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FURTHER RESULTS ON A UNIQUE RANGE SET OF MEROMORPHIC FUNCTIONS WITH DEFICIENT POLES

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Abstract. We prove the uniqueness theorem of meromorphic functions sharing one set which improves the results of Yi, Li-Yang, Fang - Hua, Lahiri and Banerjee - Majumder.

1. Introduction, definitions and results

Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, f and g have the same set of a-points with the same multiplicities, we say that f and g share the value a CM (Counting Multiplicities)and if we do not consider the multiplicities, then f and g are said to share the value a IM (Ignoring Multiplicities). We do not explain the standard notations and definitions of the value distribution theory as these are available in [6]. Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and $E_f(S) = \bigcup_{a \in S}\{z : f(z) - a = 0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity then we replace the above set by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$ we say that f and g share the set S CM. On the other hand if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM. It will be convenient to denote by E any subset of positive reals of finite measure not necessarily the same at each occurrence. For any non-constant meromorphic function h, we denote by $S(r, h)$ any quantity such that $S(r, h) = o(T(r, h))$ as $r \to \infty$, $r \notin E$. We put $T(r) = max\{T(r, f), T(r, g)\}$ and $S(r) = o(T(r))$ as $r \to \infty$, $r \notin E$.

In 1976 Gross [4] showed that there exist three finite sets S_1, S_2, S_3 such that any two entire functions f, g satisfying $E_f(S_j) \equiv E_g(S_j)$ for $j = 1, 2, 3$ must be identical. In the same paper Gross asked the following question: Can one find two (or even one) finite sets S_1 and S_2 such that any two entire functions f and g satisfying $E_f(S_j) \equiv E_g(S_j)$ for $j = 1, 2$ must be identical?
A set S for which two meromorphic functions f and g satisfying $E_f(S) \equiv E_g(S)$ become identical is called a unique range set of meromorphic functions.

In 1982, Gross and Yang [5] proved the following theorem.

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Theorem A. Let $S = \{z : e^z + z = 0\}$. If two entire functions f and g satisfy $E_f(S) = E_g(S)$ then $f \equiv g$.

Since the set $S = \{z : e^z + z = 0\}$ contains infinitely many elements, the above result does not answer the question of Gross.

In 1994 Yi[16] exhibited a finite set S containing 15 elements which is a unique range set of entire functions and provided an affirmative answer to the question of Gross .

In 1995 Yi[17] and Li and Yang [14] independently proved the following result which gives a better answer to the question of Gross .

Theorem B. Let $S = \{z : z^7 - z^6 - 1 = 0\}$. If two entire functions f and g satisfy $E_f(S) = E_g(S)$ then $f \equiv g$.

Extending Theorem B to meromorphic functions,recently Fang and Hua[2] proved the following Theorem .

Theorem C. Let $S = \{z : z^7 - z^6 - 1 = 0\}$. If two meromorphic functions f and g are such that $\Theta(\infty; f) > \frac{11}{12}$, $\Theta(\infty; g) > \frac{11}{12}$ and $E_f(S) = E_g(S)$ then $f \equiv g$.

In 2001 Lahiri introduced the notion of weighted sharing in the following way.

Definition 1.1.[8, 9] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \le k$ and $k+1$times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f and g share the value a with weight k.
The definition implies that if f, g share a value a with weight k, then z_0 is a zero of $f - a$ with multiplicity $m(\le k)$ if and only if it is a zero of $g - a$ with multiplicity $m(\le k)$ and z_0 is a zero of $f - a$ of multiplicity $m(> k)$ if and only if it is a zero of $g - a$ with multiplicity $n(> k)$ where m is not necessarily equal to n .
We write f, g share (a, k) to mean f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integers p , $0 \le p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2.[2] Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a positive integer or ∞. We denote by $E_f(S, k)$ the set $\bigcup_{a \in S} E_k(a; f)$.

Definition 1.3. For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer m we denote by $N(r, a; f \mid\ge m)$ the counting function of those a-points of $f$whose multiplicities are not less than m where each a-point is counted according to its multiplicity. We agree to write $\overline{N}(r, a; f \mid\ge m)$ to denote the corresponding reduced counting function.

Definition 1.4. We put $N_2(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\ge 2)$ and
$\delta_2(a; f) = 1 - limsup_{r\to\infty} \frac{N_2(r,a;f)}{T(r,f)}$.

