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FIXED POINTS OF WEAKLY COMPATIBLE MAPPINGS IN G-METRIC SPACES SATISFYING COMMON LIMIT RANGE PROPERTY

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Abstract. The aim of this paper is to prove an integral type fixed point theorem for a pair of weakly compatible mappings in G-metric space employing the notion of common limit range property. We extend our main result to two finite families of self mappings by using the notion of commuting pairwise. We also establish some fixed point results under ϕ-contractions. Illustrative examples are given to support our results. Our results improve, extend and generalize several previously known fixed point theorems in the existing literature.

1. Introduction

The study of common fixed point theorems satisfying contractive conditions has a wide range of applications in different areas such as, variational and linear inequality problems, optimization and parameterize estimation problems and many others. One of the simplest and most useful results in the fixed point theory is the Banach-Caccioppoli contraction principle. This theorem provides a technique for solving a variety of applied problems in mathematical sciences and engineering. Banach contraction principle has been generalized in different spaces by mathematicians over the years. Mustafa and Sims [24, 25] proposed a new class of generalized metric spaces, which are called as G-metric spaces. In this type of spaces a non-negative real number is assigned to every triplet of elements. Many mathematicians studied extensively various results on G-metric spaces by using the concept of weak commutativity, compatibility, non-compatibility and weak compatibility for single valued mappings satisfying different contractive conditions (cf. [2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 17, 18, 19, 20, 21, 22, 23, 28, 29, 30, 31, 32]).

Branciari [11] obtained a fixed point result for a single mapping satisfying an analogue of Banach's contraction principle for an integral type inequality. This influenced many authors, and consequently, a number of new results in this line

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followed (see, for example [13, 27, 34, 35, 36]). Later on, Aydi [7] proved an integral type fixed point theorem for two self mappings and extended the results of Brianciari [11] to the class of G-metric spaces. The first fixed point theorem without any continuity requirement was proved by Abbas and Rhoades [5] in which they utilized the notion of non-commuting mappings for the existence of fixed points. Shatanawi [29] proved some interesting fixed point results by using ϕ-contractive condition and generalized the results of Abbas and Rhoades [5]. Most recently, Mustafa et al. [18] defined the notion of the property (E.A) in G-metric space and proved some fixed point results (also see [37]).

In this paper, firstly we prove an integral type fixed point theorem for a pair of weakly compatible mappings in G-metric space satisfying the common limit range property which is initiated by Sintunavarat and Kumam [33]. We extend our main result to two finite families of self mappings by using the notion of pairwise commuting. We also present some fixed point results in G-metric spaces satisfying ϕ-contractions. Some related examples are furnished to support our results.

2. Preliminaries

Definition 2.1. [25] Let X be a non-empty set and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

(G-1) $G(x, y, z) = 0$ if $x = y = z$;

(G-2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;

(G-3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;

(G-4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (symmetry in all three variables);

(G-5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$, (rectangle inequality).

Then the function G is called a generalized metric, or, more specially, a G-metric on X, and the pair (X, G) is called a G-metric space.

Definition 2.2. [25] Let (X, G) be a G-metric space then for $x_0 \in X$, $r > 0$, the G-ball with center x_0 and radius r is

$$B_G(x_0, r) = \{y \in X : G(x_0, y, y) < r\}.$$

Proposition 2.1. 1 [25] *Let (X, G) be a G-metric space then for any $x_0 \in X$, $r > 0$, we have*

1. *if $G(x_0, x, y) < r$ then $x, y \in B_G(x_0, r)$,*
2. *if $y \in B_G(x_0, r)$ then there exists a $\delta > 0$ such that $B_G(y, \delta) \subseteq B_G(x_0, r)$.*

It follows from (2) of Proposition 2.1 that the family of all G-balls,

$$B = \{B_G(x, r) : x \in X, r > 0\}$$

is the base of a topology $\tau(G)$ on X, the G-metric topology.

The following are examples of G-metric spaces.

Example 2.1. [25] Let (X, d) be a usual metric space, then (X, G_s) and (X, G_m) are G-metric space, where

$$\begin{aligned} G_s(x, y, z) &= d(x, y) + d(y, z) + d(x, z), \\ G_m(x, y, z) &= \max\{d(x, y), d(y, z), d(x, z)\}, \end{aligned}$$

for all $x, y, z \in X$.

