FORCING DETOUR MONOPHONIC NUMBER OF A GRAPH

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Abstract. For a connected graph $G = (V, E)$ of order at least two, a chord of a path $P$ is an edge joining two non-adjacent vertices of $P$. A path $P$ is called a monophonic path if it is a chordless path. A longest $x - y$ monophonic path is called an $x - y$ detour monophonic path. A set $S$ of vertices of $G$ is a detour monophonic set of $G$ if each vertex $v$ of $G$ lies on an $x - y$ detour monophonic path for some elements $x$ and $y$ in $S$. The minimum cardinality of a detour monophonic set of $G$ is the detour monophonic number of $G$ and is denoted by $dm(G)$. A subset $T$ of a minimum detour monophonic set $S$ of $G$ is a forcing detour monophonic subset for $S$ if $S$ is the unique minimum detour monophonic set containing $T$. A forcing detour monophonic subset for $S$ of minimum cardinality is a minimum forcing detour monophonic subset for $S$. The forcing detour monophonic number $fdm(S)$ in $G$ is the cardinality of a minimum forcing detour monophonic subset of $S$. The forcing detour monophonic number of $G$ is $fdm(G) = \min\{fdm(S)\}$, where the minimum is taken over all minimum detour monophonic sets $S$ in $G$. We determine bounds for it and find the forcing detour monophonic number of certain classes of graphs. It is shown that for every pair $a, b$ of positive integers with $0 \leq a < b$ and $b \geq 2$, there exists a connected graph $G$ such that $fdm(G) = a$ and $dm(G) = b$.

1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of $G$ are denoted by $p$ and $q$ respectively. For basic graph theoretic terminology we refer to Harary [5]. The distance $d(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $u - v$ path in $G$. A $u - v$ path of length $d(u, v)$ is called a $u - v$ geodesic. For a vertex $v$ of $G$, the eccentricity $e(v)$ is the distance between $v$ and a vertex farthest from $v$. The minimum eccentricity among the vertices of $G$ is the radius, $rad G$ and the maximum eccentricity is its diameter, $diam G$ of $G$. Two vertices $u$ and $v$ of $G$ are called antipodal if $d(u, v) = diam G$. The neighborhood of a vertex $v$ is the set $N(v)$ consisting of all vertices $u$ which are adjacent with $v$. The closed neighborhood of a
vertex $v$ is the set $N[v] = N(v) \cup \{v\}$. A vertex $v$ is an \textit{extreme vertex} if the subgraph induced by its neighbors is complete.

The \textit{detour distance} $D(u,v)$ between two vertices $u$ and $v$ in $G$ is the length of a longest $u-v$ path in $G$. A $u-v$ path of length $D(u,v)$ is called a $u-v$ \textit{detour}. It is known that $D$ is a metric on the vertex set $V$ of $G$. The concept of detour distance was introduced in [1] and further studied in [2]. The closed detour interval $I_D[x,y]$ consists of $x$, $y$, and all the vertices in some $x-y$ detour of $G$. For $S \subseteq V$, $I_D[S]$ is the union of the sets $I_D[x,y]$ for all $x,y \in S$. A set $S$ of vertices of a graph $G$ is a \textit{detour set} if $I_D[S] = V$, and the minimum cardinality of a detour set is the \textit{detour number} $dn(G)$. The concept of detour number of a graph was introduced in [3] and further studied in [4].

A chord of a path $P$ is an edge joining two non-adjacent vertices of $P$. A path $P$ is called a \textit{monophonic path} if it is a chordless path. A longest $x-y$ monophonic path is called an $x-y$ \textit{detour monophonic path}. A set $S$ of vertices of $G$ is a \textit{detour monophonic set} if each vertex $v$ of $G$ lies on an $x-y$ detour monophonic path for some $x,y \in S$. The minimum cardinality of a detour monophonic set of $G$ is the \textit{detour monophonic number} of $G$ and is denoted by $dm(G)$. The detour monophonic set of cardinality $dm(G)$ is called $dm$-set. The detour monophonic number of a graph was introduced in [7] and further studied in [6]. There are interesting applications of these concepts to the problem of designing the route for a shuttle and communication network design.