Improving Theorem C Lahiri proved the following theorem.

Theorem D.[10] Let $S = \{z : z^7 - z^6 - 1 = 0\}$. If two meromorphic functions f and g are such that $\Theta(\infty; f) + \Theta(\infty; g) > \frac{3}{2}$ and $E_f(S, 2) = E_g(S, 2)$ then $f \equiv g$.

In 2004 Lahiri-Banerjee [11] further improved Theorem D in the following manner.

Theorem E.[11] Let $S = \{z : z^n + az^{n-1} + b = 0\}$ where $n \ge 9$ is an integer and a, b be two non-zero constants such that $z^n + az^{n-1} + b = 0$ has no multiple root.

If for two non-constant meromorphic functions f and g, $E_f(S,2) = E_g(S,2)$ and $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n-1}$ then $f \equiv g$.

Example was also cited in [11] to show that the condition $\Theta(\infty; f)+\Theta(\infty; g) > \frac{4}{n-1}$ is sharp in Theorem E.

Recently Banerjee and Majumder improved Theorem E by reducing the cardinality of the shared set S from 9 to 6 as well as by weakening the condition on ramification index which is stated as follows.

Theorem F. [1] Let $S = \{z : z^n + az^{n-1} + b = 0\}$ where $n \geq 6$ is an integer and a, b be two non-zero constants such that $z^n + az^{n-1} + b = 0$ has no multiple root. Let f and g be two non-constant meromorphic functions satisfying $E_f(S,m) = E_g(S,m)$. If
(i) $m \geq 2$ and $\Theta_f + \Theta_g > \max\{\frac{10-n}{2}, \frac{4}{n-1}\}$
(ii)or if $m = 1$ and $\Theta_f + \Theta_g > \max\{\frac{11-n}{2}, \frac{4}{n-1}\}$
(iii)or if $m = 0$, $\Theta_f + \Theta_g > \max\{\frac{16-n}{3}, \frac{4}{n-1}\}$
then $f \equiv g$ where $\Theta_f = \Theta(0; f) + \Theta(\infty; f)$ and Θ_g can be defined similarly.

It, therefore, remains an open problem that whether the degree n, of the equation defining the set S can further be reduced. In this paper we show that it is possible to reduce the degree to 4 . Note that when $n = 4$ or 5 , $\max\{\frac{10-n}{2}, \frac{4}{n-1}\}$, $\max\{\frac{11-n}{2}, \frac{4}{n-1}\}$ and $\max\{\frac{16-n}{3}, \frac{4}{n-1}\}$ are respectively $\frac{10-n}{2}$, $\frac{11-n}{2}$ and $\frac{16-n}{3}$. As a particular case we state our first theorem when $n = 4$ or 5 as follows under weaker conditions than Theorem F.

Theorem 1.1. Let S be defined as Theorem F where $n = 4$ or 5 and a, b be two non-zero constants such that $z^n + az^{n-1} + b = 0$ has no multiple root. Let f and g be two non-constant meromorphic functions satisfying $E_f(S,m) = E_g(S,m)$. If
(i) $m \geq 2$ and

$$\Theta_f + \Theta_g + \frac{1}{2}min\{\sum_{x\notin S\cup\{0,\infty\}} \delta_2(x; f), \sum_{x\notin S\cup\{0,\infty\}} \delta_2(x; g)\} > \frac{10-n}{2} \tag{1.1}$$

(ii) or if $m = 1$ and

$$\Theta_f + \Theta_g + \frac{1}{2}min\{\sum_{x\notin S\cup\{0,\infty\}} \delta_2(x; f), \sum_{x\notin S\cup\{0,\infty\}} \delta_2(x; g)\} > \frac{11-n}{2} \tag{1.2}$$

(iii) or if $m = 0$ and

$$\Theta_f + \Theta_g + \frac{1}{2}min\{\sum_{x\notin S\cup\{0,\infty\}} \delta_2(x; f), \sum_{x\notin S\cup\{0,\infty\}} \delta_2(x; g)\} > \frac{16-n}{3} \tag{1.3}$$

then $f \equiv g$ where $\Theta_f = \Theta(0; f) + \Theta(\infty; f)$ and Θ_g can be defined similarly.