Definition 2.3. [25] Let (X, G) be a G-metric space, and let (x_n) be a sequence of points of X. We say that the sequence (x_n) is G-convergent to $x \in X$ if

$$\lim_{n,m\to+\infty} G(x, x_n, x_m) = 0,$$

that is, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \geq N$. We call x the limit of the sequence and write $x_n \to x$ or $\lim_{n\to+\infty} x_n = x$.

It has been shown in [25] that the G-metric induces a Hausdorff topology and the convergence described in the above definition is relative to this topology. The topology being Hausdorff, a sequence can converge at most to one point.

Proposition 2.2. [25] *Let (X, G) be a G-metric space. The following are equivalent:*

1. *(x_n) is G-convergent to x;*
2. *$G(x_n, x_n, x) \to 0$ as $n \to +\infty$;*
3. *$G(x_n, x, x) \to 0$ as $n \to +\infty$;*
4. *$G(x_n, x_m, x) \to 0$ as $n, m \to +\infty$.*

Definition 2.4. [25] Let (X, G) be a G-metric space. A sequence (x_n) is called a G-Cauchy sequence if, for any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq N$, that is, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to 0$.

Proposition 2.3. [24] *Let (X, G) be a G-metric space. Then the following are equivalent:*

1. *the sequence (x_n) is G-Cauchy;*
2. *for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \geq N$.*

Proposition 2.4. [25] *Let (X, G) be a G-metric space. Then, the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Definition 2.5. [25] A G-metric space (X, G) is called G-complete if every G-Cauchy sequence is G-convergent in (X, G).

Proposition 2.5. [25] *Let (X, G) be a G-metric space. Then, for any $x, y, z, a \in X$ it follows that:*

1. *If $G(x, y, z) = 0$, then $x = y = z$;*
2. $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$;
3. $G(x, y, y) \leq 2G(y, x, x)$;
4. $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$;
5. $G(x, y, z) \leq \frac{2}{3}\left(G(x, y, a) + G(x, a, z) + G(a, y, z)\right)$;
6. $G(x, y, z) \leq G(x, a, a) + G(y, a, a) + G(z, a, a)$.

Definition 2.6. [1] Let f and g be self mappings of a non-empty set X. If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

Proposition 2.6. [1] *Let f and g be weakly compatible self mappings of a non-empty set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

Definition 2.7. A pair (f, g) of self mappings of a G-metric space (X, G) is said to satisfy the property (E.A) if there exists a sequence $\{x_n\}$ such that $\{fx_n\}$ and $\{gx_n\}$ G-converge to z for some $z \in X$, that is,

$$\lim_{n\to\infty} G(fx_n, fx_n, z) = \lim_{n\to\infty} G(gx_n, gx_n, z) = 0.$$

Inspired by Sintunavarat and Kumam [33], we define the "common limit range property" with respect to mapping g (denoted by (CLRg) property) in G-metric space as follows:

Definition 2.8. A pair (f, g) of self mappings of a G-metric space (X, G) is said to satisfy the (CLRg) property if there exists a sequence $\{x_n\}$ such that $\{fx_n\}$ and $\{gx_n\}$ G-converge to gu for some $u \in X$, that is,

$$\lim_{n\to\infty} G(fx_n, fx_n, gu) = \lim_{n\to\infty} G(gx_n, gx_n gu) = 0.$$

Example 2.2. Let $X = [0, +\infty)$ and $G : X \times X \times X \to R^+$ be the G-metric defined as follows:

$$G(x, y, z) = \max\{| x - y |, | y - z |, | z - x |\},$$

for all $x, y, z \in X$. Then (X, G) be a G-metric space. Define self mappings f and g on X by $f(x) = x + 2$ and $g(x) = 3x$ for all $x \in X$. Let a sequence $\{x_n\} = \{1 + \frac{1}{n}\}_{n\in\mathbb{N}}$ in X, we have

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 3 = g(1) \in X,$$

which shows that f and g satisfy the (CLRg) property.

Example 2.3. The conclusion of Example 2.2 remains true if the self mappings f and g are defined on X by $f(x) = \frac{x}{4}$ and $g(x) = \frac{x}{5}$ for all $x \in X$. Consider a sequence $\{x_n\} = \{\frac{1}{n}\}_{n\in\mathbb{N}}$ in X. Since

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0 = g(0) \in X,$$

therefore f and g satisfy the (CLRg) property.