For the graph $G$ given in Figure 1.1, $S_1 = \{z, w, v\}$, $S_2 = \{z, w, u\}$ and $S_3 = \{z, w, x\}$ are the minimum detour monophonic sets of $G$ and so $dm(G)=3$.

![Figure 1.1: Graph G](image_url)

A connected graph $G$ may contain more than one minimum detour monophonic sets. For example, the graph $G$ given in Figure 1.1 contains three minimum detour monophonic sets. For each minimum detour monophonic set $S$ in $G$ there is always some subset $T$ of $S$ that uniquely determines $S$ as the minimum detour monophonic set containing $T$. Such sets are called \textit{"forcing detour monophonic subsets}” and we discuss these sets in this paper.

The following theorems will be used in the sequel.

\textbf{Theorem 1.1.} [7] Each extreme vertex of a connected graph $G$ belongs to every detour monophonic set of $G$. 
Theorem 1.2. [7] For the complete graph $K_p$ ($p \geq 2$), $dm(K_p) = p$.

Theorem 1.3. [7] No cutvertex of a connected graph $G$ belongs to any minimum detour monophonic set of $G$.

Theorem 1.4. [7] For the cycle $C_n$ ($n \geq 3$),

\[
dm(C_n) = \begin{cases} 
2 & \text{if } n \text{ is even} \\
3 & \text{if } n \text{ is odd}
\end{cases}
\]

Throughout the paper $G$ denotes a connected graph with at least two vertices.

2. Forcing Detour Monophonic Number

Definition 2.1. Let $G$ be a connected graph and let $S$ be a minimum detour monophonic set of $G$. A subset $T$ of a minimum detour monophonic set $S$ of $G$ is a forcing detour monophonic subset for $S$ if $S$ is the unique minimum detour monophonic set containing $T$. A forcing detour monophonic subset for $S$ of minimum cardinality is a minimum forcing detour monophonic subset of $S$. The forcing detour monophonic number $fdm(S)$ in $G$ is the cardinality of a minimum forcing detour monophonic subset of $S$. The forcing detour monophonic number of $G$ is $fdm(G) = \min \{fdm(S)\}$, where the minimum is taken over all minimum detour monophonic sets $S$ in $G$.

Example 2.1. For the graph $G$ given in Figure 1.1, $S_1 = \{z, w, v\}$, $S_2 = \{z, w, u\}$ and $S_3 = \{z, w, x\}$ are the minimum detour monophonic sets of $G$. It is clear that $fdm(S_1) = 1$, $fdm(S_2) = 1$ and $fdm(S_3) = 1$ so that $fdm(G) = 1$. For the graph $G$ given in Figure 2.1, $S = \{y, v\}$ is the unique minimum detour monophonic set of $G$ and so $fdm(G) = 0$.

\[\begin{align*}
\quad & y & \quad & z & \quad & w & \quad & v \\
\quad & x & \quad & u
\end{align*}\]

Figure 2.1: Second graph $G$

The next theorem follows immediately from the definition of the detour monophonic number and forcing detour monophonic number of a graph $G$.

Theorem 2.2. For a connected graph $G$, $0 \leq f_{dm}(G) \leq dm(G) \leq p$. 
Remark 2.1. The bounds in Theorem 2.2 are sharp. For the graph $G$ given in Figure 2.1, $d_{dm}(G) = 0$. By Theorem 1.2, for the complete graph $K_p (p \geq 2)$, $d_{m}(K_p) = p$. The inequalities in Theorem 2.2 are strict. For the graph $G$ given in Figure 1.1, $d_{m}(G) = 3$ and $d_{dm}(G) = 1$. Thus $0 < d_{dm}(G) < d_{m}(G) < p$.