In our next Theorem we improve Theorem F by showing that the conclusion of Theorem F can be obtained for all $n \geq 5$ by dropping the term $\frac{4}{n-1}$, in the right hand side of the inequalities in (i), (ii) and (iii) at the cost of assuming that f and g should have no common zero.

Theorem 1.2. Let $S = \{z : z^n + az^{n-1} + b = 0\}$ where $n \geq 5$ is an integer and a, b be two non-zero constants such that $z^n + az^{n-1} + b = 0$ has no multiple root. Let f and g be two non-constant meromorphic functions having no common zero and satisfying $E_f(S, m) = E_g(S, m)$. Then any one of the conditions (1.1), (1.2) and (1.3) of (i), (ii) and (iii) of Theorem 1.1, implies that $f \equiv g$.

Following corollaries are immediate consequences of the above theorem.

Corollary 1.1. Let $S = \{z : z^n + az^{n-1} + b = 0\}$ where $n \geq 11$ is an integer and a, b be two non-zero constants such that $z^n + az^{n-1} + b = 0$ has no multiple root. Let f and g be two non-constant meromorphic functions having no common zero and satisfying $E_f(S, 2) = E_g(S, 2)$ then $f \equiv g$.

Corollary 1.2. Let $S = \{z : z^n + az^{n-1} + b = 0\}$ where $n \geq 12$ is an integer and a, b be two non-zero constants such that $z^n + az^{n-1} + b = 0$ has no multiple root. Let f and g be two non-constant meromorphic functions having no common zero and satisfying $E_f(S, 1) = E_g(S, 1)$, then $f \equiv g$.

Corollary 1.3. Let $S = \{z : z^n + az^{n-1} + b = 0\}$ where $n \geq 17$ is an integer and a, b be two non-zero constants such that $z^n + az^{n-1} + b = 0$ has no multiple root. Let f and g be two non-constant meromorphic functions having no common zero and satisfying $E_f(S, 0) = E_g(S, 0)$, then $f \equiv g$.

Note 1.1. In Theorem 1.2 and in the Corollaries above we have assumed the following:
$\{z : f(z) = 0\} \cap \{z : g(z) = 0\} = \Phi$.
And we have shown ultimately that $f \equiv g$. Therefore above condition then reduces to $\{z : f(z) = 0\} \cap \{z : f(z) = 0\} = \Phi$, implying that 0 is a Picard Exceptional value of f.

Definition 1.5.[9] Let f and g be two nonconstant meromorphic functions such that f and g share $(a, 0)$ for $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a-point of f with multiplicity p, and an a-point of g of multiplicity q. We denote by $\overline{N}_L(r, a; f)(\overline{N}_L(r, a; g))$ the reduced counting function of those a-points of $f$and g where $p > q(q > p)$. We denote by $\overline{N}_*(r, a; f, g)$the reduced counting function of those a-points of f whose multiplicities differ from the corresponding a-points of g. Clearly $\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$. We also denote by $N_E^{1)}(r, 1; f)$ the counting function of those 1-points of f and g where $p = q = 1$.

2. Lemmas

In this section we present some lemmas which will be required to establish our results. Let f and g be two nonconstant meromorphic functions and

$$F = \frac{f^{n-1}(f + a)}{-b}, G = \frac{g^{n-1}(g + a)}{-b}. \tag{2.1}$$

In the lemmas several times we use the function H defined by $H = \frac{F''}{F'} - \frac{2F'}{F-1} - \frac{G''}{G'} + \frac{2G'}{G-1}$.

Lemma 2.1.[13] Let f be a non-constant meromorphic function and let $R(f) = \frac{\sum_{k=0}^{n} a_k f^k}{\sum_{j=0}^{m} b_j f^j}$ be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$ where $a_{n\neq 0}$,$b_m \neq 0$. Then $T(r,R(f)) = dT(r,f) + S(r,f)$, where $d = max\{m,n\}$.