Definition 2.9. [14] Two families of self mappings $\{f_i\}_{i=1}^m$ and $\{g_k\}_{k=1}^n$ are said to be pairwise commuting if

1. $f_i f_j = f_j f_i$ for all $i, j \in \{1, 2, \ldots, m\}$;
2. $g_k g_l = g_l g_k$ for all $k, l \in \{1, 2, \ldots, n\}$;
3. $f_i g_k = g_k f_i$ for all $i \in \{1, 2, \ldots, m\}$ and $k \in \{1, 2, \ldots, n\}$.

3. Results

Recently, Aydi [7] proved the following result:

Theorem 3.1. [7, Theorem 3.1] *Let (X, G) be a G-metric space and $f, g : X \to X$ such that*

$$\int_0^{G(fx,fy,fz)} \varphi(t)dt \le \alpha \int_0^{G(gx,gy,gz)} \varphi(t)dt, \tag{3.1}$$

for all $x, y, z \in X$, where $\alpha \in [0, 1)$ and $\varphi : [0, +\infty) \to [0, +\infty)$ is a Lebesgue integrable mapping which is summable, non-negative and such that for each $\epsilon > 0$,

$$\int_0^{\epsilon} \varphi(t)dt > 0. \tag{3.2}$$

Assume that $f(X) \subset g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Now we prove the next result:

Theorem 3.2. *Let (X, G) be a G-metric space and the pair (f, g) of self mappings is weakly compatible satisfying conditions (3.1)-(3.2) of Theorem 3.1. If the pair (f, g) satisfies the (CLRg) property then f and g have a unique common fixed point in X.*

Proof. Since the pair (f, g) satisfies the (CLRg) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gu,$$

for some $u \in X$. We show that $fu = gu$. On using inequality (3.1), we get

$$\int_0^{G(fx_n, fx_n, fu)} \varphi(t)dt \leq \alpha \int_0^{G(gx_n, gx_n, gu)} \varphi(t)dt.$$

Letting $n \to +\infty$, we have

$$\int_0^{G(gu, gu, fu)} \varphi(t)dt \leq \alpha \int_0^{G(gu, gu, gu)} \varphi(t)dt,$$

and so

$$\int_0^{G(gu, fu, fu)} \varphi(t)dt \leq 0,$$

which implies that $G(gu, fu, fu) = 0$, hence $gu = fu$. Next, we let $w = fu = gu$. Since the pair (f, g) is weakly compatible, therefore $fw = fgu = gfu = gw$. Now we assert that $w = fw$. On using inequality (3.1), we get

$$\int_0^{G(fw, fw, fu)} \varphi(t)dt \leq \alpha \int_0^{G(gw, gw, gu)} \varphi(t)dt,$$

or, equivalently,

$$\int_0^{G(fw, fw, w)} \varphi(t)dt \leq \alpha \int_0^{G(fw, w, w)} \varphi(t)dt,$$

which holds unless $G(fw, w, w) = 0$, hence $w = fw = gw$. Therefore, w is a common fixed point of the mappings f and g. Uniqueness of the common fixed point is an easy consequence of inequality (3.1). □

Example 3.1. Let $X = [0, 1)$ and let $G : X \times X \times X \to R^+$ be the G-metric defined as follows:

$$G(x, y, z) = \max\{| x - y |, | y - z |, | z - x |\},$$

for all $x, y, z \in X$. Then (X, G) is a G-metric space. Define the self mappings f and g by

$$f(x) = \begin{cases} \frac{x}{8}, & \text{if } x \in [0, \frac{1}{2}); \\ \frac{x}{6}, & \text{if } x \in (\frac{1}{2}, 1). \end{cases} \quad g(x) = \begin{cases} \frac{x}{4}, & \text{if } x \in [0, \frac{1}{2}); \\ \frac{x}{2}, & \text{if } x \in (\frac{1}{2}, 1). \end{cases}$$

Let $\{x_n\} = \left\{\frac{1}{n}\right\}_{n\geq 2}$. We have

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0 = g(0) \in X.$$

Hence the pair (f, g) satisfy the (CLRg) property. It is noted that $f(X) = [0, \frac{1}{16}) \cup (\frac{1}{12}, \frac{1}{6}) \not\subseteq [0, \frac{1}{8}) \cup (\frac{1}{4}, \frac{1}{2}) = g(X)$ which shows that $f(X)$ and $g(X)$ are not closed subsets of X. Thus, all the conditions of Theorem 3.2 are satisfied and 0 is a unique common fixed point of the pair (f, g).