The following theorem is an easy consequence of the definitions of the detour monophonic number and forcing detour monophonic number. In fact, the theorem characterizes graphs $G$ for which the lower bound in Theorem 2.2 is attained and also graphs $G$ for which $d_{dm}(G) = 1$ and $d_{dm}(G) = d_{m}(G)$.

**Theorem 2.3.** Let $G$ be a connected graph. Then

(i) $d_{dm}(G) = 0$ if and only if $G$ has a unique minimum detour monophonic set.

(ii) $d_{dm}(G) = 1$ if and only if $G$ has at least two minimum detour monophonic sets, one of which is a unique minimum detour monophonic set containing one of its elements, and

(iii) $d_{dm}(G) = d_{m}(G)$ if and only if no minimum detour monophonic set of $G$ is the unique minimum detour monophonic set containing any of its proper subsets.

**Definition 2.4.** A vertex $v$ of a connected graph $G$ is said to be a detour monophonic vertex of $G$ if $v$ belongs to every minimum detour monophonic set of $G$.

We observe that if $G$ has a unique minimum detour monophonic set $S$, then every vertex in $S$ is a detour monophonic vertex of $G$. Also, if $x$ is an extreme vertex of $G$, then $x$ is a detour monophonic vertex of $G$. For the graph $G$ given in Figure 1.1, $w$ and $z$ are the detour monophonic vertices of $G$.

The following theorem and corollary follows immediately from the definitions of detour monophonic vertex and forcing detour monophonic subset of $G$.

**Theorem 2.5.** Let $G$ be a connected graph and let $\cup_{dm}$ be the set of relative complements of the minimum forcing detour monophonic subsets in their respective minimum detour monophonic sets in $G$. Then $\bigcap_{F \in \cup_{dm}} F$ is the set of detour monophonic vertices of $G$.

**Corollary 2.6.** Let $G$ be a connected graph and let $S$ be a minimum detour monophonic set of $G$. Then no detour monophonic vertex of $G$ belongs to any minimum forcing detour monophonic subset of $S$.

**Theorem 2.7.** Let $G$ be a connected graph and let $M$ be the set of all detour monophonic vertices of $G$. Then $d_{dm}(G) \leq d_{m}(G) - |M|$.

**Proof.** Let $S$ be any minimum detour monophonic set of $G$. Then $d_{m}(G) = |S|$, $M \subseteq S$ and $S$ is the unique minimum detour monophonic set containing $S - M$. Thus $d_{dm}(G) \leq |S - M| = |S| - |M| = d_{m}(G) - |M|$. \square

**Corollary 2.8.** If $G$ is a connected graph with $l$ extreme vertices, then $d_{dm}(G) \leq d_{m}(G) - l$. 
Remark 2.2. The bound in Theorem 2.7 is sharp. For the graph $G$ given in Figure 1.1, $dm(G) = 3$ and $f_{dm}(G) = 1$. Also, $M = \{w, z\}$ is the set of all detour monophonic vertices of $G$ and so $f_{dm}(G) = dm(G) - |M|$. Also the inequality in Theorem 2.7 can be strict. For the graph $G$ given in Figure 2.2, $S_1 = \{x, v\}$ and $S_2 = \{u, w\}$ are the minimum detour monophonic sets so that $dm(G) = 2$ and $f_{dm}(G) = 1$. Also, no vertex of $G$ is a detour monophonic vertex of $G$, we have $f_{dm}(G) < dm(G) - |M|$. 

**Theorem 2.9.** Let $G$ be a connected graph and let $S$ be a minimum detour monophonic set of $G$. Then no cutvertex of $G$ belongs to any minimum forcing detour monophonic subset of $S$.

**Proof.** Let $v$ be a cutvertex of $G$. By Theorem 1.3, $v$ does not belong to any minimum detour monophonic set of $G$. Since any minimum forcing detour monophonic subset of $S$ is a subset of $S$, the result follows from Theorem 2.5.