Lemma 2.2.[19] If F,G be two nonconstant meromorphic functions such that they share $(1,0)$ and $H \not\equiv 0$ then

$$N_E^{1)}(r,1;F \mid= 1) = N_E^{1)}(r,1;G \mid= 1) \leq N(r,H) + S(r,F) + S(r,G).$$

Lemma 2.3. [1] Let f and g be two non-constant meromorphic functions such that $E_f(S,0) = E_g(S,0)$,where S is as defined in Theorem 1.1.Also suppose that F,G be given by (4)and $H \not\equiv 0$, then

$$\begin{aligned} N(r,H) \leq\ & \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \overline{N}(r,0;nf + a(n-1)) \\ & + \overline{N}(r,0;ng + a(n-1)) + \overline{N}_*(r,1;F,G) + \overline{N}_0(r,0;f') + \overline{N}_0(r,0;g'), \end{aligned}$$

where $\overline{N}_0(r,0;f')$ denotes the reduced counting function corresponding to the zeros of f' which are not the zeros of f and $F-1$. $\overline{N}_0(r,0;g')$ is defined similarly.

Lemma 2.4. [12] If $N(r,0;f^{(k)} \mid f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$which are not the zeros of f,where a zero of $f^{(k)}$ is counted according to its multiplicity then $N(r,0;f^{(k)} \mid f \neq 0) \leq k\overline{N}(r,\infty;f) + N(r,0;f \mid< k) + k\overline{N}(r,0;f \mid\geq k) + S(r,f)$ where $N(r,0;f \mid< k)$ is the counting function of the zeros of f with multiplicity $< k$ each zero being counted according to its multiplicity.

Lemma 2.5. If Θ_f and Θ_g are defined as in Theorem 1.1 and

$$\Theta_f + \Theta_g + \frac{1}{2}min\{\sum_{x\notin S\cup\{0,\infty\}} \delta_2(x;f), \sum_{x\notin S\cup\{0,\infty\}} \delta_2(x;g)\} > 3 \tag{2.2}$$

holds, then $f^{n-1}(f+a)g^{n-1}(g+a) \not\equiv b^2$ when $n = 4$.

Proof: Assume to the contrary that

$$f^{n-1}(f+a)g^{n-1}(g+a) \equiv b^2. \tag{2.3}$$

Suppose that f has no pole. Then from (2.3)we see that g has neither zero nor $-a$-points. Hence $\Theta(\infty;f) = 1, \Theta(-a;g) = 1, \Theta(0;g) = 1$ and $\Theta(\infty;g) = 0$ and hence we obtain

$$min\{\sum_{x\notin S\cup\{0,\infty\}} \delta_2(x;f), \sum_{x\notin S\cup\{0,\infty\}} \delta_2(x;g)\} = 0.$$

Then (2.2) gives $\Theta(0;f) > 1$, which is not possible. Thus f must have poles. Similarly we can show that g must have poles .
We see that if z_0 is a zero of $f+a$ of multiplicity p then z_0 is a pole of g with

multiplicity q such that $p = nq$. Therefore $p \geq n$ and hence $\Theta(0; f + a) \geq 1 - \frac{1}{n}$. Similarly we may obtain $\Theta(0; g + a) \geq 1 - \frac{1}{n}$. If possible suppose that

$$min\{\sum_{x \notin S \cup \{0,\infty\}} \delta_2(x; f), \sum_{x \notin S \cup \{0,\infty\}} \delta_2(x; g)\} = 2.$$

Then $\Theta_f + \Theta_g = 0$ and then condition (2.2) is not satisfied . If

$$min\{\sum_{x \notin S \cup \{0,\infty\}} \delta_2(x; f), \sum_{x \notin S \cup \{0,\infty\}} \delta_2(x; g)\} = 1,$$

then the maximum value Θ_f may assume is 1. Similar is true for Θ_g also and we observe that (2.2) is not satisfied in this case too.

So let $min\{\sum_{x \notin S \cup \{0,\infty\}} \delta_2(x; f), \sum_{x \notin S \cup \{0,\infty\}} \delta_2(x; g)\} = 1 + s$, $1 > s > 0$. Then each of Θ_f and Θ_g may have maximum value as $1 - s$ and in this case (2.2) implies $1 - s + 1 - s + \frac{1}{2}(1 + s) > 3$ which implies $-\frac{1}{2} > \frac{3s}{2}$, which is not possible.