Consider $\varphi(t) = 2t$ and $\alpha = \frac{1}{2}$. Without loss of generality, take $x \leq y \leq z$. To prove (3.1), we distinguish the following cases:

Case 1: $x, y, z \in [0, \frac{1}{2})$. We have

$$\int_0^{G(fx,fy,fz)} \varphi(t)dt = \left(\frac{z}{8} - \frac{x}{8}\right)^2 \leq \frac{1}{2}\left(\frac{z}{4} - \frac{x}{4}\right)^2 = \alpha \int_0^{G(gx,gy,gz)} \varphi(t)dt.$$

Case 2: $\left(x, y \in [0, \frac{1}{2}) \text{ and } z \in \left[\frac{1}{2}, 1\right)\right)$ or $\left(x \in [0, \frac{1}{2}) \text{ and } y, z \in [\frac{1}{2}, 1)\right)$. We have

$$\int_0^{G(fx,fy,fz)} \varphi(t)dt = \left(\frac{z}{6} - \frac{x}{8}\right)^2 \leq \frac{1}{2}\left(\frac{z}{2} - \frac{x}{4}\right)^2 = \alpha \int_0^{G(gx,gy,gz)} \varphi(t)dt.$$

Case 3: $x, y, z \in \left[\frac{1}{2}, 1\right)$. We have

$$\int_0^{G(fx,fy,fz)} \varphi(t)dt = \left(\frac{z}{6} - \frac{x}{6}\right)^2 \leq \frac{1}{2}\left(\frac{z}{2} - \frac{x}{2}\right)^2 = \alpha \int_0^{G(gx,gy,gz)} \varphi(t)dt.$$

Here it is worth noting that Theorem 3.1 cannot be used in the context of this example as Theorem 3.2 never requires any condition on the containment of ranges amongst involved mappings and completeness (or closedness) of the underlying space (or subspaces).

Remark 3.1. Theorem 3.2 improves the main result of Aydi [7, Theorem 3.1].

Our next result extends Theorem 3.2 to two finite families of self mappings in G-metric space.

Corollary 3.1. *Let (X, G) be a G-metric space and $\{f_1, f_2, \ldots, f_p\}$, $\{g_1, g_2, \ldots, g_q\}$ be two finite families of self mappings such that $f = f_1 f_2 \ldots f_p$ and $g = g_1 g_2 \ldots g_q$ satisfying conditions (3.1)-(3.2) of Theorem 3.1. Suppose that the pair (f, g) satisfies the (CLRg) property.*

Moreover, if the family $\{f_i\}_{i=1}^p$ commutes pairwise with the family $\{g_i\}_{j=1}^q$, then (for all $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, q\}$) f_i and g_j have a unique common fixed point in X.

Proof. The proof of this theorem is similar to that of Theorem 2.2 contained in Imdad et al. [14], hence details are avoided. □

Remark 3.2. Corollary 3.1 extends the result of Aydi [7].

By setting $f_1 = f_2 = \ldots = f_p = f$ and $g_1 = g_2 = \ldots = g_q = g$ in Corollary 3.1, we deduce the following:

Corollary 3.2. *Let (X, G) be a G-metric space and the pair (f, g) of self mappings satisfies the (CLRg) property such that*

$$\int_0^{G(f^p x, f^p y, f^p z)} \varphi(t)dt \leq \alpha \int_0^{G(g^q x, g^q y, g^q z)} \varphi(t)dt, \tag{3.3}$$

for all $x, y, z \in X$, $\alpha \in [0, 1)$, p, q are fixed positive integers and $\varphi : [0, +\infty) \to [0, +\infty)$ is a Lebesgue integrable mapping which is summable, non-negative and such that for each $\epsilon > 0$,

$$\int_0^{\epsilon} \varphi(t)dt > 0. \tag{3.4}$$

Then f and g have a unique common fixed point in X if the pair (f^p, g^q) commutes pairwise.

