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BOUNDEDNESS OF TOEPLITZ TYPE OPERATOR ASSOCIATED TO SINGULAR INTEGRAL OPERATOR ON L^p SPACES WITH VARIABLE EXPONENT

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Abstract. In this paper, the boundedness for some Toeplitz type operator related to the singular integral operator on L^p spaces with variable exponent is obtained by using a sharp estimate of the operator.

Keywords: Toeplitz type operator; Singular integral operator; Variable L^p space; BMO.

1. Introduction

As the development of the singular integral operators(see [5][14][15]), their commutators have been well studied. In [1][12][13], the authors prove that the commutators generated by the singular integral operators and *BMO* functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. In [7][9][10], some Toeplitz type operators associated to the singular integral operators and strongly singular integral operators are introduced, and the boundedness for the operators are obtained. In the last years, a theory of L^p spaces with variable exponent has been developed because of its connections with some questions in fluid dynamics, calculus of variations, differential equations and elasticity(see [2][3][4][6][11] and their references). Karlovich and Lerner study the boundedness of the commutators of singular integral operators on L^p spaces with variable exponent(see [6]). Motivated by these papers, the main purpose of this paper is to introduce some Toeplitz type operator related to the singular integral operator and prove the boundedness for the operator on L^p spaces with variable exponent by using a sharp estimate of the operator.

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2. Preliminaries and Results

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f and $\delta > 0$, the sharp function of f is defined by

$$f_\delta^\#(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|} \int_Q |f(y) - f_Q|^\delta dy \right)^{1/\delta},$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x)dx$. It is well-known that(see [5][14])

$$f_\delta^\#(x) \approx \sup_{Q \ni x} \inf_{c \in C} \left(\frac{1}{|Q|} \int_Q |f(y) - c|^\delta dy \right)^{1/\delta}.$$

We write that $f^\# = f_\delta^\#$ if $\delta = 1$. We say that f belongs to $BMO(R^n)$ if $f^\#$ belongs to $L^\infty(R^n)$ and define $||f||_{BMO} = ||f^\#||_{L^\infty}$. Let M be the Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{Q \ni x} |Q|^{-1} \int_Q |f(y)|dy;$$

For $k \in N$, we denote by M^k the operator M iterated k times, i.e., $M^1(f)(x) = M(f)(x)$ and

$$M^k(f)(x) = M(M^{k-1}(f))(x) \text{ when } k \geqslant 2.$$

Let Φ be a Young function and $\tilde{\Phi}$ be the complementary associated to Φ, we denote that the Φ-average by, for a function f,

$$||f||_{\Phi,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \Phi\left(\frac{|f(y)|}{\lambda} \right) dy \leqslant 1 \right\}$$

and the maximal function associated to Φ by

$$M_\Phi(f)(x) = \sup_{Q \ni x} ||f||_{\Phi,Q}.$$

The Young functions to be using in this paper are $\Phi(t) = t(1 + logt)^r$ and $\tilde{\Phi}(t) = exp(t^{1/r})$, the corresponding average and maximal functions denoted by $||\cdot||_{L(logL)^r,Q}$, $M_{L(logL)^r}$ and $||\cdot||_{expL^{1/r},Q}$, $M_{expL^{1/r}}$. Following [12][13], we know the generalized Hölder's inequality:

$$\frac{1}{|Q|} \int_Q |f(y)g(y)|dy \leqslant ||f||_{\Phi,Q} ||g||_{\tilde{\Phi},Q}$$

and the following inequality, for $r, r_j \geqslant 1$, $j = 1, \cdots, l$ with $1/r = 1/r_1 + \cdots + 1/r_l$, and any $x \in R^n$, $b \in BMO(R^n)$,

$$||f||_{L(logL)^{1/r},Q} \leqslant M_{L(logL)^{1/r}}(f) \leqslant CM_{L(logL)^l}(f) \leqslant CM^{l+1}(f),$$

$$\|f - f_Q\|_{expL^r,Q} \leqslant C\|f\|_{BMO},$$
$$|f_{2^{k+1}Q} - f_{2Q}| \leqslant Ck\|f\|_{BMO}.$$

The non-increasing rearrangement of a measurable function f on R^n is defined by

$$f^*(t) = \inf\{\lambda > 0 : |\{x \in R^n : |f(x)| > \lambda\}| \leqslant t\} \ \ (0 < t < \infty).$$