So we must have

$$min\{\sum_{x \notin S \cup \{0,\infty\}} \delta_2(x; f), \sum_{x \notin S \cup \{0,\infty\}} \delta_2(x; g)\} < 1.$$

Therefore we put, $\min\{\sum_{x \notin S \cup \{0,\infty\}} \delta_2(x; f), \sum_{x \notin S \cup \{0,\infty\}} \delta_2(x; g)\} = 1 - s, s > 0$. Thus from (2.2) we observe that

$$\Theta_f + \Theta_g > 3 - \frac{1 - s}{2} = \frac{5}{2} + \frac{s}{2}.$$

Since $\Theta(0; f + a) \geq 1 - \frac{1}{n}$ and $\Theta(0; g + a) \geq 1 - \frac{1}{n}$, by the deficiency relation we get for $n = 4$,

$$\Theta(0; f) + \Theta(0; f + a) + \Theta(\infty; f) + \Theta(0; g) + \Theta(0; g + a) + \Theta(\infty; g) \leq 4$$

$$\Rightarrow \Theta_f + \Theta_g + \Theta(0; f + a) + \Theta(0; g + a) \leq 4$$

$\Rightarrow \frac{5}{2} + \frac{s}{2} + 1 - \frac{1}{4} + 1 - \frac{1}{4} \leq 4 \Rightarrow s < 0$, which is a contradiction. This proves the lemma.

Lemma 2.6.[11] Let f, g be two non-constant meromorphic functions and a, b be two nonzero constants , then $f^{n-1}(f + a)g^{n-1}(g + a) \not\equiv b^2$ where $n \geq 5$ is an integer.

Lemma 2.7.[1] Let f, g be two non-constant meromorphic functions such that $\Theta(0; f) + \Theta(\infty; f) + \Theta(0; g) + \Theta(\infty; g) > \frac{4}{n-1}$. Then $f^{n-1}(f + a) \equiv g^{n-1}(g + a)$ implies $f \equiv g$ where $n(\geq 3)$ is an integer and a is a nonzero constant.

Lemma 2.8. Let f, g be two non-constant meromorphic functions having no common zero. Then $f^{n-1}(f + a) \equiv g^{n-1}(g + a)$ implies $f \equiv g$ where $n(\geq 5)$ is an integer and a is a nonzero constant.

Proof. Suppose

$$f^{n-1}(f + a) \equiv g^{n-1}(g + a) \tag{2.4}$$

and $f \not\equiv g$. We consider the two cases .
CaseI. Let $y = \frac{g}{f}$ be a constant. Then from (2.4) it follows that $y \neq 1, y^n \neq 1$ and $y^{n-1} \neq 1$ and $f \equiv -a\frac{1-y^{n-1}}{1-y^n}$ is a constant, which is not possible .
CaseII. Let $y = \frac{g}{f}$ be a nonconstant. Then from (2.4), it follows that

$$f \equiv -a\frac{1-y^{n-1}}{1-y^n} \equiv a\left(\frac{y^{n-1}}{1+y+y^2+...+y^{n-1}} - 1\right). \tag{2.5}$$

Let z_0 be a zero of $f + a$ of multiplicity p. Then it follows from (2.4) that z_0 is either a zero of g or a zero of $g + a$. If z_0 is a zero of $g + a$, then $y(z_0) = 1$ and hence from (2.5) it follows that

$$f(z_0) = a\left(\frac{1}{1+n-1} - 1\right) = a\left(\frac{1}{n} - 1\right) \neq -a.$$

Hence z_0 must be a zero of g of multiplicity q , say. Hence $p = (n-1)q$ and hence $p \geq n-1$. It follows that

$$\Theta(0; f + a) \geq 1 - \frac{1}{n-1} = \frac{n-2}{n-1}.$$

Similarly we can show that $\Theta(0; g + a) \geq \frac{n-2}{n-1}$.
Above analysis also implies that $\{z : f(z) + a = 0\} \subseteq \{z : g(z) = 0\}$ and similarly $\{z : g(z) + a = 0\} \subseteq \{z : f(z) = 0\}$. Since the zeros of g are either the zeros of $f + a$ or the zeros of f and since f and g have no common zero it follows from above that $\{z : f(z)+a = 0\} \equiv \{z : g(z) = 0\}$. Similarly $\{z : g(z)+a = 0\} \equiv \{z : f(z) = 0\}$. Since from (2.4), it follows by Lemma 2.1, $T(r, f) = T(r, g) + O(1)$, we have $\Theta(0; g + a) = \Theta(0; f)$ and $\Theta(0; f + a) = \Theta(0; g)$. Thus

$$\Theta(0; f) + \Theta(0; g) \geq 2(\frac{n-2}{n-1}) \geq \frac{6}{n-1},$$

for $n \geq 5$. Hence by Lemma 2.7 it follows that $f \equiv g$. This leads to a contradiction that y is nonconstant.