**Theorem 2.10.** If $G$ is a connected graph with $dm(G) = 2$, then $f_{dm}(G) \leq 1$.

**Proof.** Let $dm(G) = 2$. Then by Theorem 2.2, $f_{dm}(G) \leq 2$. Suppose that $f_{dm}(G) = 2$. Then by Theorems 2.3(i) and 2.3(iii), $G$ has at least two minimum detour monophonic sets and no minimum detour monophonic set of $G$ is the unique minimum detour monophonic set containing any of its proper subsets. Since $dm(G) = 2$, there exists a unique element, say $x$, is common for any two minimum detour monophonic sets, say $S_1$ and $S_2$. Let $S_1 = \{x, u\}$ and $S_2 = \{x, v\}$. Since $S_1$ is a minimum detour monophonic set, $v$ lies on an $x - u$ detour monophonic path. Similarly, since $S_2$ is a minimum detour monophonic set, $u$ lies on an $x - v$ detour monophonic path. Then the vertices $x, u$ and $v$ lie on a cycle. Let $C$ be a longest cycle containing the vertices $x, u$ and $v$. Then the length of $C$ is more than 4. If $C$ is an even cycle, then either $S_1$ or $S_2$ is not a detour monophonic set of $G$, which is a contradiction. If $C$ is an odd cycle, then any internal vertex of an $x - u$ geodesic does not lie on an $x - u$ detour monophonic path and so $S_1$ is not a detour monophonic set of $G$, which is a contradiction. Hence $f_{dm}(G) \leq 1$.

Now, we proceed to determine the forcing detour monophonic number of certain classes of graphs.
Theorem 2.11. For any cycle $C_n (n \geq 4)$,

$$f_{dm}(C_n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$$

Proof. Let $C_n : v_1, v_2, \ldots, v_m, v_{m+1}, \ldots, v_n, v_1$ be a cycle of order $n$.

Case (i) $n$ is even. Let $n = 2m$. Then every minimum detour monophonic set of $C_n$ consists of a pair of antipodal vertices and $C_n$ has exactly $m$ minimum detour monophonic sets. Clearly every minimum detour monophonic set containing one of its elements. Then by Theorem 2.3(ii), $f_{dm}(C_n) = 1$.

Case (ii) $n$ is odd. Let $n = 2m + 1$. It is clear that no two point set will form a detour monophonic set of $C_n$. Now, $\{v_1, v_2, v_3\}$ is a minimum detour monophonic set of $C_n$ and so $dm(C_n) = 3$. We observe that any minimum detour monophonic set of $C_n$ is any one of the following.

(i) any three consecutive vertices 
(ii) a vertex and its antipodal vertices 
(iii) any three non-adjacent vertices

Then clearly no minimum detour monophonic set of $C_n$ is the unique detour monophonic set containing any of its proper subsets. Hence by Theorem 2.3(iii), $f_{dm}(C_n) = dm(G) = 3$. $\square$

Theorem 2.12. For any complete graph $G = K_p (p \geq 2)$ or any non-trivial tree $G = T$, $f_{dm}(G) = 0$.

Proof. For $G = K_p$, it follows from Theorem 1.2 that the set of all vertices of $G$ is the unique minimum detour monophonic set of $G$. Now, it follows from Theorem 2.3(i) that $f_{dm}(G) = 0$. If $G$ is a non-trivial tree, then by Theorems 1.1 and 1.3, the set of all endvertices of $G$ is the unique minimum detour monophonic set of $G$ and so by Theorem 2.3(i), $f_{dm}(G) = 0$. $\square$

Theorem 2.13. For the complete bipartite graph $G = K_{m,n} (m, n \geq 2)$,

$$f_{dm}(G) = \begin{cases} 0 & \text{if } 2 = m < n \text{ or } 3 = m < n \\ 1 & \text{if } 2 = m = n \text{ or } 3 = m = n \\ 3 & \text{if } 4 = m \leq n \\ 4 & \text{if } 5 \leq m \leq n \end{cases}$$

Proof. We prove this theorem by considering four cases. Let $U = \{u_1, u_2, \ldots, u_m\}$ and $W = \{w_1, w_2, \ldots, w_n\}$ be the bipartition of $G$, where $m \leq n$.