For $\lambda \in (0,1)$ and a measurable function f on R^n, the local sharp maximal function of f is defined by

$$M^{\#}_{\lambda}(f)(x) = \sup_{Q\ni x} \inf_{c\in C}((f - c)\chi_Q)^*(\lambda|Q|).$$

Let $p : R^n \to [1, \infty)$ be a measurable function. Denote by $L^{p(\cdot)}(R^n)$ the sets of all Lebesgue measurable functions f on R^n such that $m(\lambda f, p) < \infty$ for some $\lambda = \lambda(f) > 0$, where

$$m(f, p) = \int_{R^n} |f(x)|^{p(x)} dx.$$

The sets becomes a Banach spaces with respect to the following norm

$$\|f\|_{L^{p(\cdot)}} = \inf\{\lambda > 0 : m(f/\lambda, p) \leqslant 1\}.$$

Denote by $M(R^n)$ the sets of all measurable functions $p : R^n \to [1, \infty)$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(R^n)$ and the following holds

$$1 < p_- = ess\inf_{x\in R^n} p(x), \qquad ess\sup_{x\in R^n} p(x) = p_+ < \infty. \tag{1}$$

In recent years, the boundedness of classical operators on spaces $L^{p(\cdot)}(R^n)$ have attracted a great attention (see [4-7],[10],[19] and their references).

In this paper, we will study some integral operators as following(see [14][15]).

Definition. Let $T : S \to S'$ be a linear operator such that T is bounded on $L^{p_0}(R^n)$ for some $1 < p_0 < \infty$ and weak (L^1, L^1)-bounded and there exists a locally integrable function $K(x, y)$ on $R^n \times R^n \setminus \{(x, y) \in R^n \times R^n : x = y\}$ such that

$$T(f)(x) = \int_{R^n} K(x, y) f(y) dy$$

for every bounded and compactly supported function f, where K satisfies: for fixed $\delta > 0$,

$$|K(x, y)| \leqslant C|x - y|^{-n}$$

and

$$|K(y, x) - K(z, x)| + |K(x, y) - K(x, z)| \leqslant C|y - z|^{\delta}|x - z|^{-n-\delta}$$

if $2|y - z| \leqslant |x - z|$.

Moreover, let b be a locally integrable function on R^n. The Toeplitz type operator related to T is defined by

$$T_b = \sum_{k=1}^{m} T^{k,1} M_b T^{k,2},$$

where $T^{k,1}$ are the singular integral operator T or $\pm I$(the identity operator), $T^{k,2}$ are the linear operators for $k = 1, ..., m$ and $M_b(f) = bf$.

Note that the classical Calderón-Zygmund singular integral operator satisfies Definition(see [14][15]). Also note that the commutator is a particular operator of the Toeplitz type operator T_b. The Toeplitz type operator T_b are the non-trivial generalizations of the commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [12][13]). In recent years, the boundedness of classical operators on spaces $L^{p(\cdot)}(R^n)$ have attracted a great attention (see [2-4][6][11] and their references). The main purpose of this paper has twofold, first, we establish a sharp estimate for the operator T_b, and second, we prove the boundedness for the operator on L^p spaces with variable exponent by using the sharp estimate.

We shall prove the following theorems.

Theorem 1. Let T be the singular integral operators as Definition, $0 < \delta < 1$ and $b \in BMO(R^n)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$, then there exists a constant $C > 0$ such that for any $f \in L_0^\infty(R^n)$ and $\tilde{x} \in R^n$,

$$(T_b(f))_\delta^\#(\tilde{x}) \leqslant C\|b\|_{BMO} \sum_{k=1}^m M^2(T^{k,2}(f))(\tilde{x}).$$

Theorem 2. Let T be the singular integral operators as Definition, $p(\cdot) \in M(R^n)$ and $b \in BMO(R^n)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$ and $T^{k,2}$ are the bounded linear operators on $L^{p(\cdot)}(R^n)$ for $k = 1, ..., m$, then T_b is bounded on $L^{p(\cdot)}(R^n)$, that is

$$\|T_b(f)\|_{L^{p(\cdot)}} \leqslant C\|b\|_{BMO}\|f\|_{L^{p(\cdot)}}.$$

Corollary 1. Let $[b, T](f) = bT(f) - T(bf)$ be the commutator generated by the singular integral operator T and b. Then Theorems 1 and 2 hold for $[b, T]$.

3. Proofs of Theorems

To prove the theorems, we need the following lemmas.