Hence we must have $f \equiv g$.

3. Proofs of Theorems

Proof of Theorem 1.1. Let $w_j, j = 1, 2, ..., n$ be the distinct elements of S. From (2.1) we see that since $E_f(S, m) = E_f(S, m)$, F, G share $(1, m)$.
Case 1. Assume first $H \not\equiv 0$.
Subcase 1.1. $m \geq 1$.When $m \geq 2$, using Lemma 2.4, we obtain

$$\begin{aligned}
&\overline{N}_0(r, 0; g') + \overline{N}(r, 1; G \mid\geq 2) + \overline{N}_*(r, 1; F, G) \\
&\leq \overline{N}_0(r, 0; g') + \overline{N}(r, 1; G \mid\geq 2) + \overline{N}(r, 1; G \mid\geq 3) \\
&\leq \overline{N}_0(r, 0; g') + \sum_{j=1}^{n}\{\overline{N}(r, w_j; g \mid= 2) + 2\overline{N}(r, w_j; g \mid\geq 3)\} \\
&\leq N(r, 0; g' \mid g \neq 0) + S(r, g) \leq \overline{N}(r, 0; g) + \overline{N}(r, \infty; g) + S(r, g).
\end{aligned} \tag{3.1}$$

Let $x_1, x_2, ..., x_k$ be any k distinct complex numbers such that $x_i \notin S \cup \{0, \infty\}$.
Let us denote by $N_0^*(r, 0; f')$ the counting function of the zeros of f', which are not the zeros of $f \prod_{i=1}^{k}(f - x_i)$ and $F - 1$. By $\overline{N}_0^*(r, 0; f')$ we denote the corresponding reduced counting function. Therefore
$\overline{N}_0(r, 0; f') = \overline{N}_0^*(r, 0; f') + \sum_{i=1}^{k} \overline{N}(r, x_i; f \mid\geq 2)$.
Thus by *Lemmas 2.1* , *2.2* and *2.3* and (3.1) we obtain for $\epsilon > 0$,

$$\begin{aligned}
(n + k)T(r, f) &\leq \overline{N}(r, 0; f) + \overline{N}(r, \infty; f) + \overline{N}(r, 1; F \mid= 1) + \overline{N}(r, 1; F \mid\geq 2) \\
&+ \sum_{i=1}^{k} \overline{N}(r, x_i; f) - N_0^*(r, 0; f') + S(r, f) \\
&\leq 2\{\overline{N}(r, 0; f) + \overline{N}(r, \infty; f)\} + \overline{N}(r, \infty; g) + \overline{N}(r, 0; g) + \overline{N}(r, 0; nf + a(n - 1)) \\
&+\overline{N}(r, 0; ng + a(n - 1)) + \{\overline{N}(r, \infty; g) + \overline{N}(r, 0; g)\} + \sum_{i=1}^{k} N_2(r, x_i; f) + S(r) \\
\leq 2\{\overline{N}(r, 0; f) + \overline{N}(r, \infty; f) &+ \overline{N}(r, 0; g) + \overline{N}(r, \infty; g)\} + T(r, f) + T(r, g) + \sum_{i=1}^{k} N_2(r, x_i; f) + S(r) \\
&\leq \{10 + k - 2\Theta(0; f) - 2\Theta(\infty; f) - 2\Theta(0; g) - 2\Theta(\infty; g) \\
&- \sum_{i=1}^{k} \delta_2(x_i; f) + \epsilon\}T(r) + S(r). \qquad (3.2)
\end{aligned}$$

Similarly we may obtain

$$\begin{aligned}
&(n + k)T(r, g) \\
&\leq \{10 + k - 2\Theta(0; f) - 2\Theta(\infty; f) - 2\Theta(0; g) - 2\Theta(\infty; g) \\
&- \sum_{i=1}^{k} \delta_2(x_i; g) + \epsilon\}T(r) + S(r). \qquad (3.3)
\end{aligned}$$

