qquad R(e_3,e_1)e_3 = -e_1.
\end{aligned} \tag{2.9}
$$

3. Locally ϕ-symmetric Legendre curve on Heisenberg groups

In this section we study locally ϕ-symmetric Legendre curve on a three-dimensional Heisenberg group. The concept of local ϕ-symmetry was introduced by T. Takahashi [8]. According to Takahashi a differentiable manifold is called locally ϕ-symmetric if it satisfies

$$\phi^2(\nabla_W R)(X,Y)Z = 0, \tag{3.1}$$

where the tangent vector fields X, Y, Z are orthogonal to the unit tangent vector field ξ and R is the Riemannian curvature tensor of type $(1,3)$ of the manifold. In Sasakian geometry locally ϕ-symmetric spaces are defined by the above curvature condition, which has several geometric interpretations, E. Boeckx and L. Vanhecke have exteded the notion of locally ϕ-symmetric spaces to the broader class of contact metric manifolds using reflections with respect to characteristic curves [4]. Riemannian curvature tensor is defined on the tangent space of a differentiable manifold. A differentiable curve has 1-dimensional tangent space at any point on it. The Riemannian curvature tensor of type $(1,3)$ of a Legendre curve is given by $R(\nabla_T T, T)T$. [5]. On this curvature tensor applying ϕ^2 and following the condition (3.1) we have introduced the following definition.

Definition 3.1. *A Legendre curve γ on a three-dimensional Heisenberg group will be called locally ϕ-symmetric if it satisfies*

$$\phi^2(\nabla_T R)(\nabla_T T, T)T = 0,$$

where $T = \dot{\gamma}$.

Let us consider a locally ϕ-symmetric Legendre curve on the three-dimensional Heisenberg group. Let $T, \phi T, \xi$ be a Frenet frame on the Legendre curve. To maintain orientation let $\phi T = N$ and $\phi N = -T$. Also we take $B = \xi$. Now using Serret-Frenet formula, we get

$$R(\nabla_T T, T)T = R(k\phi T, T)T = kR(N,T)T. \tag{3.2}$$

Since T and N are orthogonal to $\xi = e_3$ (as assumed for the Heisenberg group), we can take $T = t_1e_1 + t_2e_2$ and $N = n_1e_1 + n_2e_2$. Here t_1, t_2, n_1, n_2 are scalars.

Using the definition of the curvature tensor R, the expressions of T and N and (2.9) we get after straight forward calculation

$$R(N,T)T \;=\; 3t_1(t_2n_1e_2 - n_2t_1e_2) - 3t_2(n_1t_2e_1 - n_2t_1e_1). \tag{3.3}$$

Since $T, \phi T = N$ and $\xi = e_3$ forms a right handed system, we have $t_1 n_2 - t_2 n_1 = 1$. Hence the above equation can be further simplified as

$$R(N,T)T = 3t_2 e_1 - 3t_1 e_2. \tag{3.4}$$

Combining (3.2) and (3.3), we obtain

$$R(\nabla_T T, T)T = k3t_2 e_1 - k3t_1 e_2. \tag{3.5}$$

Now

$$\begin{aligned}(\nabla_T R)(\nabla_T T, T)T &= \nabla_T R(\nabla_T T, T)T - R(\nabla_T^2 T, T)T - R(\nabla_T T, \nabla_T T)T \\ &\quad - R(\nabla_T T, T)\nabla_T T \\ &= \nabla_T R(kN, T)T - k' R(N,T)T + k^2 R(T,T)T \\ &\quad - k\tau R(B,T)T - kR(kN,T)T + k'N.\end{aligned} \tag{3.6}$$

Now

$$\begin{aligned}R(B,T)T &= R(\xi, t_1 e_1 + t_2 e_2)(t_1 e_1 + t_2 e_2) \\ &= -t_1^2 R(e_1,\xi)e_1 - t_1 t_2 R(e_2,\xi)e_1 - t_2 t_1 R(e_1,\xi)e_2 - t_2^2 R(e_2,\xi)\xi\end{aligned} \tag{3.7}$$