Lemma 1.([5, p.485]) Let $0 < p < q < \infty$. We define that, for any function $f \geqslant 0$ and $1/r = 1/p - 1/q$,

$$\|f\|_{WL^q} = \sup_{\lambda>0} \lambda|\{x \in R^n : f(x) > \lambda\}|^{1/q}, \ N_{p,q}(f) = \sup_E \|f\chi_E\|_{L^p}/\|\chi_E\|_{L^r},$$

where the sup is taken for all measurable sets E with $0 < |E| < \infty$. Then

$$\|f\|_{WL^q} \leqslant N_{p,q}(f) \leqslant (q/(q-p))^{1/p}\|f\|_{WL^q}.$$

Lemma 2.[12] Let $r_j \geqslant 1$ for $j = 1, \cdots, l$, we denote that $1/r = 1/r_1 + \cdots + 1/r_l$. Then

$$\frac{1}{|Q|}\int_Q |f_1(x)\cdots f_l(x)g(x)|dx \leqslant \|f\|_{expL^{r_1},Q} \cdots \|f\|_{expL^{r_l},Q}\|g\|_{L(logL)^{1/r},Q}.$$

Lemma 3.([6]) Let $p : R^n \to [1,\infty)$ be a measurable function satisfying (1). Then $L_0^\infty(R^n)$ is dense in $L^{p(\cdot)}(R^n)$.

Lemma 4.([6]) Let $f \in L^1_{loc}(R^n)$ and g be a measurable function satisfying

$$|\{x \in R^n : |g(x)| > \alpha\}| < \infty \quad \text{for all} \quad \alpha > 0.$$

Then

$$\int_{R^n} |f(x)g(x)|dx \leqslant C_n \int_{R^n} M^{\#}_{\lambda_n}(f)(x)M(g)(x)dx.$$

Lemma 5.([6]) Let $p : R^n \to [1,\infty)$ be a measurable function satisfying (1). If $f \in L^{p(\cdot)}(R^n)$ and $g \in L^{p'(\cdot)}(R^n)$ with $p'(x) = p(x)/(p(x)-1)$. Then fg is integrable on R^n and

$$\int_{R^n} |f(x)g(x)|dx \leqslant C||f||_{L^{p(\cdot)}}||g||_{L^{p'(\cdot)}}.$$

Lemma 6.([6]) Let $p : R^n \to [1,\infty)$ be a measurable function satisfying (1). Set

$$||f||'_{L^{p(\cdot)}} = \sup\left\{\int_{R^n} |f(x)g(x)|dx : f \in L^{p(\cdot)}(R^n), g \in L^{p'(\cdot)}(R^n)\right\}.$$

Then $||f||_{L^{p(\cdot)}} \leqslant ||f||'_{L^{p(\cdot)}} \leqslant C||f||_{L^{p(\cdot)}}$.

Lemma 7.([6][8]) Let $\delta > 0$, $0 < \lambda < 1$ and $f \in L^\delta_{loc}(R^n)$. Then

$$M^{\#}_\lambda(f)(x) \leqslant (1/\lambda)^{1/\delta} f^{\#}_\delta(x).$$

Proof of Theorem 1. It suffices to prove for $f \in L_0^\infty(R^n)$, the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q \left|T_b(f)(x) - C_0\right| dx\right)^{1/\delta} \leqslant C||b||_{BMO}\sum_{k=1}^m M^2(T^{k,2}(f))(\tilde{x}).$$

Without loss of generality, we may assume $T^{k,1}$ are $T(k = 1,\dots,m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. By $T_1(g) = 0$, we have

$$T_b(f)(x) = T_{b-b_{2Q}}(f)(x) = T_{(b-b_{2Q})\chi_{2Q}}(f)(x) + T_{(b-b_{2Q})\chi_{(2Q)^c}}(f)(x) = f_1(x) + f_2(x)$$

and

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q \left|T^b(f)(x) - f_2(x_0)\right|^\delta dx\right)^{1/\delta}\\ \leqslant\ & C\left(\frac{1}{|Q|}\int_Q |f_1(x)|^\delta dx\right)^{1/\delta} + C\left(\frac{1}{|Q|}\int_Q |f_2(x) - f_2(x_0)|^\delta dx\right)^{1/\delta} = I_1 + I_2.\end{aligned}$$