Thus from (3.2)and (3.3) we obtain,
$nT(r)$
$\leq \left(10 - 2\Theta(0; f) - 2\Theta(\infty; f) - 2\Theta(0; g) - 2\Theta(\infty; g) - \min\left\{\sum_{i=1}^{k} \delta_2(x_i; g), \sum_{i=1}^{k} \delta_2(x_i; f)\right\} + \epsilon\right) T(r)+$
$S(r)$.
This being true for any k numbers of complex numbers $x_1, x_2, ..., x_k$ not belonging to $S \cup \{0, \infty\}$ we have from above

$$\begin{aligned}
nT(r) &\leq \{10 - 2\Theta(0; f) - 2\Theta(\infty; f) - 2\Theta(0; g) - 2\Theta(\infty; g) \\
&-min\{\sum_{x \notin S \cup \{0,\infty\}} \delta_2(x; g), \sum_{x \notin S \cup \{0,\infty\}} \delta_2(x; f)\} + \epsilon\}T(r) + S(r).
\end{aligned}$$

This contradicts (1.1).

We omit the proof of the subcases for $m = 1$ and $m = 0$ since these proofs can be carried out using the techniques of our above proof of the subcase 1.1 and the proof of Theorem C[1] for subcases for $m = 1$ and $m = 0$.

Case2. $H \equiv 0$. Hence

$$F \equiv \frac{AG+B}{CG+D}, \tag{3.4}$$

where $AD-BC \neq 0$. Hence $T(r,F) = T(r,G)+O(1)$ and therefore from the definitions of F and G and *Lemma 2.1* , we have

$$T(r,f) = T(r,g) + O(1) \tag{3.5}$$

Subcase 2.1. $AC \neq 0$. Suppose that $D \neq 0$. Then from (3.4) we obtain by using *Lemma 2.1* and the second main theorem,
$T(r,G) \leq \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + \overline{N}(r,\frac{-D}{C};G) + S(r,G)$,
that is

$$nT(r,g) \leq \overline{N}(r,0;g) + \overline{N}(r,0;g+a) + \overline{N}(r,\infty;g) + \overline{N}(r,\infty;f) + S(r,g). \tag{3.6}$$

This leads to a contradiction if $n \geq 5$. So let $n = 4$. Then from (3.6) we obtain,

$$\Theta(0;g) + \Theta(\infty;g) + \Theta(0;g+a) + \Theta(\infty;f) = 0.$$

Then

$$\Theta_g = \Theta(0;g) + \Theta(\infty;g) = 0$$

and

$$\Theta_f = \Theta(0;f) + \Theta(\infty;f) = \Theta(0;f).$$

Hence from above we obtain from (1.1), with $n = 4$,

$$\Theta(0;f) + \frac{1}{2}min\{\sum \delta_2(x;f), \sum \delta_2(x;g)\} > \frac{10-4}{2} = 3,$$

which is not possible.
Similarly we may verify that (1.2) and (1.3) also lead to contradiction for $n = 4$.
If $D = 0$ then $F \equiv \alpha + \frac{\beta}{G}$, where $\alpha = \frac{A}{C}$, $\beta = \frac{B}{C}$. Since F, G share 1-points we have $\alpha + \beta = 1$. Hence

$$F \equiv 1 - \beta + \frac{\beta}{G}. \tag{3.7}$$

If $\beta \neq 1$,we obtain from (3.7), using *Lemma 2.1* and the second main theorem,
$nT(r,g) \leq \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + \overline{N}(r,\frac{\beta}{\beta-1};G) + S(r,G)$
$\leq \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \overline{N}(r,0;f) + \overline{N}(r,0;f+a) + S(r,g)$ and arguing as in

the case with (3.6) we may also arrive at a contradiction from the above inequality. If $\beta = 1$, $FG \equiv 1$ which is impossible by *Lemmas 2.5* and *2.6*.