Case 1. $2 = m = n$ or $3 = m = n$. Then $U$ and $W$ are the only minimum detour monophonic set of $G$ and so by Theorem 2.3(iii), $f_{dm}(G) = 1$.

Case 2. $2 = m < n$ or $3 = m < n$. Then $U$ is the unique minimum detour monophonic set of $G$ and so by Theorem 2.3(i), $f_{dm}(G) = 0$. 

Case 3. $4 = m \leq n$. Then $U$ is the unique minimum detour monophonic set of $G$ and so by Theorem 2.3(iii), $f_{dm}(G) = 3$.

Case 4. $5 \leq m \leq n$. Then $U$ is the unique minimum detour monophonic set of $G$ and so by Theorem 2.3(iii), $f_{dm}(G) = 4$.
Case 3. $4 = m \leq n$. If $4 = m = n$, then the minimum detour monophonic sets of $G$ are $U$, $W$ and any set got by choosing any two elements from each of $U$ and $W$. Clearly, neither an 1-element nor a 2-element subset of any minimum detour monophonic set is a forcing subset and any 3-element subset of $U$ is a forcing subset for $U$. Hence $f_{dm}(G) = 3$. If $4 = m < n$, then the minimum detour monophonic sets of $G$ are $U$ and any set got by choosing any two elements from each of $U$ and $W$. Then similar to the above argument, we have $f_{dm}(G) = 3$.

Case 4. $5 \leq m \leq n$. Then any minimum detour monophonic set is got by choosing any two elements from each of $U$ and $W$, and $G$ has at least two minimum detour monophonic sets. Hence $dm(G) = 4$. Clearly, no minimum detour monophonic set of $G$ is the unique minimum detour monophonic set containing any of its proper subsets. Then by Theorem 2.3(iii), we have $f_{dm}(G) = dm(G) = 4$. □

Theorem 2.14. For every pair $a$, $b$ of positive integers with $0 \leq a < b$ and $b \geq 2$, there exists a connected graph $G$ such that $f_{dm}(G) = a$ and $dm(G) = b$.

Proof. If $a = 0$, let $G = K_b$. Then by Theorem 2.12, $f_{dm}(G) = 0$ and by Theorem 1.2, $dm(G) = b$. Thus we assume that $0 < a < b$. We consider four cases.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{G.png}
\caption{G}
\end{figure}

Case 1. $a = 1$. If $b = 2$, then for any even cycle $G$, $f_{dm}(G) = a$ by Theorem 2.11 and $dm(G) = b$ by Theorem 1.4. So, we assume that $b \geq 3$. Let $G$ be the graph obtained from the cycle $C_5 : v_1, v_2, v_3, v_4, v_5, v_1$ of order 5 by adding $b - 2$ new vertices $u_1, u_2, \ldots, u_{b-2}$ and joining each $u_i (1 \leq i \leq b - 2)$ to $v_3$; and joining two more edges $v_2v_5$ and $v_2v_4$. The graph $G$ is shown in Figure 2.3. Let $S = \{u_1, u_2, \ldots, u_{b-2}, v_1\}$ be the set of all extreme vertices of $G$. By Theorem 1.1, every detour monophonic set of $G$ contains $S$. It is clear that $S$ is not a detour monophonic set of $G$. It is easily verified that $S_1 = S \cup \{v_3\}$, $S_2 = S \cup \{v_2\}$ and $S_3 = S \cup \{v_4\}$ are the minimum detour monophonic sets of $G$. Hence $dm(G) = b$. Moreover, since $S_1$ is the unique minimum detour monophonic set containing $\{v_3\}$, it follows that $f_{dm}(S_1) = 1$ and so $f_{dm}(G) = 1$.