For I_1, by the weak (L^1, L^1) boundedness of T and Lemma 1 and 2, we obtain

$$\left(\frac{1}{|Q|}\int_Q |T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(x)|^\delta dx\right)^{1/\delta}$$

$$
\begin{aligned}
&\leqslant |Q|^{-1}\frac{\|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)\chi_Q\|_{L^\delta}}{|Q|^{1/\delta-1}}\\
&\leqslant C|Q|^{-1}\|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)\|_{WL^1}\\
&\leqslant C|Q|^{-1}\|M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)\|_{L^1}\\
&\leqslant C|Q|^{-1}\int_{2Q}|b(x)-b_{2Q}||T^{k,2}(f)(x)|dx\\
&\leqslant C\|b-b_{2Q}\|_{expL,2Q}\|T^{k,2}(f)\|_{L(logL),2Q}\\
&\leqslant C\|b\|_{BMO}M^2(T^{k,2}(f))(\tilde{x}),
\end{aligned}
$$

thus

$$
\begin{aligned}
I_1 &\leqslant C\sum_{k=1}^{m}\left(\frac{1}{|Q|}\int_Q|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(x)|^\delta dx\right)^{1/\delta}\\
&\leqslant C\|b\|_{BMO}\sum_{k=1}^{m}M^2(T^{k,2}(f))(\tilde{x}).
\end{aligned}
$$

For I_2, we get, for $x\in Q$,

$$
\begin{aligned}
&|T^{k,1}M_{(b-b_{2Q})\chi_{(2Q)^c}}T^{k,2}(f)(x)-T^{k,1}M_{(b-b_{2Q})\chi_{(2Q)^c}}T^{k,2}(f)(x_0)|\\
\leqslant&\int_{(2Q)^c}|b(y)-b_{2Q}|K(x,y)-K(x_0,y)||T^{k,2}(f)(y)|dy\\
\leq& C\sum_{j=1}^{\infty}\int_{2^jd\leq|y-x_0|<2^{j+1}d}|b(y)-b_{2Q}|\frac{|x-x_0|^\delta}{|x_0-y|^{n+\delta}}|T^{k,2}(f)(y)|dy\\
\leq& C\sum_{j=1}^{\infty}2^{-j\delta}\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|b(y)-b_{2Q}||T^{k,2}(f)(y)|dy\\
\leqslant& C\sum_{j=1}^{\infty}2^{-j\delta}\|b-b_{2Q}\|_{expL,2^{j+1}Q}\|T^{k,2}(f)\|_{L(logL),2^{j+1}Q}\\
\leqslant& C\sum_{j=1}^{\infty}j2^{-j\delta}\|b\|_{BMO}M^2(T^{k,2}(f))(\tilde{x})\\
\leq& C\|b\|_{BMO}M^2(T^{k,2}(f))(\tilde{x}),
\end{aligned}
$$

thus

$$
\begin{aligned}
I_2 &\leqslant \frac{C}{|Q|}\int_Q\sum_{k=1}^{m}|T^{k,1}M_{(b-b_{2Q})\chi_{(2Q)^c}}T^{k,2}(f)(x)-T^{k,1}M_{(b-b_{2Q})\chi_{(2Q)^c}}T^{k,2}(f)(x_0)|dx\\
&\leqslant C\|b\|_{BMO}\sum_{k=1}^{m}M^2(T^{k,2}(f))(\tilde{x}).
\end{aligned}
$$

This completes the proof of Theorem 1.

Proof of Theorem 2. By Lemma 3-6, we get, for $f \in L_0^\infty(R^n)$ and $g \in L^{p'(\cdot)}(R^n)$,

$$\begin{aligned}
\int_{R^n} |T_b(f)(x)g(x)|dx &\leqslant C\int_{R^n} M^{\#}_{\Lambda_n}(T_b(f))(x)M(g)(x)|dx \\
&\leqslant C\int_{R^n} (T_b(f))^{\#}_{\delta}(x)M(g)(x)dx \\
&\leqslant C||b||_{BMO}\sum_{k=1}^{m}\int_{R^n} M^2(T^{k,2}(f))(x)M(g)(x)dx \\
&\leqslant C||b||_{BMO}\sum_{k=1}^{m}||M^2(T^{k,2}(f))||_{L^{p(\cdot)}}||M(g)||_{L^{p'(\cdot)}} \\
&\leqslant C||b||_{BMO}\sum_{k=1}^{m}||T^{k,2}(f)||_{L^{p(\cdot)}}||M(g)||_{L^{p'(\cdot)}} \\
&\leqslant C||b||_{BMO}||f||_{L^{p(\cdot)}}||g||_{L^{p'(\cdot)}},
\end{aligned}$$

thus, by Lemma 7,

$$||T_b(f)||_{L^{p(\cdot)}} \leqslant ||f||_{L^{p(\cdot)}}.$$

This completes the proof of Theorem 2.

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