On taking $\varphi(t) = 1$ in Theorem 3.2, we get the following result:

Corollary 3.3. *Let (X, G) be a G-metric space and the pair (f, g) of self mappings is weakly compatible such that*

$$G(fx, fy, fz) \leq \alpha G(gx, gy, gz), \tag{3.5}$$

for all $x, y, z \in X$ and $\alpha \in [0, 1)$. If the pair (f, g) satisfies the (CLRg) property then f and g have a unique common fixed point in X.

Remark 3.3. Corollary 3.3 improves the result of Choudhury et al. [12, Corollary 3.1].

Now we prove our next result satisfying ϕ-contraction integral type conditions in G-metric spaces.

Following by Matkowski [16], let Φ be the set of all functions ϕ such that $\phi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing function with $\lim_{n\to+\infty} \phi^n(t) = 0$ for all $t \in (0, +\infty)$. If $\phi \in \Phi$, then ϕ is called a Φ-mapping. If ϕ is a Φ-mapping, then it is easy matter to show that:

1. $\phi(t) < t$ for all $t \in (0, +\infty)$,

2. $\phi(0) = 0$.

In the rest of this paper, by ϕ we mean a Φ-mapping. Now, we prove our next result.

Theorem 3.3. *Let (X,G) be a G-metric space and the pair (f,g) of self mappings is weakly compatible such that*

$$\int_0^{G(fx,fy,fz)} \varphi(t)dt \leq \phi\left(\int_0^{\mathcal{L}(x,y,z)} \varphi(t)dt\right), \tag{3.6}$$

for all $x,y,z \in X$, $\phi \in \Phi$ and $\varphi : [0,+\infty) \to [0,+\infty)$ is a Lebesgue integrable mapping which is summable, non-negative and such that for each $\epsilon > 0$,

$$\int_0^{\epsilon} \varphi(t)dt > 0,$$

where

$$\mathcal{L}(x,y,z) = \max\left\{ \begin{array}{c} G(gx,gy,gz), G(gx,fx,fx), \\ G(gy,fy,fy), G(gz,fz,fz) \end{array} \right\}, \tag{3.7}$$

or

$$\mathcal{L}(x,y,z) = \max\left\{ \begin{array}{c} G(gx,gy,gz), G(gx,gx,fx), \\ G(gy,gy,fy), G(gz,gz,fz) \end{array} \right\}. \tag{3.8}$$

If the pair (f,g) satisfies the (CLRg) property then f and g have a unique common fixed point in X.

Proof. If the pair (f,g) satisfies the (CLRg) property, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gu,$$

for some $u \in X$. We show that $fu = gu$. Suppose that $fu \neq gu$, then using inequality (3.5), we get

$$\int_0^{G(fx_n,fx_n,fu)} \varphi(t)dt \leq \phi\left(\int_0^{\mathcal{L}(x_n,x_n,u)} \varphi(t)dt\right), \tag{3.9}$$

where

$$\mathcal{L}(x_n,x_n,u) = \max\left\{ \begin{array}{c} G(gx_n,gx_n,gu), G(gx_n,fx_n,fx_n), \\ G(gx_n,fx_n,fx_n), G(gu,fu,fu) \end{array} \right\}$$

Taking limit as $n \to +\infty$ in (3.9), we have

$$\int_0^{G(gu,gu,fu)} \varphi(t)dt \leq \phi\left(\lim_{n\to+\infty} \int_0^{\mathcal{L}(x_n,x_n,u)} \varphi(t)dt\right), \tag{3.10}$$

where

$$\begin{aligned}\lim_{n\to+\infty}\mathcal{L}(x_n,x_n,u) &= \max\left\{\begin{array}{c}G(gu,gu,gu),G(gu,gu,gu),\\ G(gu,gu,gu),G(gu,fu,fu)\end{array}\right\}\\ &= G(gu,fu,fu).\end{aligned}$$

So, from (3.10), we get

$$\int_0^{G(gu,gu,fu)}\varphi(t)dt \le \phi\left(\int_0^{G(gu,fu,fu)}\varphi(t)dt\right).$$

Therefore,

$$\int_0^{G(gu,gu,fu)}\varphi(t)dt < \int_0^{G(gu,fu,fu)}\varphi(t)dt. \tag{3.11}$$

Similarly, one can obtain

$$\int_0^{G(gu,fu,fu)}\varphi(t)dt < \int_0^{G(gu,gu,fu)}\varphi(t)dt. \tag{3.12}$$