Case 2. $a = 2$. Then $b \geq 3$. Let $G$ be the graph obtained from the cycle $C_5 : v_1, v_2, v_3, v_4, v_5, v_1$ of order 5 by adding $b - 2$ new vertices $u_1, u_2, \ldots, u_{b-2}$ and joining each $u_i (1 \leq i \leq b - 2)$ to $v_3$. The graph $G$ is shown in Figure 2.4. Let $S = \{u_1, u_2, \ldots, u_{b-2}\}$ be the set of all extreme vertices of $G$. By Theorem 1.1, every
detour monophonic set of $G$ contains $S$. Clearly $S$ is not a detour monophonic set of $G$. Also $S \cup \{x\}$, where $x \in V(G) - S$, is not a detour monophonic set of $G$. It is easily verified that $S_1 = S \cup \{v_1, v_3\}$, $S_2 = S \cup \{v_1, v_4\}$, $S_3 = S \cup \{v_2, v_4\}$, $S_4 = S \cup \{v_1, v_2\}$, $S_5 = S \cup \{v_4, v_5\}$ and $S_6 = S \cup \{v_2, v_5\}$ are the minimum detour monophonic sets of $G$. Hence $d_m(G) = b$. If $x$ is an element of $S_i (1 \leq i \leq 6)$, then $\{x\}$ is a subset of at least two minimum detour monophonic sets of $G$. Hence it follows from Theorem 2.3(i) and (ii) that $f_{d_m}(G) \geq 2$. Since $S_1$ is the unique minimum detour monophonic set containing $\{v_1, v_3\}$, we have $f_{d_m}(G) = 2$.

**Figure 2.4: G**

**Case 3.** $a \geq 3$ and $b = a + 1$. For each integer $i$ with $0 \leq i \leq b$, let $F_i : u_i, v_i$ be a path of order 2. Let $G$ be the graph obtained from $F_i (0 \leq i \leq b)$ by adding 2$b$ edges $u_0u_i, v_0v_i$ for all $j$ with $1 \leq j \leq b$. The graph $G$ is shown in Figure 2.5. First we show that $d_m(G) = b$. Let $U = \{u_1, u_2, \ldots, u_b\}$ and $W = \{v_1, v_2, \ldots, v_b\}$. We observe that a set $S$ of vertices of $G$ is a minimum detour monophonic set only if $S$ has the following two properties: (1) $S$ contains exactly one vertex from each set $\{u_i, v_j\} (1 \leq j \leq b)$, and (2) $S \cap U \neq \emptyset$ and $S \cap W \neq \emptyset$. Then (1) implies that $d_m(G) \geq b$. Since $S' = \{u_2, u_3, \ldots, u_b, v_1\}$ is a detour monophonic set of $G$ with $|S'| = b$, it follows that $d_m(G) = b = a + 1$.