Using (2.9) in (3.7), we get

$$R(B,T)T = (t_1^2 + t_2^2)e_3. \tag{3.8}$$

Again

$$\nabla_T R(kN,T)T = (k'3t_2 - kt_2 3t_2 e_3 - k'3t_1)e_2 - kt_1 3t_1 e_3. \tag{3.9}$$

Using (3.7), (3.8) and (3.9) in (3.6) we have after simplification

$$\begin{aligned}(\nabla_T R)(\nabla_T T, T)T &= -kt_2 3t_2 e_3 - kt_1 3t_1 e_3 - k\tau((t_1^2 + t_2^2)e_3 \\ &\quad - k^2(3t_2 e_1 - 3t_1 e_2) + k'(n_1 e_1 + n_2 e_2).\end{aligned} \tag{3.10}$$

By (2.1) and (2.2), the above equation yields

$$\phi^2(\nabla_T R)(\nabla_T T, T)T = k^2(3t_2 e_1 - 3t_1 e_2) - k'(n_1 e_1 + n_2 e_2). \tag{3.11}$$

Let the Legendre curve be locally ϕ-symmetric. Then by definition

$$k^2(3t_2 e_1 - 3t_1 e_2) - k'(n_1 e_1 + n_2 e_2) = 0. \tag{3.12}$$

In both sides of (3.12) taking inner product with e_1, we get

$$-k^2 3t_2 + k' n_1 = 0. \tag{3.13}$$

If the curve is a circle, then $k =$ a positive constant and $\tau = 0$. Then (3.13) gives

$$k = 0.$$

The above equation contradicts that $k =$ a positive constant. Hence we can conclude the following:

Theorem 3.1. *A locally ϕ-symmetric Legendre curve on a three-dimensional Heisenberg group is not a circle.*

4. Biharmonic Legendre curves on Heisenberg groups

In this section we study biharmonic Legendre curves on a three-dimensional Heisenberg group.

Definition 4.1. *A Legendre curve on a three-dimensional Heisenberg group will be called biharmonic [5] if it satisfies the biharmonic equation*

$$\nabla_T^3 T + R(\nabla_T T, T)T = 0, \tag{4.1}$$

where $T = \dot{\gamma}$.

Using Serret-Frenet formula, by direct computations, we have

$$\begin{aligned} \nabla_T^3 T &= -3kk'T + (k'' - k^3 - k\tau^2)N + (2\tau k' + k\tau')B \\ &= -3kk'(t_1e_1 + t_2e_2) + (k'' - k^3 - k\tau^2)(n_1e_1 + n_2e_2) \\ &+ (2\tau k' + k\tau')e_3. \end{aligned} \tag{4.2}$$

In view of (3.5) and (4.1), it follows that

$$\begin{aligned} \nabla_T^3 T + R(\nabla_T T, T)T &= -3kk'(t_1e_1 + t_2e_2) + (k'' - k^3 - k\tau^2)(n_1e_1 + n_2e_2) \\ &+(2\tau k' + k\tau')e_3 + k3t_2e_1 - k3t_1e_2. \end{aligned} \tag{4.3}$$

Consider that the Legendre curve is biharmonic. Then by definition

$$\begin{aligned} 0 &= -3kk'(t_1e_1 + t_2e_2) + (k'' - k^3 - k\tau^2)(n_1e_1 + n_2e_2) \\ &+(2\tau k' + k\tau')e_3 + k3t_2e_1 - k3t_1e_2. \end{aligned} \tag{4.4}$$

In both sides of (4.4) taking inner product with e_3 we obtain

$$2\tau k' + k\tau' = 0.$$

The above equation can be written as

$$2\frac{\tau}{\tau'} + \frac{k}{k'} = 0,$$

which gives

$$k\tau^2 = \text{an arbitrary constant.} \tag{4.5}$$

If possible let the Legendre curve is a circle. Then k = a positive constant and $\tau = 0$, which contradicts (4.5). Thus we are in a position to state the following:

Theorem 4.1. *A biharmonic Legendre curve on a three-dimensional Heisenberg group is not a circle.*

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