From (3.11) and (3.12), we have

$$\int_0^{G(gu,gu,fu)}\varphi(t)dt < \int_0^{G(gu,fu,fu)}\varphi(t)dt < \int_0^{G(gu,gu,fu)}\varphi(t)dt,$$

which is contradiction, hence $fu = gu$. Suppose that $w = fu = gu$. Since the pair (f,g) is weakly compatible and $w = fu = gu$, therefore $fw = fgu = gfu = gw$. Finally, we assert that $w = fw$. Let, on contrary $w \neq fw$, then inequality (3.5) implies

$$\int_0^{G(fw,fw,fu)}\varphi(t)dt \le \phi\left(\int_0^{\mathcal{L}(w,w,u)}\varphi(t)dt\right), \tag{3.13}$$

where

$$\begin{aligned}\mathcal{L}(w,w,u) &= \max\{G(gw,gw,gu),G(gw,fw,fw),G(gw,fw,fw),G(gu,fu,fu)\}\\ &= \max\{G(fw,fw,w),G(fw,fw,fw),G(fw,fw,fw),G(w,w,w)\}\\ &= G(fw,fw,w).\end{aligned}$$

Therefore (3.13) implies

$$\begin{aligned}\int_0^{G(fw,fw,w)}\varphi(t)dt &\le \phi\left(\int_0^{G(fw,fw,w)}\varphi(t)dt\right)\\ &< \int_0^{G(fw,fw,w)}\varphi(t)dt,\end{aligned}$$

which is a contradiction, thus $w = fw = gw$. Therefore, w is a common fixed point of the mappings f and g. The proof is similar for condition (3.8), hence the details are omitted. Uniqueness of the common fixed point is easy consequences of inequalities (3.5)-(3.6). $\square$

Remark 3.4. Define the function $\phi : [0, +\infty) \to [0, +\infty)$ by $\phi(t) = \alpha t$, for all $t \geq 0$, where $\alpha \in [0, 1)$. Then Example 3.1 also holds all the conditions of Theorem 3.3 and have a unique common fixed point at $x = 0$.

Theorem 3.4. *Let (X, G) be a G-metric space and the pair (f, g) of self mappings is weakly compatible such that*

$$\int_0^{G(fx,fy,fz)} \varphi(t)dt \leq \alpha \int_0^{\mathcal{L}(x,y,z)} \varphi(t)dt, \tag{3.14}$$

for all $x, y, z \in X$, $\alpha \in [0, 1)$ and $\varphi : [0, +\infty) \to [0, +\infty)$ is a Lebesgue integrable mapping which is summable, non-negative and such that for each $\epsilon > 0$,

$$\int_0^{\epsilon} \varphi(t)dt > 0,$$

where

$$\mathcal{L}(x, y, z) = \max \left\{ \begin{array}{c} G(gx, gy, gz), G(gx, fx, fx), \\ G(gy, fy, fy), G(gz, fz, fz) \end{array} \right\}, \tag{3.15}$$

or

$$\mathcal{L}(x, y, z) = \max \left\{ \begin{array}{c} G(gx, gy, gz), G(gx, gx, fx), \\ G(gy, gy, fy), G(gz, gz, fz) \end{array} \right\}. \tag{3.16}$$

If the pair (f, g) satisfies the (CLRg) property then f and g have a unique common fixed point in X.

Proof. On setting $\phi(t) = \alpha t$, where $\alpha \in [0, 1)$ in Theorem 3.3, the result follows. $\square$

Remark 3.5. Theorems 3.3-3.4 generalize the results of Shatanawi [31, Theorem 3.1] and Abbas and Rhoades [5, Theorems 2.3-2.4].

Remark 3.6. The conclusions of Theorems 3.3-3.4 remain true if $\mathcal{L}(x, y, z)$ is replaced by one of the following: for all $x, y, z \in X$

$$\mathcal{L}(x, y, z) = \max \{G(gx, fx, fx), G(gy, fy, fy), G(gz, fz, fz)\}. \tag{3.17}$$

$$\mathcal{L}(x, y, z) = \max \{G(gx, gx, fx), G(gy, gy, fy), G(gz, gz, fz)\}. \tag{3.18}$$

$$\mathcal{L}(x, y, z) = aG(gx, gy, gz) + bG(gx, fx, fx) + cG(gy, fy, fy) + dG(gz, fz, fz), \tag{3.19}$$

where $a + b + c + d < 1$.

$$\mathcal{L}(x, y, z) = aG(gx, gy, gz) + bG(gx, gx, fx) + cG(gy, gy, fy) + dG(gz, gz, fz), \tag{3.20}$$

where $a + b + c + d < 1$.