Now, we prove that $f_{d_m}(G) = a$. First assume that a minimum detour monophonic set contains at least one vertex from $U$ and $W$. Without loss of generality, let $S_1 = \{u_2, u_3, \ldots, u_b, v_1\}$ be a minimum detour monophonic set of $G$. We claim that $f_{d_m}(G) = b - 1$. Let $T$ be a subset of $S_1$ such that $|T| \leq b - 2$. Then there exist at least two vertices, say $x, y \in S_1$, such that $x, y \notin T$. Suppose that $x = v_1$ and $y = u_j$ for some $j (2 \leq j \leq b)$. Now, $S_2 = (S_1 - \{v_1, u_j\}) \cup \{u_1, v_j\}$ satisfies (1) and (2) and so $S_2$ is a minimum detour monophonic set such that $T \subseteq S_2$. Therefore $S_1$ is not the unique minimum detour monophonic set containing $T$ and so $T$ is not a forcing subset of $S_1$. Suppose that $x = u_i$ for some $i (2 \leq i \leq b)$ and $y = u_j$ for some $j (2 \leq j \leq b)$ and $i \neq j$. Now, $S_3 = (S_1 - \{u_i, u_j\}) \cup \{v_1, v_j\}$ satisfies (1) and (2) and so $S_3$ is a minimum detour monophonic set containing $T$. Hence $T$ is not a forcing subset of $S_1$ and so $f_{d_m}(S_1) \geq b - 1$. Now, it is clear that $S_1$ is the unique minimum detour monophonic set containing $\{u_2, u_3, \ldots, u_b\}$ so that $f_{d_m}(S_1) = b - 1$. Hence it follows that $f_{d_m}(G) = b - 1 = a$. 
Case 4. \(a \geq 3\) and \(b \geq a + 2\). Let \(F_i : l_i, m_i, n_i, o_i, p_i, q_i, r_i, \ell_i (1 \leq i \leq a)\) be "\(a\)" number of copies of \(C_7\). Let \(G\) be the graph obtained from \(F_i (1 \leq i \leq a)\) by identifying the vertices \(o_{i-1}\) of \(F_{i-1}\) and \(l_i\) of \(F_i (2 \leq i \leq a)\); and adding \(b - a\) new vertices \(z_1, z_2, \ldots, z_{b-a-1}, z\) and joining each \(z_i (1 \leq i \leq b - a - 1)\) to \(l_1\); and joining each \(m_i, n_i (1 \leq i \leq a)\) to the vertices \(r_i, q_i, p_i (1 \leq i \leq a)\); and joining the vertex \(z\) to \(a_0\).

The graph \(G\) is shown in Figure 2.6. Let \(S = \{z_1, z_2, \ldots, z_{b-a-1}, z\}\) be the set of all extreme vertices of \(G\). Then by Theorem 1.1, every detour monophonic set of \(G\) contains \(S\). Clearly, \(S\) is not a detour monophonic set of \(G\). We observe that every minimum detour monophonic set contains exactly one vertex from \(\{m_i, n_i\}\) for every \(i (1 \leq i \leq a)\). Thus \(dm(G) \geq b\). Since \(S_1 = S \cup \{m_1, m_2, \ldots, m_a\}\) is a detour monophonic set of \(G\), it follows that \(dm(G) = b\).

Next we show that \(fdm(G) = a\). Since every minimum detour monophonic set of \(G\) contains \(S\), it follows from Theorem 2.7 that \(fdm(G) \leq dm(G) - |S| = b - (b - a) = a\). Now, since \(dm(G) = b\) and every minimum detour monophonic set of \(G\) contains \(S\), it is easily seen that every minimum detour monophonic set \(S'\) of \(G\) is of the form \(S' = S \cup \{x_1, x_2, \ldots, x_a\}\), where \(x_i \in \{m_i, n_i\}\) for every \(i (1 \leq i \leq a)\). Let \(T\) be any proper subset of \(S'\) with \(|T| < a\). Then there is a vertex \(x \in S' - S\) such that \(x \notin T\). If \(x = m_i (1 \leq i \leq a)\), then \(S'' = (S' - \{m_i\}) \cup \{n_i\}\) is a minimum detour monophonic set containing \(T\). Similarly, if \(x = n_j (1 \leq j \leq a)\), then \(S''' = (S' - \{n_j\}) \cup \{m_j\}\) is a minimum detour monophonic set containing \(T\). Thus \(S'\) is not the unique minimum detour monophonic set containing \(T\) and so \(T\) is not a forcing subset of \(S'\). This is true for all minimum detour monophonic sets of \(G\) and so \(fdm(G) = a\). \(\Box\)
Figure 2.6: G

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