Subcase 2.2. $A = 0$, $C \neq 0$. Then from (3.4) we have $F = \frac{\eta}{G+\gamma}$, where $\eta = \frac{B}{C}$. Since F, G share 1-points , $\eta = 1 + \gamma$. Thus

$$F = \frac{1+\gamma}{G+\gamma} \tag{3.8}$$

We see that $\gamma \neq 0$, for $\gamma = 0 \Rightarrow FG \equiv 1$, which is not allowed by *Lemmas 2.5* and *2.6*. By the second main theorem we obtain, from above

$nT(r,g) \leq \overline{N}(r,0;G) + \overline{N}(r,-\gamma;G) + \overline{N}(r,\infty;G) + S(r,G)$

$\leq \overline{N}(r,0;g) + \overline{N}(r,0;g+a) + \overline{N}(r,\infty;g) + \overline{N}(r,\infty;f) + S(r,g)$, this is previously obtained inequality (3.6) and therefore we arrive at a contradiction as in the Subcase 2.1

Subcase 2.3 $A \neq 0$, $C = 0$. Hence from (3.4) , we note that $F \equiv \eta G + \gamma$, where $\eta = \frac{A}{D}, \gamma = \frac{B}{D}$. Since F, G share 1-points, $\gamma + \eta = 1$.

Hence

$$F \equiv \eta G + 1 - \eta. \tag{3.9}$$

From (3.9) we observe that $T(r,f) = T(r,g) + O(1)$. Suppose $\eta \neq 1$. Then from (3.9) we obtain by the second main theorem

$$\begin{aligned} nT(r,g) &\leq \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + \overline{N}(r,\frac{\eta-1}{\eta};G) + S(r,G) \\ &\leq \overline{N}(r,0;g) + \overline{N}(r,0;g+a) + \overline{N}(r,\infty;g) + \overline{N}(r,0;f) + \overline{N}(r,0;f+a) + S(r,g) \\ &\leq \{5 - \Theta(0;g) - \Theta(0;g+a) - \Theta(\infty;g) - \Theta(0;f) - \Theta(0;f+a) + \epsilon\}T(r,g), \end{aligned}$$

where, $\epsilon > 0$ is arbitrary. This leads to a contradiction if

$$n > 5 - \Theta(0;g) - \Theta(0;g+a) - \Theta(\infty;g) - \Theta(0;f) - \Theta(0;f+a).$$

So let

$$n \leq 5 - \Theta(0;g) - \Theta(0;g+a) - \Theta(\infty;g) - \Theta(0;f) - \Theta(0;f+a),$$

that is

$$\Theta(0;g) + \Theta(0;g+a) + \Theta(\infty;g) + \Theta(0;f) + \Theta(0;f+a) \leq 5 - n.$$

Hence

$$\Theta_f + \Theta_g = 5 - n - \xi + \Theta(\infty;f),$$

for some $\xi \geq 0$. Thus from (1.1) we obtain

$$5 - n - \xi + \Theta(\infty;f) + \frac{1}{2}min\{\sum_{x \notin S \cup \{0,\infty\}} \delta_2(x;f), \sum_{x \notin S \cup \{0,\infty\}} \delta_2(x;g)\} > \frac{10-n}{2},$$

which implies that

$$\Theta(\infty; f) + \frac{1}{2}min\{\sum_{x\notin S\cup\{0,\infty\}} \delta_2(x; f), \sum_{x\notin S\cup\{0,\infty\}} \delta_2(x; g)\} > \frac{n+2\xi}{2},$$

which leads to contradiction for $n \geq 4$. Similarly we may arrive at contradiction from (1.2) and (1.3). Hence $\eta = 1$. Therefore $F \equiv G$. Now from (1.1), we have

$$\Theta_f + \Theta_g > \frac{10-n}{2} - \frac{1}{2}min\{\sum_{x\notin S\cup\{0,\infty\}} \delta_2(x; f), \sum_{x\notin S\cup\{0,\infty\}} \delta_2(x; g)\}$$
$$\geq \frac{10-n}{2} - 1 > \frac{4}{n-1},$$

for $n = 4, 5$.
Similarly we can verify that the inequalities (1.2) and (1.3) also imply $\Theta_f + \Theta_g > \frac{4}{n-1}$, for $n = 4, 5$. Thus our theorem follows from *Lemma 2.7.*

We omit the proof of Theorem 1.2 as it can be carried out exactly as in the above proof except in the Subcase 2.3 when we obtain $F \equiv G$ we should apply *Lemma 2.8* to obtain our desired result instead of *Lemma 2.7* as in the previous case.

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