Remark 3.7. Theorems 3.3-3.4 (under choices (3.17-3.20)) improve and generalize the results of Abbas and Rhoades [5, Theorem 2.4] and Mustafa et al. [21].

Remark 3.8. The results similar to Corollaries 3.1-3.2 can be outlined in respect of Theorem 3.3 together with Remarks 3.6-3.7, hence the details of all possible corollaries are not included here.

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REFERENCES

1. M. Abbas, A. R. Khan and T. Nazir: *Coupled common fixed point results in two generalized metric spaces*. Appl. Math. Comput. **217** (2011), 6328–6336.
2. M. Abbas, S. H. Khan and T. Nazir: *Common fixed points of R-weakly commuting maps in generalized metric spaces*. Fixed Point Theory Appl. 2011, 2011:41, 11 pp. MR283114
3. M. Abbas, T. Nazir and S. Radenović: *Some periodic point results in generalized metric spaces*. Appl. Math. Comput. **217**(8) (2010), 4094–4099. MR2739653
4. M. Abbas, T. Nazir and S. Radenović: *Common fixed point of generalized weakly contractive maps in partially ordered G-metric spaces*. Appl. Math. Comput. **218**(18) (2012), 9383–9395. MR2923035
5. M. Abbas and B. E. Rhoades: *Common fixed point results for noncommuting mappings without continuity in generalized metric spaces*. Appl. Math. Comput. **215**(1) (2009), 262–269. MR2568327
6. H. Aydi: *A fixed point result involving a generalized weakly contractive condition in G-metric spaces*. Bull. Math. Anal. Appl. **3**(4) (2011), 180–188.
7. H. Aydi: *A common fixed point of integral type contraction in generalized metric spaces*. J. Adv. Math. Stud. **5**(1) (2012), 111–117.
8. H. Aydi, B. Damjanović, B. Samet and W. Shatanawi: *Coupled fixed point theorems for nonlinear contractions in partially ordered G-metric spaces*. Math. Comput. Model. **54**(9-10) (2011), 2443–2450. MR2834646
9. H. Aydi, M. Postolache and W. Shatanawi: *Coupled fixed point results for (ψ,ϕ)-weakly contractive mappings in ordered G-metric spaces*. Comput. Math. Appl. **63**(1) (2012), 298–309. MR2863501
10. H. Aydi, W. Shatanawi and C. Vetro: *On generalized weakly G-contraction mapping in G-metric spaces*. Comput. Math. Appl. **62**(11) (2011), 4222–4229.
11. A. Branciari: *A fixed point theorem for mappings satisfying a general contractive condition of integral type*. Int. J. Math. Math. Sci. **29**(9) (2002), 531–536. MR1900344 (2003c:54075)
12. B. S. Choudhury, S. Kumar, Asha and K. Das: *Some fixed point theorems in G-metric spaces*. Math. Comput. Model. **54** (2011), 73–79. MR2801866

13. A. Djoudi and A. Aliouche: *Common fixed point theorems of Gregus type for weakly compatible mappings satisfying contractive conditions of integral type*. J. Math. Anal. Appl. **329**(1) (2007), 31–45. MR2306786 (2008c:47094)

14. M. Imdad, J. Ali and M. Tanveer: *Coincidence and common fixed point theorems for nonlinear contractions in Menger PM spaces*. Chaos, Solitons & Fractals 42(5) (2009), 3121–3129.

15. M. A. Khan and S. Chauhan: *Coincidence and fixed points in G-metric spaces*. Bull. Math. Anal. Appl. **4**(4) (2012), 67–75. MR3015652

16. J. Matkowski: *Fixed point theorems for mappings with contractive iterate at point*. Proc. Amer. Math. Soc. **62**(2) (1977), 344–348. MR0436113 (55 #9063)

17. Z. Mustafa: *A new structure for generalized metric Spaces with applications to fixed point theory*. Ph. D. Thesis, The University of Newcastle, Australia, 2005.

18. Z. Mustafa, H. Aydi and E. Karapinar: *On common fixed points in G-metric spaces using (E.A) property*. Comput. Math. Appl. **64**(6) (2012), 1944–1956. MR2960817

19. Z. Mustafa, M. Khandaqji and W. Shatanawi: *Fixed point results on complete G-metric spaces*. Studia Sci. Math. Hungar. **48**(3) (2011), 304–319. MR2883242 (2012i:47074)

20. Z. Mustafa and H. Obiedat: *A fixed point theorem of Reich in G-metric spaces*. CUBO **12**(1) (2010), 83–93. MR2730587 (2011i:54041)

21. Z. Mustafa, H. Obiedat and F. Awawdeh: *Some fixed point theorem for mappings on complete G-metric spaces*. Fixed Point Theorey Appl. **2008**, Article ID 18970, 12 pages. MR2438713 (2009h:54060)

22. Z. Mustafa, W. Shatanawi and M. Bataineh: *Fixed point theorem on uncomplete G-metric spaces*. J. Math. Stat. **2009**, Article ID. 283028, 10 pages.

23. Z. Mustafa, W. Shatanawi and M. Bataineh: *Existence of fixed points results in G-metric spaces*. Internat. J. Math. Math. Sci. **2009**, Article ID. 283028, 10 pages.

24. Z. Mustafa and B. Sims: *Some remarks concerning D-metric spaces*. Proceedings of International Conference on Fixed Point Theory and Applications, Yokoham Publishers, Valencia Spain, July 13-19, (2004), 189–198. MR2140217

25. Z. Mustafa and B. Sims: *A new approach to a generalized metric spaces*. J. Nonlinear Convex Anal. **7**(2) (2006), 289–297. MR2254125 (2007f:54049)

26. Z. Mustafa and B. Sims: *Fixed point theorems for contractive mappings in complete G-metric spaces*. Fixed Point Theory Appl. 2009, Art. ID 917175, 10 pp. MR2520267 (2010g:54050)

27. B. E. Rhoades: *Two fixed-point theorems for mappings satisfying a general contractive condition of integral type*. Internat. J. Math. Math. Sci., **2003**, 63, 4007–4013. MR2030391 (2005b:54074)

28. R. Saadati, S. M. Vaezpour, P. Vetro and B. E. Rhoades, *Fixed Point Theorems in generalized partially ordered G-metric spaces*. Math. Comput. Model. **52**(5-6) (2010), 797–801. MR2661765

29. W. Shatanawi: *Fixed point theory for contractive mappings satisfying ϕ-maps in G-metric spaces*. Fixed Point Theory Appl. **2010**, Article ID 181650, 9pp. MR2661198 (2011f:47100)

30. W. Shatanawi: *Some fixed point theorems in ordered G-metric spaces and application*. Abstr. Appl. Anal. **2011**, Article ID 126205, 11 pp. MR2802831

31. W. Shatanawi: *Common fixed point results for two self-maps in G-metric spaces*. Mat. Vesnik **65**(2) (2013), 143–150.

32. W. Shatanawi, S. Chauhan, M. Postolache, M. Abbas and S. Radenović: *Common fixed points for contractive mappings of integral type in G-metric spaces*. J. Adv. Math. Stud. **6**(1) (2013), 53–72.

33. W. Sintunavarat and P. Kumam: *Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces*. J. Appl. Math. **2011**, Article ID 637958, 14 pages, 2011. MR2822403

34. T. Suzuki: *Meir-Keeler contractions of integral type are still Meir-Keeler contractions*. Internat. J. Math. Math. Sci. **2007**, Article ID 39281, 6 pages, 2007. MR2285999 (2007k:54049)

35. C. Vetro: *On Branciari's theorem for weakly compatible mappings*. Appl. Math. Lett. **23**(6) (2010), 700–705. MR2609801 (2011d:47136)

36. P. Vijayaraju, B. E. Rhoades and R. Mohanraj: *A fixed point theorem for a pair of maps satisfying a general contractive condition of integral type*. Int. J. Math. Math. Sci. **2005**, 15, 2359–2364. MR2184475 (2006g:54050)

37. W. Long, M. Abbas, T. Nazir and S. Radenović, *Common fixed point for two pairs of mappings satisfying (E.A) property in generalized metric spaces*. Abstr. Appl. Anal. **2012**, Article ID 394830, (2012), 15 pages.

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